Relativistic effects on electron plasma waves

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The effect of relativistic electron-mass variation on the propagation of electron plasma waves is considered. Including thermal dispersive effects, a variational principle is used to find the possible final states of the linearly unstable modes. It is shown that dispersion will limit the nonlinear steepening predicted by the cold-fluid approach.

Intense laser radiation in plasmas, as needed for, e.g., laser-fusion experiments, require a nonlinear description. It has been shown[1] that in such situations, when the oscillatory velocity of electrons approaches the speed of light, the resulting variations in the electron-mass produce a contribution to the nonlinear refractive index which can be more important than that due to ponderomotive force effects. The propagation of an incoming electromagnetic wave, penetrating into the region of plasma, where its frequency is close to the local plasma frequency, can then be described by nonlinear optics.[2]

It is well known[3] that an electromagnetic wave obliquely incident on a density gradient will undergo mode conversion into an electron plasma wave at the critical density. Since the convection of the plasma wave is weak, the amplitude of the electrostatic mode can be even larger than that of the incoming electromagnetic wave. Recently,[4] the role of relativistic electron-mass variation in the generation of plasma waves by linear mode conversion has been investigated. For the initial stages of the mode conversion, the electromagnetic source has been incorporated into the relativistic equations for the excited plasma wave. In a more detailed publication,[5] the latter effect has been investigated thoroughly, showing that wave breaking accompanied by strong plasma heating saturates the amplitude of the mode-converted plasma wave at a much lower level than previously predicted.

For the mode-converted electron plasma wave, a linear instability has been found.[6] The nonlinear stage of the instability and the effect of ion motion on the instability have been investigated by computer simulation. In addition, the nonlinear evolution of the electron plasma waves has been discussed[4] analytically within the cold-fluid approach. Numerical solutions of the basic equations demonstrate shock formation and subsequent wave breaking.

In this paper, we study the nonlinear evolution of the large-amplitude mode-converted plasma wave by using a variational principle known from nonlinear optics. This approach allows us to predict the nonlinear evolution of the large-amplitude plasma wave as has been demonstrated before by numerical calculations. Furthermore, we show that inclusion of thermal dispersive effects will limit the nonlinear steepening found so far within the cold-fluid approach.

The dynamical behavior of a nonlinear electrostatic oscillation $\vec{E}=\vec{E}_0\cos\Phi$, where $\Phi=\omega t-\vec{k}\cdot\vec{r}$, can be determined from a variational principle which is widely used[7] in nonlinear optics. This principle uses the averaged Lagrangian L which in our case for a system of (relativistic) electrons in the electrostatic field $\vec{E}$ is

$$L=\frac{\epsilon_0}{2}\left(\frac{\partial\varphi}{\partial\vec{r}}\right)^2-nmc^2\left(1-\frac{(d\vec{\xi}/dt)^2}{c^2}\right)^{1/2}+ne\varphi-\tfrac{1}{2}\gamma\pi_0(\operatorname{div}\vec{\xi})^2. \qquad (1)$$

Here, n is the electron number density; m is the electron rest mass, $\vec{\xi}$ is the fluid displacement, related with the electron velocity $\vec{v}$ through $\vec{v}=d\vec{\xi}/dt$; and φ is the scalar potential. The last term in Eq. (1) represents the potential energy density associated with the pressure force,[8] where γ is the adiabaticity index and π_0 is the zeroth-order scalar pressure. We note that with application to the weak relativistic limit the thermal dispersive effects will balance weak amplitude nonlinearities and thus in the following the treatment of thermal effects as higher order contributions is justified.

The variational principle can be written in the form[7]

$$\delta\mathfrak{L}\equiv\delta\int\int L(-\Phi_{\vec{r}},\Phi_t,a)\,d\vec{r}\,dt=0, \qquad (2)$$

where a is the amplitude parameter and Φ is the phase determining the slowly varying wave vector $\vec{k}=-\Phi_{\vec{r}}$ and frequency $\omega=\Phi_t$.

From Eq. (2) together with Eq. (1) the basic relativistically invariant equations can be recovered by taking the appropriate variations with respect to $\vec{\xi}$, φ, etc. The pressure term $\pi=-\gamma\pi_0\operatorname{div}\vec{\xi}$ follows from the time-integrated pressure balance.

In the following we anticipate a one-dimensional model which can be easily generalized to higher dimensions.

Using the electrostatic approximation

$$\epsilon_0 \partial^2\varphi/\partial x \partial t = -env\,, \tag{3}$$

and Poisson's equation

$$\partial^2\varphi/\partial x^2 = e(n-n_0)/\epsilon_0, \tag{4}$$

we obtain, up to the third order in the amplitude E_0, the consistent expansions

$$\varphi = \frac{a}{k}\sin\Phi - \frac{a_2}{2k}\cos 2\Phi + \frac{a_3}{3k}\sin 3\Phi\,, \tag{5}$$

$$\frac{n}{n_0} = 1 - a\delta k\sin\Phi + 2a_2\delta k\cos 2\Phi - 3a_3\delta k\sin 3\Phi\,, \tag{6}$$

$$\xi = (\delta a + 2\delta^2 k a_2 a + \tfrac{1}{2}\delta^3 k^2 a^3)\cos\Phi + (\delta a_2 + \tfrac{1}{2}\delta^2 k a^2)\sin 2\Phi$$
$$- (\tfrac{1}{6}\delta^3 k^2 a^3 + \tfrac{2}{3}\delta^2 k a a_2 - \delta a_3)\cos 3\Phi\,, \tag{7}$$

where $\delta = \epsilon_0/en_0$.

Inserting the expansions (5)–(7) into Eq. (1) and collecting the zeroth harmonic contributions we obtain

$$L = \tfrac{1}{4}\epsilon_0(a^2+a_2^2) + \tfrac{1}{4}n_0 m\omega^2\delta^2[a^2 + 5\delta k a_2 a^2 + \delta^2 k^2 a^4 + 4a_2^2 + \tfrac{3}{16}(\omega^2/c^2)\delta^2 a^4] - \tfrac{1}{2}en_0\delta(a^2+a_2^2)$$
$$- \tfrac{1}{4}\gamma\pi_0 k^2\delta^2 a^2(1 + 8\delta k a_2 + 2\delta^2 k^2 a^2) - \gamma\pi_0 k^2\delta^2 a_2^2 - \tfrac{1}{4}\gamma\pi_0\delta^2\left(\frac{\partial a}{\partial x}\right)^2. \tag{8}$$

As expected, L is independent of the coefficient a_3 if we collect terms up to the fourth order in the amplitude parameter.

The coefficient a_2 can be eliminated by taking a variation with respect to a_2. We get the result

$$a_2 \approx -\tfrac{5}{6}\delta k a^2\,, \tag{9}$$

and thus the Lagrangian appropriate for the variation prescribed by Eq. (2) is

$$L = -\tfrac{1}{4}\epsilon_0 a^2(1 - \tfrac{25}{12}\delta^2 k^2 a^2)$$
$$+ \frac{\omega^2}{4\omega_p^2}\epsilon_0 a^2[1 - \tfrac{19}{6}\delta^2 k^2 a^2 + \tfrac{3}{16}(\omega^2/c^2)\delta^2 a^2]$$
$$- \tfrac{1}{4}\gamma\pi_0 k^2\delta^2 a^2 - \tfrac{1}{4}\gamma\pi_0\delta^2\left(\frac{\partial a}{\partial x}\right)^2. \tag{10}$$

The Euler equations corresponding to the variation with respect to a and Φ are

$$\frac{\partial}{\partial t}L_{a_t} + \frac{\partial}{\partial x}L_{a_x} - L_a = 0, \tag{11}$$

and

$$\frac{\partial}{\partial t}L_\omega - \frac{\partial}{\partial x}L_k = 0. \tag{12}$$

We note that Eq. (11) yields the nonlinear dispersion equation

$$1 - \frac{\omega^2}{\omega_p^2} + \frac{\gamma\pi_0 k^2}{m\omega_p^2} + [\tfrac{13}{6}\delta^2 k^2 - \tfrac{3}{8}(\omega^2/c^2)\delta^2]a^2 = 0, \tag{13}$$

if higher-order dispersive effects are neglected and where the zeroth-order dispersion relation $\omega^2 = \omega_p^2(1+\gamma k^2\lambda_e^2) \approx \omega_p^2$ has been used in evaluating the higher-order contributions.

In the following we use the ansatz

$$\Phi = \omega_p t - \epsilon^2 s\left(t, \frac{x}{\epsilon}\right), \tag{14}$$

for the nonlinear phase, The smallness parameter ϵ has been introduced in Eq. (14) in order to demonstrate the different scaling for the time and space variations. In the following, we shall not explicitly indicate this ordering anymore.

Introducing Eqs. (10) and (14) into Eqs. (11) and (12) we obtain the coupled set of equations

$$-\frac{\gamma\pi_0}{m\omega_p n_0}\frac{\partial^2 a}{\partial x^2} + a\left[2\frac{\partial s}{\partial t} + \frac{\gamma\pi_0}{m\omega_p n_0}\left(\frac{\partial s}{\partial x}\right)^2 - [\tfrac{3}{8}(\omega^2/c^2)\delta^2 - \tfrac{13}{6}\delta^2 k^2]a^2\right] = 0, \tag{15}$$

and

$$\frac{m\omega_p n_0}{\gamma\pi_0}\frac{\partial a^2}{\partial t} + \frac{\partial s}{\partial x}\frac{\partial a^2}{\partial x} + \frac{\partial^2 s}{\partial x^2}a^2 = 0. \tag{16}$$

The last two equations can be combined after introducing the complex envelope

$$\psi = ae^{is}\,, \tag{17}$$

to yield

$$2i\frac{\partial\psi}{\partial t} + \frac{\gamma\pi_0}{m\omega_p n_0}\frac{\partial^2\psi}{\partial x^2} + [\tfrac{3}{8}(\omega^2/c^2)\delta^2 - \tfrac{13}{6}\delta^2 k^2]\omega_p|\psi|^2\psi = 0. \tag{18}$$

Equation (18) is the cubic nonlinear Schrödinger equation. The general solution of Eq. (18) has been found by applying the inverse scattering method. For details we refer to the original paper by Zakharov and Shabat.[9]

Here, we want to stress the attention to some aspects of Eq. (18) which might explain the behavior of large amplitude electron plasma waves reported previously. First, by linearizing Eq. (18) one gets an instability if the potential is attractive ($k^2 < 9\omega^2/26c^2$). The corresponding growth rate

$$\gamma \approx \tfrac{3}{16}\omega_p(v_E^2/c^2), \tag{19}$$

where $v_E = eE_0/m\omega$, agrees with that found previously by a different method.

Secondly, for the case $k^2 > \frac{9}{26}(\omega^2/c^2)$, no modulational instability exists. Instead of this, wave breaking can take place if thermal dispersive effects are neglected. To demonstrate the possible nonlinear steepening in the latter case, we show that Eq. (18) can be transformed into a Korteweg-de Vries equation by introducing ρ and σ through

$$\psi = \rho^{1/2}\exp\left(i\int\frac{\sigma dx}{2p}\right), \tag{20}$$

where $p = \gamma\pi_0/2m\omega_p n_0$. For the expansion in terms of a small parameter μ about the constant state

$$\rho = \rho_0 + \mu\rho_1 + \mu^2\rho_2, \tag{21}$$

$$\sigma = \sigma_0 + \mu\sigma_1 + \mu^2\sigma_2, \tag{22}$$

the stretching[10]

$$x' = \mu^{1/2}\{x - [\sigma_0 + (-2pq\rho_0)^{1/2}]t\}, \tag{22}$$

$$t' = \mu^{3/2}t, \tag{23}$$

where $q = \frac{3}{16}(\omega^2/c^2)\delta^2 - \frac{13}{12}\delta^2k^2 < 0$, transforms the nonlinear Schrödinger equation into a Korteweg-de Vries equation. The nonlinear steepening and wave breaking then follows[11] from the latter description if thermal dispersive terms are neglected. The inclusion of dissipative effects would introduce some asymmetry into the above equations producing shock-like solutions.[7]

Finally, we want to mention that the process discussed here occurs at a rather fast time scale and may dominate over the ponderomotive force effects on the mode-converted electron plasma waves[12] for large laser intensities.

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[1]N. L. Tsintsadze, Zh. Eksp. Theor. Fiz. 59, 1251 (1971) [Sov. Phys.-JETP 32, 684 (1971)]; C. Max and F. Perkins, Phys. Rev. Lett. 27, 1342 (1971); P. Kaw and J. Dawson, Phys. Fluids 13, 472 (1970).

[2]S. A. Akmanov, L. P. Sukhorukov, and R. V. Khokhlov, Usp. Fiz. Nauk 93, 19 (1967) [Sov. Phys.-Uspekhi 10, 609 (1967)]; K. H. Spatschek, J. Plasma Physics (to be published).

[3]V. L. Ginzburg, *Propagation of Electromagnetic Waves in Plasma* (Gordon and Breach, New York, 1961), p. 377.

[4]J. F. Drake, Y. C. Lee, K. Nishikawa, and N. L. Tsintsadze, Phys. Rev. Lett. 36, 196 (1976).

[5]J. F. Drake and Y. C. Lee, Phys. Fluids 19, 1772 (1976).

[6]A. T. Lin and N. L. Tsintsadze, Phys. Fluids 19, 708 (1976).

[7]G. B. Whitham, *Linear and Nonlinear Waves* (Academic, New York, 1974).

[8]B. B. Kadomtsev, in *Reviews of Plasma Physics*, edited by M. A. Leontovich (Consultants, New York, 1966), Vol. 2, p. 153.

[9]V. E. Zakharov and A. B. Shabat, Zh. Eksp. Theor. Fiz. 61, 118 (1971) [Sov. Phys.-JETP 34, 62 (1972)].

[10]T. Taniuti and C. C. Wei, J. Phys. Soc. Jpn. 24, 941 (1968).

[11]R. C. Davidson, *Methods in Nonlinear Plasma Theory* (Academic, New York, 1972), Chap. 2.

[12]H. H. Chen and C. S. Liu, Phys. Rev. Lett. 37, 693 (1976); and University of Maryland Technical Report Number 77-006 (1977) (unpublished).