Multidetector photon statistics

Tadashi Aoki
Department of Pure and Applied Sciences, University of Tokyo, Meguro-ku, Tokyo, Japan
(Received 6 June 1977)

It is theoretically shown that detector quantum-noise-free measurement is feasible by a joint moment (JM) method which utilizes many photodetectors. The quantum/shot noise disappears in the statistical moments of JM photon counting. The statistical error in the JM measurement is calculated for light with Gaussian statistics and a Lorentzian spectrum and compared with that in the usual factorial moment method which utilizes a single photodetector. The JM measurement is shown to be useful for the measurement of very weak light with a wide spectrum.

I. INTRODUCTION

The probability distribution of photoelectrons counted during some time interval has been shown[1] to reflect the probability density of light intensity during the count interval. Much experimental work has been done on the photon statistics of various types of light such as a scattered light,[2] mixed light,[3] and laser light,[4] and several summaries and reviews have been made on the statistical studies of optical fields.[5] These experimental measurements of the fluctuation of light intensity are usually based on the factorial moments of photoelectrons counted by using a single photodetector. In this paper we consider a new method for the study of photon statistics by using many photodetectors.

When one measures the statistical properties of a light field, a photon counting method is usually employed. The probability $P(n)$ that n photoelectrons are detected in a time interval T is given by the Poisson transform of the probability density $P(W)$ of the integrated light intensity W,[1]

$$P(n)=\int_0^\infty \frac{(\alpha W)^n}{n!}\, e^{-\alpha W}P(W)\,dW, \qquad (1)$$

where $W=\int_0^T V^*(t)V(t)\,dt$, and α is a constant characteristic of a photodetector. $V(t)$ is the analytic signal[6] of a light field. From the above basic equation, we obtain directly the relation between the kth-order moment of W and the kth-order factorial moment (FM) of n,

$$\langle W^k\rangle=\langle n(n-1)(n-2)\ldots(n-k+1)\rangle\,; \qquad (2)$$

where $\langle\ldots\rangle$ represents a statistical average and we assumed α to be unity for simplicity. Usually one calculates the factorial moments of n from the measured $P(n)$ and thus obtains the moments of W. When one is given a set of the moments of W, one is allowed to identify the light field by the uniqueness theorem for the probability distribution and its moments.[7]

The dispersion of counted photoelectrons is given by

$$\langle n^2\rangle-\langle n\rangle^2=\langle n\rangle+(\langle W^2\rangle-\langle W\rangle^2)\,.$$

It is well known[8] that the first term on the right-hand side of the equation represents quantum/shot noise and the second term represents excess noise. As can be seen from Eq. (2), in the FM method the moments of W are obtained by evaluating the factorial moments of n. Each moment suffers from an experimental error caused by both the excess noise and the quantum/shot noise. The experimental error due to the quantum/shot noise, however, becomes more serious when we study weak light by the FM method. In the following sections we consider the joint-moment (JM) method with many photodetectors and it will be shown to be free from the quantum/shot noise. We apply these two methods to Gaussian light and calculate the experimental accuracies of these methods.

II. TWO-DETECTOR MEASUREMENT

It will be helpful to consider the following two-detector measurement before we discuss the generalized case of the JM measurement with k photodetectors. Consider that the light beam is split by a half-silvered mirror and the divided light beams strike two detectors as shown in Fig. 1. In the following, the two photodetectors are assumed to have the same efficiency of photodetection. Let n_1, n_2, and n denote the numbers of the photoelectrons counted by photodetectors 1 and 2, and 1 with the mirror removed, respectively. The average number of photoelectrons counted by photodetector 1 with the mirror removed should be the sum of the average numbers of photoelectrons counted by the photodetectors 1 and 2, i.e., the conservation

$$\langle n\rangle=\langle n_1\rangle+\langle n_2\rangle \qquad (3)$$

holds assuming that the half-silvered mirror only splits the incident light beam into a pair of beams of equal intensity and never makes any other disturbance to the light beams. The statistical dispersions ψ^2 and φ^2 of n and n_1+n_2 are defined as

$$\psi^2=\langle n^2\rangle-\langle n\rangle^2, \tag{4}$$

$$\begin{aligned}\varphi^2&=\langle (n_1+n_2)^2\rangle-\langle n_1+n_2\rangle^2\\&=\langle n_1^2\rangle-\langle n_1\rangle^2+\langle n_2^2\rangle-\langle n_2\rangle^2\\&\quad+2(\langle n_1n_2\rangle-\langle n_1\rangle\langle n_2\rangle).\end{aligned} \tag{5}$$

When the incident light beam is coherent, the probability distribution of n, n_1 and n_2 are given by a Poisson distribution.[1] ψ^2 and φ^2 are, therefore, given by

$$\psi^2=\langle n\rangle, \tag{6}$$

$$\varphi^2=\langle n_1\rangle+\langle n_2\rangle+2(\langle n_1n_2\rangle-\langle n_1\rangle\langle n_2\rangle). \tag{7}$$

Since $\psi^2=\varphi^2$, the reduced JM ψ_{12}, defined below, vanishes:

$$\psi_{12}=\langle n_1n_2\rangle-\langle n_1\rangle\langle n_2\rangle=0, \tag{8}$$

from Eqs. (3), (6), and (7). This implies that the quantum/shot noise does not appear in the joint moments. On the other hand, when Gaussian light beams strike the photodetectors, ψ^2 and φ^2 are given by[1]

$$\psi^2=\langle n\rangle+\langle n\rangle^2, \tag{9}$$

$$\begin{aligned}\varphi^2&=\langle n_1\rangle+\langle n_1\rangle^2+\langle n_2\rangle+\langle n_2\rangle^2+\langle n_2\rangle^2\\&\quad+2(\langle n_1n_2\rangle-\langle n_1\rangle\langle n_2\rangle),\end{aligned} \tag{10}$$

where, for simplicity, the count duration is assumed to be much shorter than the correlation time of the fluctuation. Again from the requirement $\psi^2=\varphi^2$, ψ_{12} is given by

$$\psi_{12}=\langle n_1n_2\rangle-\langle n_1\rangle\langle n_2\rangle=\langle n_1\rangle\langle n_2\rangle, \tag{11}$$

by using Eqs. (3), (9), and (10). It is thus evident that the reduced JM ψ_{12} has nonzero value only when the light intensity fluctuates, in contrast to the dispersion of n. It will be shown later that the reduced JM ψ_{12} represents only the excess noise of light intensity.

III. MULTIDETECTOR MEASUREMENT

Now let us develop the previous two-detector measurement to a k-detector measurement and make a more detailed discussion of the JM measurement. We suppose the measuring setup shown in Fig. 2. We use k photodetectors whose efficiencies of photodetection are, for simplicity, assumed to be unity. It is again assumed that the beam splitters do not cause any disturbance to the light field other than splitting the light field into k light fields whose fraction of the incident field amplitude is, for example, β_j for the jth beam. The analytic signal at the jth detector at r_j is, therefore, given by

$$V_j(r_j,t)=\beta_jV(r_k,t-\tau_j), \tag{12}$$

at time t, assuming linear polarization. Here, the analytic signal at the kth detector with all the beam splitters removed is denoted by $V(r_k,t)$, which is equal to the analytic signal of the incident light field at the first beam splitter except for its phase. The time difference τ_j is entered in order to adjust the phase difference of the analytic signals at r_j and r_k, and is assumed to be zero in the following by adjusting the location of the detectors so that all the transit times of light from the light source to the detectors are equal. The conservation of light intensity holds in general,

$$\sum_{j=1}^{k}|\beta_j|^2=1. \tag{13}$$

The average number of photoelectrons registered by the jth detector during T is proportional to the light intensity W_j, integrated during T, at the jth detector,

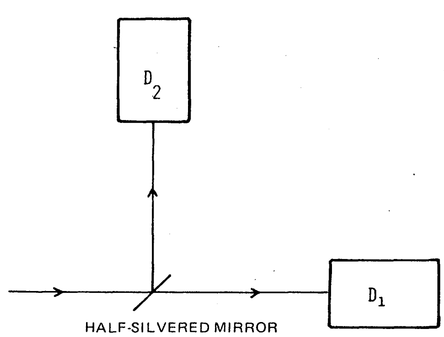

FIG. 1. Two-detector scheme.

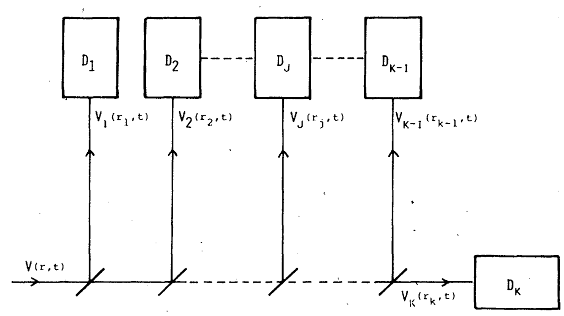

FIG. 2. k-detector scheme.

$$\langle n_j \rangle = \langle W_j \rangle , \tag{14}$$

and

$$\langle W_j \rangle = \left\langle \int_t^{t+T} V_j^*(r_j, t) V_j(r_j, t)\, dt \right\rangle .$$

Then the conservation of the average number of photoelectrons,

$$\langle n \rangle = \sum_{j=1}^{k} \langle n_j \rangle \tag{15}$$

and

$$\langle n_j \rangle = |\beta_j|^2 \langle n \rangle , \tag{16}$$

where

$$\langle n \rangle = \left\langle \int_t^{t+T} V^*(r_k, t) V(r_k, t)\, dt \right\rangle ,$$

holds in general. Using the joint probability density $P(W_1, W_2, \ldots, W_k)$ for the integrated light intensities $W_1, W_2, \ldots,$ and W_k, the joint probability distribution $P(n_1, n_2, \ldots, n_k)$ is expressed as[9]

$$P(n_1, n_2, \ldots, n_k) = \int \cdots \int P(W_1, W_2, \ldots, W_k) \prod \frac{W_j^{n_j}}{n_j!} e^{-W_j}\, dW_j . \tag{17}$$

The joint moment is, therefore, given by

$$\begin{aligned}\langle n_1 n_2 \cdots n_k \rangle &= \sum_{\text{all } n_j} n_1 n_2 \ldots n_k P(n_1, n_2, \ldots, n_k) \\ &= \int \cdots \int P(W_1, W_2, \ldots, W_k) \\ &\quad \times \prod W_j\, dW_j ,\end{aligned} \tag{18}$$

from Eq. (17). Since the beam splitters do not affect the statistical properties of the light field, $P(W_1, W_2, \ldots, W_k) \Pi\, dW_j$ can be reduced to the probability density $P(W)\, dW$ for W. Here W is the light intensity integrated during T at the kth detector with all the beam splitters removed. We obtain the reduced form

$$\begin{aligned}P(W_1, W_2, \ldots, W_k)\, dW_1\, dW_2 \ldots dW_k &= P(W)\delta(W_1 - |\beta_1|^2 W)\delta(W_2 - |\beta_2|^2 W) \ldots \delta(W_k - |\beta_k|^2 W) \\ &\quad \times dW_1\, dW_2 \ldots dW_k\end{aligned} \tag{19}$$

from Eq. (12). Here δ is the Dirac's delta function. Finally we obtain the simple expression for $\langle n_1 n_2 \ldots n_k \rangle$ as

$$\langle n_1 n_2 \ldots n_k \rangle = \left(\prod |\beta_j|^2 \right) \langle W^k \rangle , \tag{20}$$

by substituting Eq. (19) into Eq. (18).

For Gaussian light,[9] $\langle W^k \rangle$ is proportional to $\langle W \rangle^k$,

$$\langle W^k \rangle = F(k) \langle W \rangle^k . \tag{21}$$

Here, $F(k)$ represents the kth-order normalized factorial moment of n,

$$F(k) = \frac{\langle n(n-1)(n-2) \ldots (n-k+1) \rangle}{\langle n \rangle^k} . \tag{22}$$

Now we are given two basic formulas, Eqs. (2) and (20), for the FM and the JM measurements of the kth-order moment of light intensity. One can see from Eqs. (2) and (20) that the joint-moment $\langle n_1 n_2 \ldots n_k \rangle$ in the JM measurement is substantially equivalent to the factorial moment $\langle n(n-1)(n-2) \ldots (n-k+1) \rangle$ in the FM measurement. By substituting Eq. (21) into Eq. (20) and using Eq. (16), Eq. (20) is rewritten for Gaussian light as

$$\langle n_1 n_2 \ldots n_k \rangle = F(k) \langle n_1 \rangle \langle n_2 \rangle \ldots \langle n_k \rangle . \tag{23}$$

When $k = 2$, we obtain

$$\psi_{12} = \langle n_1 n_2 \rangle - \langle n_1 \rangle \langle n_2 \rangle = |\beta_1|^2 \cdot |\beta_2|^2 (\langle W^2 \rangle - \langle W \rangle^2)$$

from Eq. (20). It is, therefore, evident that the reduced JM ψ_{12} involves only the excess noise; i.e., the quantum/shot noise does not appear in the joint moments. In this respect, the JM method becomes advantageous to the FM method for weak light.

IV. STATISTICAL ERROR IN JM AND FM MEASUREMENTS

In order to obtain the kth-order moment of light intensity, we measure $\langle n_1 n_2 \ldots n_k \rangle$ in the JM measurement and calculate $\langle n(n-1)(n-2) \ldots (n-k+1) \rangle$ in the FM measurement. Assuming the number of data to be unity, we estimate the ratios of the amounts of these two experimental averages to their errors. First we calculate the error of the experimental value of $\langle n_1 n_2 \ldots n_k \rangle$ in the JM measurement. From Eqs. (17), (19), and (20),

$$\begin{aligned}\langle (n_1 n_2 \ldots n_k)^2 \rangle &= \left[\prod |\beta_j|^4 \right] \langle W^{2k} \rangle + \left[\prod |\beta_j|^2 \right] \langle W^k \rangle \\ &= F(2k) \langle n_1 \rangle^2 \langle n_2 \rangle^2 \ldots \langle n_k \rangle^2 \\ &\quad + F(k) \langle n_1 \rangle \langle n_2 \rangle \ldots \langle n_k \rangle .\end{aligned} \tag{24}$$

The dispersion σ_k^2, defined by

$$\sigma_k^2 = \langle (n_1 n_2 \ldots n_k)^2 \rangle - \langle n_1 n_2 \ldots n_k \rangle^2 , \tag{25}$$

is, therefore, given by

$$\sigma_k^2 = [F(2k) - F(k)^2] \prod \langle n_j \rangle^2 + F(k) \prod \langle n_j \rangle , \tag{26}$$

from Eqs. (23) and (24). Then S_{multi} defined by the ratio of the kth joint moment to its statistical error, $S_{\text{multi}} = \langle n_1 n_2 \ldots n_k \rangle / (\sigma_k^2)^{1/2}$, is

$$S_{\text{multi}} = \frac{F(k) \Pi \langle n_j \rangle}{\{[F(2k) - F(k)^2] \Pi \langle n_j \rangle^2 + F(k) \Pi \langle n_j \rangle\}^{1/2}} . \tag{27}$$

Next we consider the FM measurement of the kth-order moment of W. When one studies the kth-order moment of W by the FM measurement, one calculates $\langle n(n-1)(n-2) \ldots (n-k+1) \rangle$ and then $\langle W^k \rangle$ with the aid of Eq. (2). The factorial moment is evaluated from the measured $P(n)$ or directly obtained by the acquisition of the experimental value of $n(n-1)(n-2) \ldots (n-k+1)$. In both ways, the original quantity measured is the photoelectron count n. The finite number of samples inevitably introduces an error in the obtained distribution of the photoelectron count. Such a dispersion of the experimental value of $P(n)$, in turn, causes an error in the average of the factorial moment, which results in an error in the calculated moment of W. It is noted that these errors are almost solely attributed to the quantum/shot noise when $\langle n \rangle$ is much smaller than unity. It is at this point that the FM method is limited and replaced by the JM method. In this case, each experimental moment of n has an error of magnitude of same order as the lowest moment. Therefore, the dispersion of the factorial moment of n will be approximately determined by the statistical dispersion of n. We can at least obtain the lower limit of the dispersion of the factorial moment in the FM measurement in this way. For Gaussian light, the dispersion of n, which is denoted by σ_1^2, is given by

$$\sigma_1^2 = \langle n^2 \rangle - \langle n \rangle^2 = \langle n \rangle + [F(2) - 1] \langle n \rangle^2 , \tag{28}$$

from Eq. (22). Then we write the approximate formula for the ratio of the factorial moment to its error as

$$\begin{aligned} S_{\text{single}} &= \frac{\langle n(n-1)(n-2) \ldots (n-k+1) \rangle}{(k-1)! \, (\sigma_1^2)^{1/2}} \\ &= \frac{F(k) \langle n \rangle^k}{(k-1)! \, (\sigma_1^2)^{1/2}} = \frac{F(k) \langle n \rangle^{k-1/2}}{(k-1)!} , \end{aligned} \tag{29}$$

where we used the condition $\langle n \rangle \ll 1$. For weak light with a broad spectrum, S_{single} is small since $\langle n \rangle$ is small and $F(k)$ is reduced by the averaging out of the high speed fluctuation. This disadvantage becomes more serious for higher-order moments.

Now let us calculate the ratio between S_{multi} and S_{single}. From Eqs. (27) and (29), the ratio $S_k = S_{\text{multi}} / S_{\text{single}}$ is given by

$$S_k = \frac{(k-1)! \, \Pi \langle n_j \rangle}{\langle n \rangle^{k-1/2} \{[F(2k) - F(k)^2] \Pi \langle n_j \rangle^2 + F(k) \Pi \langle n_j \rangle\}^{1/2}} . \tag{30}$$

When we adjust the reflectivities of the beam splitters so that

$$\langle n_1 \rangle = \langle n_2 \rangle = \ldots = \langle n_k \rangle = \langle n \rangle / k$$

regarding the conservation of photoelectrons [Eq. (15)], then

$$S_k = \frac{(k-1)!}{\{[F(2k) - F(k)^2] \langle n \rangle^{2k-1} + F(k) k^k \langle n \rangle^{k-1}\}^{1/2}} . \tag{31}$$

If $\langle n \rangle = 0.1$, Eq. (31) is well approximated by the simple form

$$S_k = \frac{(k-1)!}{[F(k) k^k \langle n \rangle^{k-1}]^{1/2}} . \tag{32}$$

By using the value of $F(k)$ calculated for Gaussian light with a Lorentzian spectrum in our previous work,[10] S_k was plotted in Fig. 3 from $k = 2$ to 6 as a function of γT for $\langle n \rangle = 0.01$ and 0.1. Here γ denotes the spectral width of the optical field. The advantage of the JM method is quite apparent for the smaller value of $\langle n \rangle$ and for the greater value of γT from Fig. 3. The statistical error in the data average will be reduced inversely proportionally to the square root of the number of samples. Therefore the value of S_3 at $\gamma T = 10$

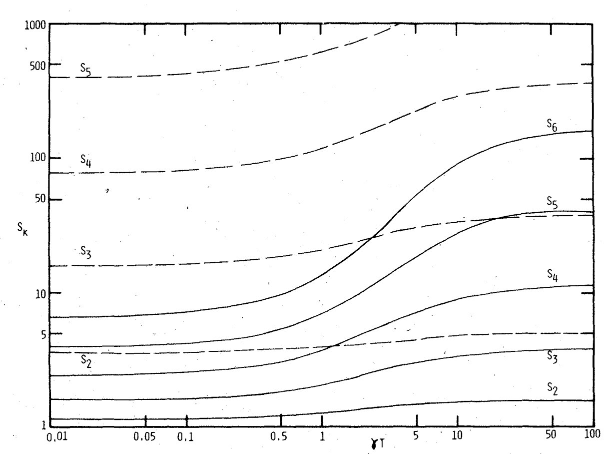

FIG. 3. Plots of the value of S_k for the Gaussian-Lorentzian light. The solid lines: $\langle n \rangle = 0.1$. Broken lines: $\langle n \rangle = 0.01$.

($\langle n\rangle = 0.1$), for instance, implies that we can obtain the third-order moment with the same accuracy by a JM measurement by measuring for an order-of-magnitude shorter time interval than would be required for a FM measurement.

The bunching property of photons due to the fluctuation of an optical field will be an alternative candidate utilized to reduce the effect of the quantum/shot noise. Here, we consider some remarks of the conditional probability (CP) method.[11] The conditional probability $P_c(n_1, \tau{:}n)$ describes the event that n photocount is registered τ seconds after n_1 photocount was registered, and is straightforwardly calculated from $P(n_1)$ and a joint probability $P(n_1, \tau{:}n)$. For $n_1 = 1$ and $\tau = 0$, for example, $P_c(1{:}n)$ is related to an unconditional probability $P(n)$ by the relation

$$P_c(1{:}n) = (1+\langle n\rangle)\left[\frac{1+\langle n\rangle}{1+2\langle n\rangle}\right]^{n+2}(n+1)P(n),$$

for Gaussian light.[12] The kth conditional factorial moment $F_1(k)$ defined by

$$F_1(k) = \sum_0^\infty n(n-1)(n-2)\ldots(n-k+1)P_c(1{:}n),$$

is easily related to $F(k)$ as

$$F_1(k) = (k+1)F(k), \quad (\langle n\rangle \ll 1),$$

for Gaussian light with low intensity. The kth factorial moment is enhanced by a factor of $(k+1)$ while the quantum/shot noise is unchanged. One of the disadvantages of the CP method is the decrease of a sampling rate. The sampling rate becomes $P(n_1)$ times the rate which would be in the unconditional counting measurement. It is evident that the correlation between the two successive events vanishes and such enhancement can not be expected for τ greater than the correlation time of the optical field. The details of the dependence of the enhancement factor on the time delay τ, the general form of $F_{n_1}(k)$ for various values of n_1, and a comparison with the JM method are under study and will be presented elsewhere.

V. CONCLUSION

The statistical average of the product of the quantum/shot noises from photodetectors vanishes while that of the self-product of the quantum/shot noise from a single photodetector does not disappear. By standing upon this distinction, the joint-moment measurement was shown to be advantageous for the study of the statistics of weak light. The joint-moment method could be successfully applied to the investigation of photon statistics of extremely faint light scattered from dispersed matter.

[1]L. Mandel, E. C. G. Sudershan, and E. Wolf, Proc. Phys. Soc. Lond. 84, 435 (1964).

[2]E. Jakeman, C. J. Oliver, and E. R. Pike, J. Phys. A 1, 406 (1968); B. Crosignani, B. Daino, and P. Di Porto, J. Appl. Phys. 42, 399 (1971); T. Aoki, Y. Okabe, and K. Sakurai, Phys. Rev. A 10, 259 (1974); P. N. Pusey and E. Jakeman, J. Phys. A 8, 392 (1975); Y. Okabe, T. Aoki, and K. Sakurai, J. Phys. Soc. Jpn. 40, 798 (1976); H. Shimizu and G. Matsumoto, Opt. Commun. 16, 197 (1976).

[3]W. Martienssen and E. Spiller, Phys. Rev. 145, 285 (1966); T. Aoki, Y. Endo, H. Takayanagi, and K. Sakurai, Phys. Rev. A 13, 853 (1976).

[4]C. F. Freed and H. A. Haus, IEEE J. Quantum Electron. 2, 90 (1966); E. Jakeman, C. J. Oliver, E. R. Pike, M. Lax, and M. Zwanziger, J. Phys. A 3, L52 (1970); F. T. Arecchi and V. Degiorgio, Phys. Rev. A 3, 1108 (1971); D. Meltzer and L. Mandel, Phys. Rev. A 3, 1763 (1971); J. A. Abate, H. J. Kimble, and L. Mandel, Phys. Rev. A 14, 788 (1976).

[5]B. Chu, *Laser Light Scattering* (Academic, New York, 1974); H. Z. Cummins and E. R. Pike, *Photon Correlation and Light Beeting Spectroscopy* (Plenum, New York, 1974); B. Crosignani, P. Di Porto, and M. Bertolotti, *Statistical Properties of Scattered Light* (Academic, New York, 1975); L. Mandel, *The Case for and against Semiclassical Radiation Theory*, in *Progress in Optics*, edited by E. Wolf (North-Holland, Amsterdam, 1976).

[6]L. Mandel and E. Wolf, Rev. Mod. Phys. 37, 231 (1965).

[7]H. Cramer, *Mathematical Methods of Statistics* (Princeton University, Princeton, N.J., 1963).

[8]L. Mandel, E. C. G. Sudershan, and E. Wolf, Proc. Roy. Soc. Lond. 84, 435 (1964).

[9]C. L. Mehta, in *Progress in Optics*, edited by E. Wolf (North-Holland, Amsterdam, 1970).

[10]Aoki, Okabe, and Sakurai, Ref. 2. γ denotes a spectral width of the optical field as used in the previous paper.

[11]L. Mandel, *Proceedings of the Symposium on Modern Optics, Microwave Resonance and Instrumentation Symposia Ser.* (Polytechnic, New York, 1967), Vol. 17, p. 143–165.

[12]Evaluated from formula given in F. T. Arecchi, A. Berne, and A. Sona, Phys. Rev. Lett. 17, 260 (1966).