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Three-particle scattering rates and singularities of the T matrix. II.

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We evaluate the contributions of double-scattering singularities to three-particle scattering rates. It is shown that these contributions are finite and consistent with the physical interpretation of double-scattering processes. The "double-scattering regions" of momentum space where these processes contribute are calculated, and it is shown that in these regions the double-scattering contributions dominate and the regular part of the T matrix can be neglected.

I. INTRODUCTION

This is the second of two papers about counting rates in nonrelativistic three-particle collisions. In the previous paper[1] we set up a quantum ensemble to describe two beams of particles, called α and β, incident on a third, heavy particle that is fixed at the origin. We derived expressions for various different counting densities. The two-particle density $n_{\alpha\beta}(\vec{p},\vec{q})$ was defined as the density for finding the particles α and β scattered with momenta in the neighborhoods of $\vec{p}$ and $\vec{q}$. Provided the beams are switched on for a sufficiently long time T, the density $n_{\alpha\beta}(\vec{p},\vec{q})$ is proportional to T (as one would expect) and it is natural to work with the *rate* $r_{\alpha\beta}(\vec{p},\vec{q}) = n_{\alpha\beta}(\vec{p},\vec{q})/T$. The *one-particle* rate $r_\alpha(\vec{p})$ for observing particle α with momentum near $\vec{p}$ irrespective of the momentum of β is obtained by integrating over all values of $\vec{q}$,

$$r_\alpha(\vec{p}) = \int d^3q\, r_{\alpha\beta}(\vec{p},\vec{q}) . \tag{1.1}$$

The main purpose of both of these papers is to examine the behavior of the scattering rates in the neighborhood of the well-known disconnected and double-scattering singularities of the three-particle T matrix. In Ref. 1 we showed that if the observed final momenta $\vec{p}$ and $\vec{q}$ are chosen to avoid these singularities, then the rate $r_{\alpha\beta}$ is proportional to the modulus squared of the T matrix and is

$$r_{\alpha\beta}(\vec{p},\vec{q}) = (2\pi)^7 \rho_\alpha \rho_\beta \delta(E_{pq} - E_{p_0 q_0}) |\langle \vec{p},\vec{q}|T|\vec{p}_0,\vec{q}_0\rangle|^2 , \tag{1.2}$$

where $\vec{p}_0$ and $\vec{q}_0$ are the mean momenta of the two incident wave packets.

If the observed final momenta include a disconnected or double-scattering singularity, then the scattering rate cannot be given by (1.2) since the latter becomes badly infinite. In Ref. 1 we considered the case of the disconnected singularities, which contribute on certain "shells" in final momentum space.[2] We found that on these shells the scattering count remains finite. Furthermore, in the immediate neighborhood of the shells, the count is completely dominated by the singular terms and all other terms can be neglected.

In the present paper we consider the case of the double-scattering singularities. We show that, for these too, the observed scattering count remains perfectly finite and is entirely consistent with the physical interpretation of the double-scattering process. When the final momenta lie in the "double-scattering regions" where these processes occur the scattering rate is proportional to the macroscopic linear size of the incident beams. This means that in the double-scattering regions, the corresponding singularities dominate and all other terms can be neglected.

In Sec. II we review briefly some of the notation and results of Ref. 1. We also introduce the simplifying assumption that the incident wave packets are Gaussian. This assumption can be avoided but lets us perform several integrals explicitly and considerably simplifies our subsequent calculations. In Sec. III we derive a general formula for the contribution of any of the four[3] double-scattering processes. In Sec. IV we use this formula to analyze each of the double-scattering contributions in detail. Sec. V summarizes the main results of both Ref. 1 and the present paper.

II. PRELIMINARIES

The experiment we consider consists of two beams, one of particles α, the other of particles β, incident on a fixed heavy particle at the origin 0. The beams are cylindrical, with axes passing through 0 and cross sectional areas $\mathcal{S}_\alpha = \pi R_\alpha^2$ and $\mathcal{S}_\beta = \pi R_\beta^2$. They are switched on at time $t = 0$ and off again at $t = T$. The densities of particles in the beams we denote by ρ_α and ρ_β.

A typical wave packet in the beam α is denoted

$$\psi_{\vec{a}}(\vec{p}) = \exp(-i\vec{a}\cdot\vec{p})\psi(\vec{p}),$$

where $\psi(\vec{p})$ is a wave function centered at $\vec{p}_0$ in momentum space and at the origin in coordinate space. The vector $\vec{a}$ is the position of the packet within the beam. The corresponding packet for particles β is

$$\phi_{\vec{b}}(\vec{q}) = \exp(-i\vec{b}\cdot\vec{q})\phi(\vec{q}).$$

Provided the duration T of the beams is long enough, the counting density $n_{\alpha\beta}$ is proportional to T and we therefore work with the rate $r_{\alpha\beta} = n_{\alpha\beta}/T$. In Ref. 1, Eq. (3.20), we showed that this rate is given by the integral[4]

$$r_{\alpha\beta}(\vec{p},\vec{q}) = 2\pi\rho_\alpha\rho_\beta\delta(E_{pq} - E_{p_0q_0})\int_{S_\alpha\times\infty} d^3a \int_{S_\beta\times\infty} d^3b \left|\int d^3p' \int d^3q' \langle\vec{p},\vec{q}|T|\vec{p}\,',\vec{q}\,'\rangle\psi_{\vec{a}}(\vec{p}\,')\phi_{\vec{b}}(\vec{q}\,')\right|^2. \tag{2.1}$$

Here the integrals over $\vec{a}$ and $\vec{b}$ run over infinite cylinders of cross sections S_α and S_β.

To simplify our calculations we now suppose our wave functions $\psi(\vec{p})$ and $\phi(\vec{q})$ are both Gaussians,

$$\psi(\vec{p}) = (\pi\gamma^2)^{-3/4}\exp[-(\vec{p}-\vec{p}_0)^2/2\gamma^2] \tag{2.2}$$

and

$$\phi(\vec{q}) = (\pi\gamma^2)^{-3/4}\exp[-(\vec{q}-\vec{q}_0)^2/2\gamma^2]. \tag{2.3}$$

We assume, as usual, that the width[5] γ is much smaller than p_0 and q_0, and that the spatial width $1/\gamma$ is much smaller than the beam size, but much larger than the interaction radius.

The advantage of the Gaussian form is that the three-dimensional wave function can be factored as the product of three one-dimensional functions of any conveniently chosen orthogonal coordinates. We can immediately exploit this factoring of the Gaussian to simplify the integral (2.1). We decompose the vectors $\vec{a}$ and $\vec{p}\,'$ as the sums of their components, $\vec{a}_\parallel + \vec{a}_\perp$ and $\vec{p}\,'_\parallel + \vec{p}\,'_\perp$ parallel and normal to $\vec{p}_0$, and then write

$$\psi_{\vec{a}}(\vec{p}\,') = \exp(-ia_\parallel p'_\parallel)\psi_\parallel(p'_\parallel)\exp(-i\vec{a}_\perp\cdot\vec{p}\,'_\perp)\psi_\perp(\vec{p}\,'_\perp).$$

The integral over $\vec{a}$ in (2.1) can be rewritten as

$$\int d^3a = \int_{-\infty}^{\infty} da_\parallel \int_{S_\alpha} d^2a_\perp.$$

If we make corresponding decompositions of $\vec{b}$, $\vec{q}\,'$, and $\phi_{\vec{b}}(\vec{q}\,')$, then the integrations over $a_\parallel$ and $b_\parallel$ can be performed in the familiar way. Taking advantage of the well peaked nature of $\psi_\parallel(p'_\parallel)$ and $\phi_\parallel(q'_\parallel)$ we obtain (dropping the primes from $\vec{p}\,'_\perp$ and $\vec{q}\,'_\perp$)

$$r_{\alpha\beta}(\vec{p},\vec{q}) = (2\pi)^3\rho_\alpha\rho_\beta\delta(E_{pq} - E_{p_0q_0})\int_{S_\alpha} d^2a_\perp \int_{S_\beta} d^2b_\perp \left|\int d^2p_\perp \int d^2q_\perp \langle\vec{p},\vec{q}|T|\vec{p}_0+\vec{p}_\perp,\vec{q}_0+\vec{q}_\perp\rangle \times\exp[-i(\vec{a}_\perp\cdot\vec{p}_\perp + \vec{b}_\perp\cdot\vec{q}_\perp)]\psi_\perp(\vec{p}_\perp)\phi_\perp(\vec{q}_\perp)\right|^2. \tag{2.4}$$

If $\vec{p}$ and $\vec{q}$ are chosen so as to avoid any singularities of the T matrix, then (as discussed in Ref. 1) it is easily seen that (2.4) reduces to the familiar result (1.2). However, our main interest is to see how (2.4) behaves when $\vec{p}$ and $\vec{q}$ are near singularities. The case of the disconnected singularities was treated in Ref. 1; that of the double-scattering singularities will be analyzed in what follows. In Secs. III and IV we shall always suppose that $\vec{p}$ and $\vec{q}$ avoid the shells where the disconnected processes contribute. In Sec. V we shall review the results of both papers and discuss the contributions of all processes for all values of the final momenta.

III. GENERAL EXPRESSION FOR DOUBLE-SCATTERING CONTRIBUTIONS

A. Singularities

The double-scattering singularities that we wish to discuss are conveniently represented by diagrams as in Fig. 1. These four diagrams illustrate clearly the interpretation of the singularities. For example, the singularity corresponding to process I is the pole

$$\langle\vec{p},\vec{q}|T_{\rm I}|\vec{p}_0,\vec{q}_0\rangle = \frac{t_{\alpha\beta}t_\beta}{E_{p_0q_0} - E'' + i0}, \tag{3.1}$$

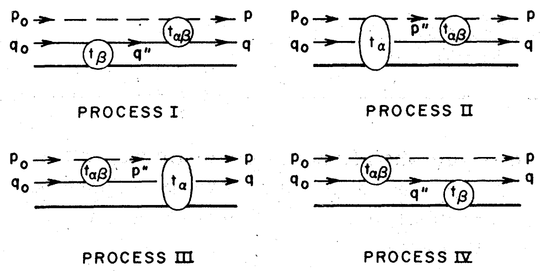

FIG. 1. The four double-scattering processes.

where E'' is the intermediate energy (which is defined more precisely below) and $t_{\alpha\beta}$ and t_β are the two-particle T matrices for scattering of α's off β's and β's off the fixed center. This pole arises because particle β can scatter off the fixed target, propagate freely, and then scatter off particle α. This sequence of two separate binary collisions is what the diagram illustrates. The other three singularities have similar interpretations.

It should be mentioned here that in the general three-body problem there are two more double-scattering processes, in which first α and then β scatter independently off the third particle, and *vice versa*. In the special case (which we consider) that the third particle is fixed, these two poles cancel to give a single delta function, which we have included in the "separate-scattering term" treated in Ref. 1. Therefore we do not need to consider these last two double-scattering processes here.

All four of the double-scattering poles that we discuss can be written in the general form

$$\langle \vec{p}, \vec{q}\,|T_i|\vec{p}_0, \vec{q}_0\rangle = \frac{\tilde{T}_i(\vec{p}, \vec{q}, \vec{p}_0, \vec{q}_0)}{E_{p_0q_0} - E'' + i0} \tag{3.2}$$

with i=I, II, III, IV. Here $\tilde{T}_i$ is the product of two two-particle T matrices and is a smooth function of its arguments. The energy E'' is the kinetic energy of the two intermediate particles, the momentum of the particle that partakes in both collisions being determined by conservation of momentum in the α-β collision. For example, in process I, E'' is

$$E'' = \frac{p_0^2}{2m_\alpha} + \frac{q''^2}{2m_\beta},$$

with $\vec{q}''$ determined by conservation of momentum as

$$\vec{q}'' = \vec{p} + \vec{q} - \vec{p}_0 .$$

In all cases E'' is a well-defined function of the external momenta $\vec{p}, \vec{q}, \vec{p}_0, \vec{q}_0$.

To evaluate the double-scattering contributions to the counting rate we must substitute (3.2), with $\vec{p}_0$ and $\vec{q}_0$ replaced by $\vec{p}_0 + \vec{p}_\perp$ and $\vec{q}_0 + \vec{q}_\perp$, into the integral (2.4) for $r_{\alpha\beta}$. Because the wave functions $\psi_\perp(\vec{p}_\perp)$ and $\phi_\perp(\vec{q}_\perp)$ are sharply localized around $\vec{p}_\perp = \vec{q}_\perp = 0$ we can remove the smooth numerator $\tilde{T}_i$ from the integrals with $\vec{p}_\perp$ and $\vec{q}_\perp$ replaced by zero. We *cannot* do the same with the denominator if $\vec{p}$ and $\vec{q}$ are anywhere near the points where it can vanish. On the other hand we can make a Taylor expansion of E'' as

$$\begin{aligned} E''(\vec{p}, \vec{q}, \vec{p}_0 + \vec{p}_\perp, \vec{q}_0 + \vec{q}_\perp) &\approx E''(\vec{p}, \vec{q}, \vec{p}_0, \vec{q}_0) + \frac{\partial E''}{\partial \vec{p}_0}\cdot \vec{p}_\perp + \frac{\partial E''}{\partial \vec{q}_0}\cdot \vec{q}_\perp \\ &\equiv E''(\vec{p}, \vec{q}, \vec{p}_0, \vec{q}_0) + \vec{v}\cdot\vec{p}_\perp + \vec{u}\cdot\vec{q}_\perp \end{aligned} \tag{3.3}$$

where the velocities $\vec{v}$ and $\vec{u}$ are just the derivatives of E'' with respect to $\vec{p}_0$ and $\vec{q}_0$ (and depend on which of the four double-scattering processes is being considered).

If we substitute (3.3) into the pole (3.2) we obtain (for small $p_\perp$ and $q_\perp$)

$$\langle \vec{p}, \vec{q}\,|T_i|\vec{p}_0 + \vec{p}_\perp, \vec{q}_0 + \vec{q}_\perp\rangle \approx \frac{\tilde{T}_i(\vec{p}, \vec{q}, \vec{p}_0, \vec{q}_0)}{\Delta_i - \vec{v}_\perp\cdot\vec{p}_\perp - \vec{u}_\perp\cdot\vec{q}_\perp + i0}. \tag{3.4}$$

Here we have introduced

$$\begin{aligned} \Delta_i &= \Delta_i(\vec{p}, \vec{q}, \vec{p}_0, \vec{q}_0) \\ &= E_{p_0q_0} - E''(\vec{p}, \vec{q}, \vec{p}_0, \vec{q}_0), \end{aligned} \tag{3.5}$$

which measures the distance of the external momenta from the pole. If Δ_i is large, we are observing far from the singularity, which is therefore harmless; if Δ_i is small or zero, we are close to the singularity. In writing (3.4) we have also replaced $\vec{v}\cdot\vec{p}_\perp$ and $\vec{u}\cdot\vec{q}_\perp$ in (3.3) by $\vec{v}_\perp\cdot\vec{p}_\perp$ and $\vec{u}_\perp\cdot\vec{q}_\perp$ as is obviously legitimate.

B. Contribution to the scattering rate

We are now ready to start calculating the contribution of any one of the double-scattering poles to the observed counting rate. Substituting (3.4) into (2.4) we find for the ith double-scattering contribution,

$$\begin{aligned} r_{\alpha\beta}^{(i)}(\vec{p}, \vec{q}) = (2\pi)^3\rho_\alpha\rho_\beta\,\delta(E_{pq} - E_{p_0q_0}) \\ \times |\tilde{T}_i(\vec{p}, \vec{q}, \vec{p}_0, \vec{q}_0)|^2\,\xi(\Delta_i), \end{aligned} \tag{3.6}$$

where

$$\xi(\Delta)=\int_{S_\alpha} d^2a_\perp \int_{S_\beta} d^2b_\perp \left| \int d^2p_\perp \int d^2q_\perp \frac{\exp[-i(\vec{p}_\perp\cdot\vec{a}_\perp+\vec{q}_\perp\cdot\vec{b}_\perp)]\psi_\perp(\vec{p}_\perp)\phi_\perp(\vec{q}_\perp)}{\vec{v}_\perp\cdot\vec{p}_\perp+\vec{u}_\perp\cdot\vec{q}_\perp-\Delta-i0}\right|^2 . \tag{3.7}$$

In (3.6) the factors multiplying $\xi(\Delta)$ are all expected and well behaved. All of the interesting effects are contained in the function $\xi(\Delta)$, which we must now evaluate. In particular, we want to know how $\xi(\Delta)$ behaves as Δ approaches 0.

To evaluate the integrals in (3.7) it is convenient to abbreviate our notation and introduce three four-dimensional vectors defined as the pairs

$$\vec{A}=(\vec{a}_\perp,\vec{b}_\perp),\quad \vec{P}=(\vec{p}_\perp,\vec{q}_\perp),\quad \vec{V}=(\vec{v}_\perp,\vec{u}_\perp). \tag{3.8}$$

We define a scalar product for these four-dimensional vectors in the natural way, such that

$$\vec{A}\cdot\vec{A}'=\vec{a}_\perp\cdot\vec{a}_\perp'+\vec{b}_\perp\cdot\vec{b}_\perp' .$$

The product wave function in (3.7) can be written as

$$\Psi(\vec{P})=\psi_\perp(\vec{p}_\perp)\phi_\perp(\vec{q}_\perp),$$

and is just a Gaussian in the four-dimensional variable $\vec{P}$,

$$\Psi(\vec{P})=(\gamma^2\pi)^{-1}\exp(-P^2/2\gamma^2). \tag{3.9}$$

The whole integral (3.7) now takes the form

$$\xi(\Delta)=\int_{S_\alpha\times S_\beta} d^4A \left| \int d^4P\, \frac{\exp(-i\vec{A}\cdot\vec{P})\Psi(\vec{P})}{\vec{V}\cdot\vec{P}-\Delta-i0}\right|^2 . \tag{3.10}$$

To isolate the effects of the denominator in (3.10) it is clearly convenient to decompose our four-vectors $\vec{A}$ and $\vec{P}$ into their components parallel and normal to $\vec{V}$,

$$\vec{A}=\vec{A}_\parallel+\vec{A}_\perp,\quad \vec{P}=\vec{P}_\parallel+\vec{P}_\perp .$$

With this notation, $\vec{V}\cdot\vec{P}=VP_\parallel$, where V denotes the magnitude of the four-vector $\vec{V}$,

$$V=(\vec{v}_\perp^{\,2}+\vec{u}_\perp^{\,2})^{1/2} .$$

The wave function $\Psi(\vec{P})$ can be factored as $\Psi_\parallel(P_\parallel)\Psi_\perp(\vec{P}_\perp)$ and the integral (3.10) can be broken down as

$$\xi(\Delta)=\int d^4A \left| \int d^3P_\perp \exp(-i\vec{A}_\perp\cdot\vec{P}_\perp)\Psi_\perp(\vec{P}_\perp) \times \int dP_\parallel \frac{\exp(-iA_\parallel P_\parallel)\Psi_\parallel(P_\parallel)}{VP_\parallel-\Delta-i0}\right|^2 . \tag{3.11}$$

Here the three-dimensional integral over $\vec{P}_\perp$ is just the Fourier transform $\tilde{\Psi}_\perp$ of $\Psi_\perp$. All effects of the dangerous denominator are contained in the one-dimensional integral over $P_\parallel$, which we can rewrite as

$$\begin{aligned}\int_{-\infty}^{\infty} dP_\parallel \frac{\exp(-iA_\parallel P_\parallel)}{VP_\parallel-\Delta-i0}\Psi_\parallel(P_\parallel) &= \int_{-\infty}^{\infty} dP_\parallel \frac{\exp(-iA_\parallel P_\parallel)}{VP_\parallel-\Delta-i0}\{[\Psi_\parallel(P_\parallel)-\Psi_\parallel(\Delta/V)]+\Psi_\parallel(\Delta/V)\} \\ &= \int_{-\infty}^{\infty} dP_\parallel \exp(-iA_\parallel P_\parallel)g(P_\parallel)+\Psi_\parallel(\Delta/V)\int_{-\infty}^{\infty} dP_\parallel \frac{\exp(-iA_\parallel P_\parallel)}{VP_\parallel-\Delta-i0} .\end{aligned} \tag{3.12}$$

In (3.12) the first term is the Fourier transform $\tilde{g}$ of the function

$$g(P_\parallel)=[\Psi_\parallel(P_\parallel)-\Psi_\parallel(\Delta/V)]/(VP_\parallel-\Delta) \tag{3.13}$$

which is continuous at $P_\parallel=\Delta/V$. The singularity is contained in the second term, which can be evaluated explicitly and is proportional to the step function $\Theta(-A_\parallel)$. Inserting these into (3.11) we obtain

$$\xi(\Delta)=(2\pi)^3\int_{S_\alpha\times S_\beta} d^4A\,|\tilde{\Psi}_\perp(\vec{A}_\perp)|^2 \left|(2\pi)^{1/2}\tilde{g}(A_\parallel)+\frac{2\pi i}{V}\Theta(-A_\parallel)\Psi_\parallel(\Delta/V)\exp(-iA_\parallel\Delta/V)\right|^2 \tag{3.14}$$

$$=\xi_1(\Delta)+\xi_2(\Delta)+\xi_3(\Delta), \tag{3.15}$$

where ξ_1 denotes the term involving $|\tilde{g}|^2$, ξ_2 is that involving $|\Psi_{\parallel}|^2$, and ξ_3 is the cross term involving $g^*\Psi_{\parallel}$.

C. Evaluation of ξ_1, ξ_2, ξ_3

The first term in (3.14) is

$$\xi_1(\Delta) = (2\pi)^4 \int_{s_\alpha \times s_\beta} d^4A\, |\tilde{\Psi}_\perp(\vec{A}_\perp)|^2 |\tilde{g}(A_\parallel)|^2 .$$

The important point is that $g(P_\parallel)$ is free of singularities and so that this integral is unchanged if we extend the region of integration to include all $\vec{A}$. When we do this, the integration over $\vec{A}_\perp$ disappears as a normalization integral and, using the Parseval identity, we find

$$\xi_1(\Delta) = (2\pi)^4 \int_{-\infty}^{\infty} dP_\parallel\, |g(P_\parallel)|^2 .$$

The function $g(P_\parallel)$ is given by (3.13). If we recall that $\Psi_\parallel(P_\parallel)$ is well peaked about $P_\parallel = 0$ with width γ, we see immediately that when Δ is large,

$$\xi_1(\Delta) \approx (2\pi)^4/\Delta^2 \quad (\Delta \gg V\gamma) . \tag{3.16}$$

Thus, when Δ is large and we are far away from the pole, $\xi_1(\Delta)$ shows precisely the expected contribution from the "tail" of the pole of the T matrix. On the other hand, when Δ is small $\xi_1(\Delta)$ does *not* have any singularity. In fact $\xi_1(0)$ is easily evaluated explicitly, and one finds that as $\Delta \to 0$,

$$\xi_1(\Delta) \to 2(\sqrt{2} - 1)(2\pi)^4/V^2\gamma^2 . \tag{3.17}$$

We shall see directly that when Δ is small the second term ξ_2 is much larger than ξ_1. Thus we can combine (3.16) and (3.17) and write

$$\xi_1(\Delta) = (2\pi)^4/(\Delta^2 + V^2\gamma^2) . \tag{3.18}$$

For large Δ ($\Delta \gg V\gamma$) this is an excellent approximation; for small Δ (where we are going to see that ξ_1 is unimportant anyway), it gives the correct order of magnitude.

The second term in (3.14) is

$$\xi_2(\Delta) = (2\pi)^5 \frac{|\Psi_\parallel(\Delta/V)|^2}{V^2} \int_{s_\alpha \times s_\beta} d^4A\, \Theta(-A_\parallel) \times |\tilde{\Psi}_\perp(\vec{A}_\perp)|^2 . \tag{3.19}$$

This integral grows linearly with the beam radius and can certainly not be extended to infinity. We must examine carefully the region of integration, which is defined by the conditions

$$a_\perp \leq R_\alpha \quad \text{and} \quad b_\perp \leq R_\beta , \tag{3.20}$$

where R_α and R_β are the radii of the two beams. Our first step is to rewrite the integral in (3.19) as

$$\int_{s_\alpha \times s_\beta} d^4A\, \Theta(-A_\parallel) |\tilde{\Psi}_\perp(A_\perp)|^2 = \int_{-A_{\max}}^{0} dA_\parallel \int_{\Omega(A_\parallel)} d^3A_\perp\, |\tilde{\Psi}_\perp(\vec{A}_\perp)|^2 , \tag{3.21}$$

where $\Omega(A_\parallel)$ is the region of integration in $\vec{A}_\perp$ for fixed $A_\parallel$.

Because our spatial wave packets are localized at $\vec{A}_\perp = 0$ and are much smaller than the beam, the integration over $\vec{A}_\perp$ is just

$$\int_{\Omega(A_\parallel)} d^3A_\perp\, |\tilde{\Psi}(\vec{A}_\perp)|^2 = 1 \quad \text{if } \vec{A}_\perp = 0 \in \Omega(A_\parallel)$$
$$= 0 \quad \text{otherwise.}$$[6]

Now, if $\vec{A}_\perp$ is zero, it is easily seen that the condition (3.20) is[7]

$$|A_\parallel v_\perp/V| \leq R_\alpha \quad \text{and} \quad |A_\parallel u_\perp/V| \leq R_\beta . \tag{3.22}$$

Thus the integrand in (3.21) is 1 when both of these conditions are satisfied and zero otherwise. That is, the integral (3.21) is given by V times $\min\{R_\alpha/v_\perp, R_\beta/u_\perp\}$. Substitution in (3.19) gives

$$\xi_2(\Delta) = (2\pi)^5 |\Psi_\parallel(\Delta/V)|^2 V^{-1} \min\{R_\alpha/v_\perp, R_\beta/u_\perp\} . \tag{3.23}$$

As a function of Δ, $\xi_2(\Delta)$ behaves quite differently from $\xi_1(\Delta)$. The wave function $\Psi_\parallel$ in (3.23) is sharply peaked around 0 with width γ. Thus for $\Delta \gg V\gamma$, $\xi_2(\Delta)$ is exponentially small and negligible compared to $\xi_1(\Delta)$. On the other hand, in the immediate neighborhood of $\Delta = 0$, the term $\xi_2(\Delta)$ is much larger than $\xi_1(\Delta)$. Specifically,

$$\xi_2(0) = (2\pi)^5 |\Psi_\parallel(0)|^2 V^{-1} \min\{R_\alpha/v_\perp, R_\beta/u_\perp\}$$
$$\sim R_{\rm beam}/V^2\gamma$$

where $R_{\rm beam}$ is a typical beam radius (R_α or R_β). On the other hand, from (3.17),

$$\xi_1(0) \approx 1/V^2\gamma^2 = R_{\rm pac}/V^2\gamma$$

where $R_{\rm pac} = 1/\gamma$ denotes the spatial size of the incident wave packets. Thus $\xi_2(0)/\xi_1(0)$ is of the order of $R_{\rm beam}/R_{\rm pac}$, which we have assumed all along is a very large number. The behavior of $\xi_1(\Delta)$ and $\xi_2(\Delta)$ is therefore as shown schematically in Fig. 2, with $\xi_2(\Delta)$ much narrower and taller than $\xi_1(\Delta)$. In fact, we have taken for granted all along that the incident packets are very narrow compared to the experimental resolution of energy; since $\xi_2(\Delta)$ is proportional to $|\Psi_\parallel(\Delta/V)|^2$, this means we can now replace $|\Psi_\parallel(\Delta/V)|^2$ in (3.23) by the delta function $\delta(\Delta/V)$ and we obtain as our final expression for $\xi_2(\Delta)$,

$$\xi_2(\Delta) = (2\pi)^5 \delta(\Delta) \min\{R_\alpha/v_\perp, R_\beta/u_\perp\} . \tag{3.24}$$

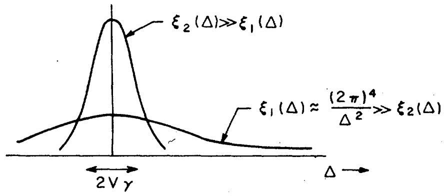

FIG. 2. For Δ small, the term $\xi_2(\Delta)$ dominates; for $\Delta \gg V\gamma$, the term $\xi_1(\Delta)$ dominates.

Finally, the cross term $\xi_3(\Delta)$ must be discussed. Since it is proportional to $\Psi_{\parallel}(\Delta/V)$ it is certainly negligible compared to $\xi_1(\Delta)$ for large values of Δ. For small Δ, it can be estimated as being of order $1/V^2\gamma^2$ and is therefore much less than ξ_2. Therefore the cross term $\xi_3(\Delta)$ can be neglected for all values of Δ.

D. Scattering rates

We can now return to (3.6) for the contribution of the ith double-scattering process (i=I, II, III, IV) and substitute the results (3.18) and (3.24) in $\xi = \xi_1 + \xi_2$. This gives

$$r^{(i)}_{\alpha\beta}(\vec{p},\vec{q}) = (2\pi)^7 \rho_\alpha \rho_\beta\, \delta(E_{pq} - E_{p_0q_0}) \,|\, \tilde{T}_i(\vec{p},\vec{q},\vec{p}_0,\vec{q}_0)|^2 \times\left[\frac{1}{\Delta_i^2 + V_i^2\gamma^2} + 2\pi\,\delta(\Delta_i)\min\left(\frac{R_\alpha}{v_{\perp i}}, \frac{R_\beta}{u_{\perp i}}\right)\right]. \tag{3.25}$$

This is just the contribution of the ith double-scattering pole. To write down the complete scattering rate we write the three-particle T matrix as

$$\langle \vec{p},\vec{q}\,|T|\,\vec{p}_0,\vec{q}_0\rangle = \sum_{i=\mathrm{I}}^{\mathrm{IV}} \frac{\tilde{T}_i(\vec{p},\vec{q},\vec{p}_0,\vec{q}_0)}{\Delta_i(\vec{p},\vec{q},\vec{p}_0,\vec{q}_0) + i0} + T_{\mathrm{reg}}(\vec{p},\vec{q},\vec{p}_0,\vec{q}_0), \tag{3.26}$$

where the remainder $T_{\mathrm{reg}}(\vec{p},\vec{q},\vec{p}_0,\vec{q}_0)$ is regular at all four double-scattering poles.[8] With this notation it is easily seen that the complete scattering rate is (omitting most arguments)

$$r_{\alpha\beta}(\vec{p},\vec{q}) = (2\pi)^7 \rho_\alpha\rho_\beta\,\delta(E_{pq} - E_{p_0q_0}) \times\left[\left|\sum_i \frac{\tilde{T}_i}{\Delta_i + iV_i\gamma} + T_{\mathrm{reg}}\right|^2 + 2\pi\sum_i \delta(\Delta_i)\min\left(\frac{R_\alpha}{v_{\perp i}}, \frac{R_\beta}{u_{\perp i}}\right)|\tilde{T}_i|^2\right]. \tag{3.27}$$

If we choose to observe final momenta such that none of the four energy denominators Δ_i is zero (more precisely $\Delta_i \gg V_i\gamma$, for all i), then the four delta functions in (3.27) are zero; furthermore the quantity that appears in the modulus signs is just the complete T matrix; therefore, in this case (3.27) reduces to the familiar answer (1.2). In particular, for observations where the Δ_i do not vanish we need to know the full three-particle T matrix before we can calculate the rate (3.27). On the other hand, if our region of observation is such that any of the denominators Δ_i vanishes, then one or more of the delta function terms in (3.27) contribute. Further, since the delta functions are multiplied by the macroscopic beam radius, these contributions always dominate. Thus for any observations where one or more of the denominators Δ_i vanish, we can entirely neglect the three-particle T matrix and the counting rate is given by the relevant product, or products, $\tilde{T}_i$ of two-particle T matrices.

In Sec. IV we shall discuss in detail the contributions of the four different double-scattering singularities. Before we do so we should perhaps remind the reader that we are, at the moment, taking for granted that the momenta $\vec{p}, \vec{q}$ for which we monitor are chosen to avoid the shells on which the more severe disconnected singularities occur. In Sec. V we shall remove this restriction and discuss *all* of the singularities and their relative importance.

IV. CONTRIBUTIONS OF THE FOUR DOUBLE-SCATTERING PROCESSES

In this section we discuss in detail the contribution (3.25) for each of the four double-scattering processes in turn and discuss the regions of momentum space where the processes can occur. It should, perhaps, be emphasized that the double-scattering poles contribute to the scattering rate for all momenta. If none of the denominators Δ_i vanish, then the poles contribute through their "tails," which must be added to the regular part of T as in (3.27); if one or more of the denominators Δ_i vanish, then the poles contribute through the delta functions in (3.27). The difference between these cases is that when Δ_i vanishes the corresponding double-scattering process can actually occur as a physical process with the appropriate conservation of energy and momentum in each of the separate collisions; and when this happens the pole term dominates and the regular parts of the T matrix can be neglected. It is this case, where the Δ_i vanish and the corresponding double-scattering processes can actually occur, that we shall be interested in here.

A. Process I

We begin by discussing the process I of Fig. 1, which we show in more detail in Fig. 3. For this

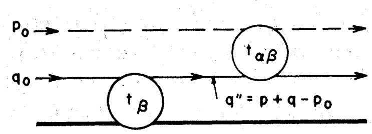

FIG. 3. Double-scattering process I.

process the intermediate energy is

$$E''=p_0^2/2m_\alpha+q''^2/2m_\beta\,, \tag{4.1}$$

with

$$\vec{q}\,''=\vec{p}+\vec{q}-\vec{p}_0\,. \tag{4.2}$$

Thus the two derivatives of (3.3) are

$$\vec{v}\equiv\partial E''/\partial\vec{p}_0=\vec{p}_0/m_\alpha-\vec{q}\,''/m_\beta$$

and

$$\vec{u}\equiv\partial E''/\partial\vec{q}_0=0\,. \tag{4.3}$$

In particular the component of $\vec{v}$ normal to $\vec{p}_0$ is

$$\vec{v}_\perp=-(\vec{q}\,''/m_\beta)_\perp=-\vec{v}\,''_\perp\,.$$

Thus $\vec{v}_\perp$ is minus the component of the intermediate velocity $\vec{v}\,''=\vec{q}\,''/m_\beta$ normal to the axis of the beam α; while $\vec{u}_\perp$ is zero.

We can now substitute these velocities into the expression (3.25) for the scattering contribution

$$r^{\rm I}_{\alpha\beta}(\vec{p},\vec{q})=(2\pi)^8\rho_\alpha\rho_\beta(R_\alpha/v''_\perp)\times\delta(E_{pq}-E_{p_0q_0})\delta(\Delta_{\rm I})|t_{\alpha\beta}t_\beta|^2\,. \tag{4.4}$$

This is the contribution of the double-scattering process I in any domain where it can actually occur; that is, where the arguments of the two δ functions can vanish.

From (4.4) we can evaluate the one-particle rate $r^{\rm I}_\alpha(\vec{p})$ for counting particles α irrespective of the momenta of the particles β. (As argued in Ref. 1, this rate is likely to be the most interesting in practice.) This is

$$r^{\rm I}_\alpha(\vec{p})=(2\pi)^8\rho_\alpha\rho_\beta R_\alpha\int\frac{d^3q}{v''_\perp}\,\delta(E_{pq}-E_{p_0q_0})\delta(\Delta_{\rm I})|t_{\alpha\beta}t_\beta|^2 \tag{4.5}$$

for all $\vec{p}$ in the "double-scattering region" where the arguments of the two delta functions can vanish (for some $\vec{q}$ in the region of integration).

The important feature of the answer (4.5) is that it is precisely the rate one would predict for particles α to emerge from two successive two-body collisions as shown in Fig. 4. This rate is calculated in two steps as follows: The rate at which the incident particles β are scattered by the fixed center to the intermediate momentum $\vec{q}\,''$ is

$$r_\beta(\vec{q}\,''\leftarrow\vec{q}_0)=(2\pi)^4\rho_\beta\delta(E_{q''}-E_{q_0})|t_\beta(\vec{q}\,'',\vec{q}_0)|^2\,. \tag{4.6}$$

Each intermediate particle β, with momentum $\vec{q}\,''$, starts from the origin 0 and must pass through a thickness R_α of the beam α, which has density ρ_α and momentum $\vec{p}_0$. For each such intermediate particle the probability of a collision producing an α with momentum $\vec{p}$ is easily seen to be[9]

$$w_\alpha(\vec{p}\leftarrow\vec{p}_0,\vec{q}\,'')=(2\pi)^4\rho_\alpha(R_\alpha/v''_\perp)\delta(E_{pq}-E_{p_0q''})\times|t_{\alpha\beta}(\vec{\pi},\vec{\pi}\,'')|^2\,, \tag{4.7}$$

where $\vec{\pi}$ and $\vec{\pi}\,''$ are the relevant relative momenta.

The total rate at which particles α emerge with momentum near $\vec{p}$ is obtained by integrating (over all intermediate $\vec{q}\,''$) the product of the rate (4.6) and the probability (4.7):

$$\begin{aligned}r^{\rm I}_\alpha(\vec{p})&=\int d^3q''\,w_\alpha(\vec{p}\leftarrow\vec{p}_0,\vec{q}\,'')r_\beta(\vec{q}\leftarrow\vec{q}_0)\\&=(2\pi)^8\rho_\alpha\rho_\beta R_\alpha\int\frac{d^3q''}{v''_\perp}\,\delta(E_{pq}-E_{p_0q''})\\&\qquad\times\delta(E_{q''}-E_{q_0})|t_{\alpha\beta}t_\beta|^2\,.\end{aligned} \tag{4.8}$$

By making the change of variables from $\vec{q}\,''$ to

$$\vec{q}=\vec{q}\,''+\vec{p}_0-\vec{p} \tag{4.9}$$

one can easily check that this is exactly the answer (4.5).

By examining the two δ functions in (4.8) we can determine those momenta for which the double-scattering process I can actually occur. The second δ function requires that the intermediate momentum $\vec{q}\,''$ lie on the sphere $q''=q_0$. The first δ function requires that $E_{pq}=E_{p_0q_0}$; or, if we substitute (4.9) for $\vec{q}$,

$$\frac{p^2}{m_\alpha}+\frac{(\vec{q}\,''+\vec{p}_0-\vec{p})^2}{m_\beta}=\frac{p_0^2}{m_\alpha}+\frac{q_0^2}{m_\beta}\,. \tag{4.10}$$

Since $q''=q_0$ this simplifies to

$$2\vec{q}\,''\cdot(\vec{p}-\vec{p}_0)=\mu(p^2-p_0^2)+(\vec{p}-\vec{p}_0)^2$$

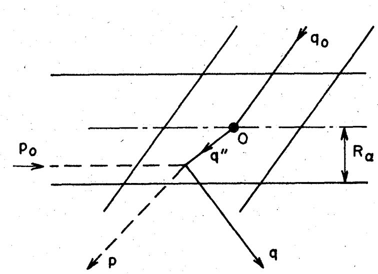

FIG 4. The two successive collisions corresponding to the double-scattering process I of Fig. 3.

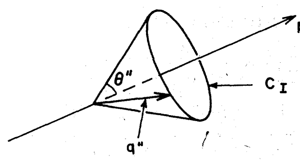

FIG. 5. The double-scattering process I can occur only if the intermediate momentum $\vec{q}''$ lies on the circle C_{I} where the cone of angle θ'' intersects the sphere $q''=q_0$.

where we have introduced the mass ratio $m_\beta/m_\alpha = \mu$. If we call θ'' the angle between $\vec{q}''$ and $(\vec{p}-\vec{p}_0)$, this requires that

$$\cos\theta'' = \frac{\mu(p^2-p_0^2)+(\vec{p}-\vec{p}_0)^2}{2q_0|\vec{p}-\vec{p}_0|}\,; \tag{4.11}$$

that is, $\vec{q}''$ must lie on a cone with axis $(\vec{p}-\vec{p}_0)$ and half angle θ'' given by (4.11). Since $\vec{q}''$ must also lie on the sphere $q''=q_0$ we conclude that the double-scattering process I can occur if and only if $\vec{q}''$ lies on the circle C_{I} in which this cone and sphere intersect, as shown in Fig. 5.

If we measure the two-particle rate $r_{\alpha\beta}(\vec{p},\vec{q})$ and monitor for a definite value of $\vec{p}$ then the double-scattering process I will contribute only if $\vec{q}$ (which equals $\vec{q}''+\vec{p}_0-\vec{p}$) lies on the circle obtained from C_{I} by rigid translation through $\vec{p}_0-\vec{p}$. Of course the process can only occur if the angle θ'' which defines this circle is a real angle. That is, $|\cos\theta''|$ as given by (4.11) must be less than or equal to 1. This obviously requires that

$$|\mu(p^2-p_0^2)+(\vec{p}-\vec{p}_0)^2| \le 2q_0|\vec{p}-\vec{p}_0|\,. \tag{4.12}$$

If $\vec{p}$ satisfies the condition (4.12), then the double-scattering process I can lead to final states in which particle α has momentum $\vec{p}$; if $\vec{p}$ does not satisfy (4.12) then process I cannot produce particles α with momentum $\vec{p}$. For this reason we call the set of momenta $\vec{p}$ satisfying (4.12) *the double-scattering region* I. The boundary of this region is a fourth-order surface whose precise shape depends on the values of the parameters q_0/p_0 and $\mu=m_\beta/m_\alpha$. In all cases the region is a solid of revolution obtained by rotation about $\vec{p}_0$. If $m_\beta/m_\alpha > q_0/p_0$, the solid has a hole in its interior; if $m_\beta/m_\alpha \le q_0/p_0$, it does not. In Fig. 6 we show the former possibility for the particular choice of parameters $m_\beta/m_\alpha=3.3$ and $q_0/p_0=1.25$.

If $\vec{p}$ lies in the double-scattering region I, then the contribution of process I to the one-particle rate $r_\alpha(\vec{p})$ is given by the integral (4.8), and the contributions of the regular parts of the T matrix can be neglected. We can now simplify this integral, taking advantage of the two delta functions, which fix the magnitude $q''=q_0$ and the angle θ'' between $\vec{q}''$ and $\vec{p}-\vec{p}_0$. Some simple algebra shows that the three-dimensional integral (4.8) reduces to a one-dimensional integral over the azimuth φ'' of $\vec{q}''$ (that is, around the circle C_{I} of Fig. 5),

$$r_\alpha^{\text{I}}(\vec{p}) = (2\pi)^8\rho_\alpha\rho_\beta\frac{R_\alpha m_\beta^2}{|\vec{p}-\vec{p}_0|}\int_0^{2\pi}\frac{d\varphi''}{v''_\perp}\,|t_{\alpha\beta}t_\beta|^2\,. \tag{4.13}$$

Clearly we cannot simplify this rate any further without knowing the details of the interactions and calculating $t_{\alpha\beta}$ and t_β.

It is clear from (4.13) that the rate $r_\alpha^{\text{I}}(\vec{p})$ has a singularity if $v''_\perp=0$. This is easily understood with reference to Fig. 4. Since $\vec{v}''_\perp$ is the component of the intermediate velocity normal to the beam α, it can only vanish if the intermediate momentum $\vec{q}''$ points directly up or down the beam α. When this happens the intermediate particle β has an infinite time in which to interact with the beam of particles α and the rate $r_\alpha^{\text{I}}(\vec{p})$ naturally becomes infinite. However, it is easily seen that this singularity is logarithmic, hence integrable, and therefore entirely harmless.

B. Process II

One can continue to analyze the remaining three double-scattering processes in a similar way. Obviously we need not spell out all of the details here. In particular, the process II of Fig. 1 is very similar to the process I just discussed (with a corresponding interpretation), except that in process II it is the particle α that participates in both collisions.

One can evaluate the two derivatives $\vec{v}_\perp$ and $\vec{u}_\perp$ and finds

$$\vec{v}_\perp = 0 \quad\text{and}\quad \vec{u}_\perp = -\vec{p}''/m_\alpha = -\vec{u}''_\perp\,;$$

that is, $\vec{u}_\perp$ is minus the component of the velocity

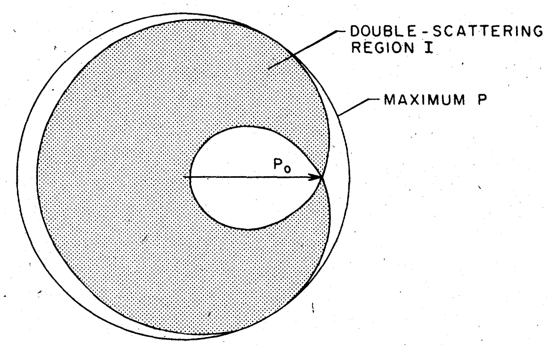

FIG. 6. The space of the final momentum $\vec{p}$ showing the sphere which defines the maximum possible p and the double-scattering region I (shaded). Those $\vec{p}$ inside this region are accessible via the double-scattering process I. (Curves calculated for $m_\beta/m_\alpha=3.3$ and $q_0/p_0=1.25$.)

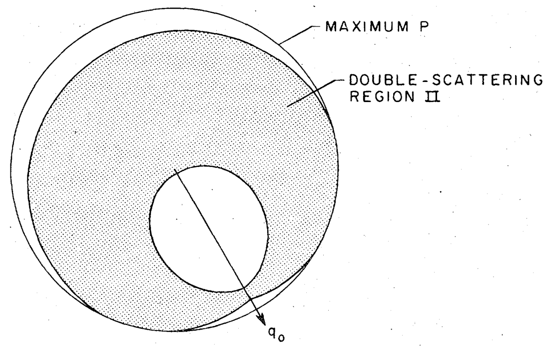

FIG. 7. The space of the final momentum $\vec{p}$ showing the double-scattering region II (for the same initial parameters as Fig. 6).

$\vec{u}''$ of the intermediate α, normal to the axis of the beam β. It is then a simple matter to write down expressions analogous to (4.4) and (4.8) for the rates $r^{II}_{\alpha\beta}(\vec{p},\vec{q})$ and $r^{II}_{\alpha}(\vec{p})$. The two delta functions which appear in these expressions let one determine the double-scattering region II which is accessible to process II. One of the δ functions requires that $E_{p''}=E_{p_0}$; that is, the intermediate momentum $\vec{p}''$ must lie on the sphere $p''=p_0$ if process II is to be possible. The other δ function requires that $E_{pq}=E_{p_0q_0}$; if we substitute

$$\vec{q}=\vec{p}''+\vec{q}_0-\vec{p}\,, \tag{4.14}$$

this constraint is seen to determine the angle θ'' between $\vec{p}''$ and the vector $\vec{p}-\vec{q}_0$ as

$$\cos\theta''=\frac{(\mu+1)p^2+(\mu-1)p_0^2-2\vec{p}\cdot\vec{q}_0}{2p_0\,|\vec{p}-\vec{q}_0|}\,. \tag{4.15}$$

These two conditions mean that $\vec{p}''$ must lie on the circle C_{II} where the cone defined by (4.15) meets the sphere $p''=p_0$. It follows from (4.14) that, if we measure the two-particle rate $r_{\alpha\beta}(\vec{p},\vec{q})$ and monitor for a definite value of $\vec{p}$, then the double-scattering process II will contribute only for those $\vec{q}$ lying on the circle obtained from C_{II} by translation through $\vec{q}_0-\vec{p}$.

The process II can actually occur only if the angle defined by (4.15) is real; that is $|\cos\theta''|\le 1$. This condition defines the double-scattering region II, comprising those $\vec{p}$ that are accessible via process II. The region is a solid of revolution with axis along $\vec{q}_0$. Its precise shape depends on the initial parameters, but it is qualitatively similar to the region I of Fig. 6.[10] In Fig. 7 we show the region II for the same values of m_β/m_α and q_0/p_0 as shown in Fig. 6.

As long as $\vec{p}$ is chosen in region II the contribution of process II is given by the delta function term in (3.25), and the contribution of all regular parts of the T matrix can be neglected. The integral giving the contribution $r^{II}_{\alpha}(\vec{p})$ can be simplified and reduces to [cf. Eq. (4.13)]

$$r^{II}_{\alpha}(\vec{p})=(2\pi)^8\rho_\alpha\rho_\beta\frac{R_\beta m_\alpha m_\beta}{|\vec{p}-\vec{q}_0|}\int_0^{2\pi}\frac{d\varphi''}{u''_\perp}\,|t_{\alpha\beta}t_\alpha|^2\,. \tag{4.16}$$

This can be seen to be exactly the rate one would predict from the appropriate succession of two two-body collisions. Finally we remark that the contribution (4.16) has a weak (logarithmic) singularity when the intermediate momentum $\vec{p}''$ can be parallel to the axis of the beam β and $u''_\perp$ can vanish.

C. Process III

The last two processes differ slightly from those just considered, in that the collision with the fixed target occurs last. As we shall see this changes some of the constraints.

For process III of Fig. 1 the momentum of the intermediate particle α is $\vec{p}''=\vec{p}_0+\vec{q}_0-\vec{q}$ and the relevant derivatives of the intermediate energy are

$$\vec{v}_\perp=(\partial E''/\partial\vec{p}_0)_{\perp\alpha}=(\vec{p}''/m_\alpha)_{\perp\alpha}\equiv\vec{v}''_{\perp\alpha}\,,$$

$$\vec{u}_\perp=(\partial E''/\partial\vec{q}_0)_{\perp\beta}=(\vec{p}''/m_\alpha)_{\perp\beta}\equiv\vec{v}''_{\perp\beta}\,.$$

Here we have introduced subscripts $\perp\alpha$ and $\perp\beta$ to denote components normal to the axes of beams α and β, since the two derivatives turn out to be the components $\vec{v}''_{\perp\alpha}$ and $v''_{\perp\beta}$ of the same velocity, namely the velocity $\vec{v}''=\vec{p}''/m_\alpha$ of the intermediate particle α.

The two delta functions that appear in (3.25) determine the momenta for which process III can actually occur. It is easily seen that they require that the intermediate momentum $\vec{p}''$ lie on the sphere $p''=p$ and a cone with axis $\vec{p}_0+\vec{q}_0$ and half angle given by

$$\cos\theta''=\frac{(\mu+1)p^2-\mu p_0^2-q_0^2+(\vec{p}_0+\vec{q}_0)^2}{2p\,|\vec{p}_0+\vec{q}_0|}\,. \tag{4.17}$$

The double-scattering region III, consisting of those $\vec{p}$ that are accessible via process III, is determined by the requirement that $|\cos\theta''|\le 1$. It will be seen that this condition involves the magnitude but not the direction of $\vec{p}$. Thus the region III is spherically symmetric and is in fact the space between two spheres as illustrated in Fig. 8 for the same values of the parameters as used in Figs. 6 and 7. [Since (4.17) involves the magnitude of $\vec{p}_0+\vec{q}_0$ it depends on the angle between $\vec{p}_0$ and $\vec{q}_0$. Figure 8 shows region III for the case that this angle is 60°.]

Within the region III the dominant term in $r^{III}_{\alpha}(\vec{p})$ can be written down and simplified as above to give

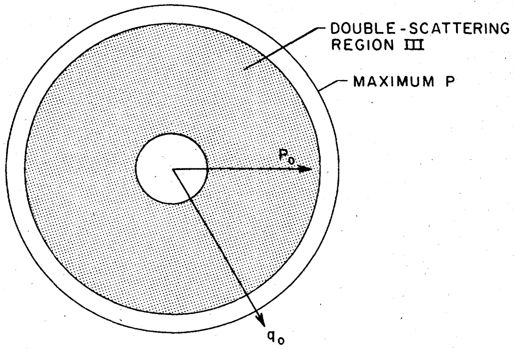

FIG. 8. The space of the final momentum $\vec{p}$ showing the double-scattering region III (for the case that $m_\beta/m_\alpha = 3.3$, $q_0/p_0 = 1.25$, and the angle between $\vec{p}_0$ and $\vec{q}_0$ is 60°).

$$r_\alpha^{\rm III}(\vec{p}) = (2\pi)^8 \rho_\alpha \rho_\beta \frac{m_\alpha m_\beta}{|\vec{p}_0 + \vec{q}_0|} \times \int d\varphi'' \, |t_\alpha t_{\alpha\beta}|^2 \min\left(\frac{R_\alpha}{v''_{\perp\alpha}}, \frac{R_\beta}{v''_{\perp\beta}}\right) . \quad (4.18)$$

This can be shown to be the rate one would predict for production of particles α in two separate collisions, the first between α and β and the second between α and the fixed target. It will be seen that because the two components $v''_{\perp\alpha}$ and $v''_{\perp\beta}$ in (4.18) cannot both vanish[11] this rate (unlike that for processes I and II) has no singularities.

D. Process IV

The contribution $r_\alpha^{\rm IV}(\vec{p})$ from process IV of Fig. 1 can be analyzed in a similar way. However, if we choose to measure only the one-particle rate $r_\alpha(\vec{p})$ no analysis is necessary. This is because the double-scattering region IV where process IV contributes coincides exactly with the "α-β scattering shell" where the disconnected process of Ref. 1 contributes. Since the disconnected process dominates all of the contributions discussed here (including that of process IV) there is no point in evaluating $r_\alpha^{\rm IV}(\vec{p})$ at all.

To see that this is so we have only to examine the two delta functions which determine when process IV can occur. As usual, one of these fixes the magnitude of the intermediate momentum $q'' = q$; the other requires that $E_{pq''} = E_{p_0 q_0}$ or (since $\vec{q}'' = \vec{p}_0 + \vec{q}_0 - \vec{p}$)

$$\frac{p^2}{m_\alpha} + \frac{(\vec{p}_0 + \vec{q}_0 - \vec{p})^2}{m_\beta} = \frac{p_0^2}{m_\alpha} + \frac{q_0^2}{m_\beta} .$$

This condition is precisely the condition that defines the α-β scattering shell of Ref. 1, Eq. (4.4) and therefore requires that $\vec{p}$ lie on the same sphere with center at $(\vec{p}_0 + \vec{q}_0)/(1+\mu)$ and passing through the point $\vec{p} = \vec{p}_0$.

We see that process IV contributes to $r_\alpha(\vec{p})$ only on the α-β scattering shell. On this shell the contribution of the disconnected process is proportional to the macroscopic volume $\mathcal{V}_{\alpha\beta}$ of intersection of the two beams (as we saw in Ref. 1), while that of process IV is proportional only to the radius (R_α or R_β) of the beams. It follows that the disconnected process always dominates the double-scattering process IV wherever the latter can occur, and there is no point in calculating the contribution $r_\alpha^{\rm IV}(\vec{p})$.

Of course, if we measure the correlated two-particle rate $r_{\alpha\beta}(\vec{p}, \vec{q})$ we *can* distinguish the contribution of process IV. For given $\vec{p}$ (on the appropriate shell) the disconnected process contributes for just one value of $\vec{q}$, namely $\vec{q} = \vec{p}_0 + \vec{q}_0 - \vec{p}$; on the other hand, process IV contributes for all $\vec{q}$ with this magnitude (i.e., for all $\vec{q}$ on a certain sphere). One can, of course, write down an expression analogous to (4.4) for the contribution $r_{\alpha\beta}^{\rm IV}(\vec{p}, \vec{q})$ to such a measurement.

V. CONCLUSION

We have seen in Ref. 1 and the present paper that there are several processes that contribute to the three-particle scattering rates. These various processes contribute in markedly different subsets of momentum space and with strikingly different orders of magnitude. Loosely speaking, we have found that the larger the region in which a process contributes, the smaller the size of its contribution. This means that in the region where a given process contributes we can neglect all processes which contribute in a larger region.

In Ref. 1 we examined first what we called the *separate-scattering terms*, which arise because particles α and β can scatter independently off the fixed target. Such processes contribute to the one-particle rate $r_\alpha(\vec{p})$ only on the "α-scattering shell", the sphere defined by conservation of energy, $E_p = E_{p_0}$, for particle α alone. On this shell their contribution is proportional to the density ρ_α, as one would expect, and dominates all real three-body effects.

We next considered the disconnected process in which particles α and β collide, but do not interact with the target. This process contributes on the "α-β scattering shell", the sphere with center $(\vec{p}_0 + \vec{q}_0)/(1+\mu)$ passing through $\vec{p}_0$. On this shell, its contribution is proportional to $\rho_\alpha \rho_\beta \mathcal{V}_{\alpha\beta}$ (where $\mathcal{V}_{\alpha\beta}$ is the macroscopic volume of intersection of the two beams) and dominates all real three-body effects.

It should be emphasized that the "separate-scat-

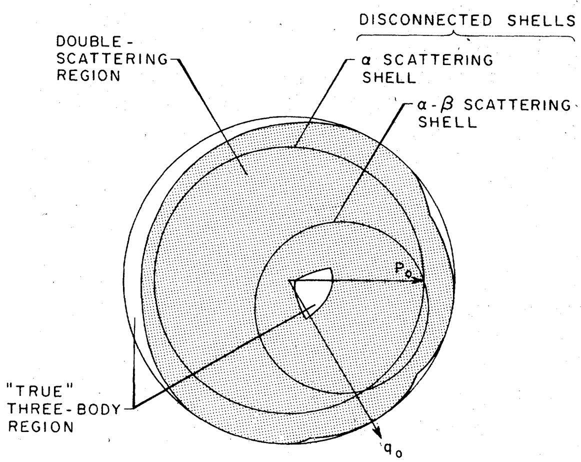

FIG. 9. The space of the final momentum $\vec{p}$ showing the regions in which different processes can contribute (for the same external parameters as in Figs. 6–8). The double-scattering region is the union of the three regions of Figs. 6–8.

tering" processes are really just disconnected processes in which particle α collides with the target while β is undeflected, or *vice versa.* Our different treatment of these disconnected processes reflects our unsymmetrical treatment of the three particles themselves, with one infinitely heavy.[12] For the present discussion it is convenient to recognize that these processes are all basically similar. In Fig. 9, which shows the various different regions in p space, we have therefore grouped together the "α scattering" and "α-β scattering" shells as "disconnected shells".

In this paper we have discussed the double-scattering processes. We have seen that, as far as the rate $r_\alpha(\vec{p})$ is concerned, there are just three processes to consider, each of which occurs in its own region (as shown in Figs. 6–8). The *union* of these regions we call the *double-scattering region.* This region is shown (for the same choice of parameters as in Figs. 6–8) shaded in Fig. 9. Any $\vec{p}$ in this region is accessible by one or more double-scattering processes; the contribution of these processes is proportional to $\rho_\alpha\rho_\beta R$, where R is the macroscopic radius of one of the beams. Therefore, for any $\vec{p}$ in the double-scattering region we can neglect the contribution of the regular parts of the three-particle T matrix.

Finally, for any $\vec{p}$ that is not in the double-scattering region,[13] the scattering is given by the familiar formula

$$r_\alpha(\vec{p}) = (2\pi)^7 \rho_\alpha\rho_\beta \times \int d^3q\, \delta(E_{pq} - E_{p_0q_0}) |\langle \vec{p}, \vec{q} | T | \vec{p}_0, \vec{q}_0\rangle|^2 .$$

If we wish to predict the rate for $\vec{p}$ in this "true three-body region" we must know the full three-particle T matrix, $\langle \vec{p}, \vec{q} | T | \vec{p}_0, \vec{q}_0\rangle$. Conversely, if we wish to measure the contribuion of the full three-particle T matrix to the rate $r_\alpha(\vec{p})$, then we must make our measurements in this "true three-body region."

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*Work supported in part by I.R.E.X.

[1] V. S. Potapov and J. R. Taylor, preceding paper [Phys. Rev. A 16, 2264 (1977).

[2] In the terminology of Ref. 1 there are the "separate-scattering" processes (which contribute on the "α-scattering shell" and "β-scattering shell") and the disconnected process (which contributes on the "α-β scattering shell"). As we shall discuss in Sec. V these are all really disconnected processes, and we shall refer to the corresponding shells as disconnected shells here.

[3] As we discuss below, there are actually six double-scattering processes, but two have already been treated as part of the "separate-scattering" term in Ref. 1.

[4] In Ref. 1 the T matrix in this equation was written as $T^{(3)}$ to denote the three-particle T matrix *minus* its "separate-scattering" part (i.e., its value when $V_{\alpha\beta} = 0$). In the present paper we need not distinguish between T and $T^{(3)}$ since we shall usually avoid the shell where the separate-scattering part contributes.

[5] There is no loss of generality in our taking the widths of ψ and ϕ to be the same, since these can always be adjusted by an appropriate scale change.

[6] With a Gaussian wave function the integral is not *exactly* zero but is, rather, exponentially small.

[7] If $\vec{A}_\perp = 0$ then $\vec{A} = A_\parallel \vec{V}/V = (A_\parallel/V)(\vec{v}_\perp, \vec{u}_\perp)$. But $\vec{A} = (\vec{a}_\perp, \vec{b}_\perp)$, so $a_\perp = A_\parallel v_\perp/V$ and $b_\perp = A_\parallel u_\perp/V$.

[8] Strictly speaking T_{reg} as defined by (3.26) contains the disconnected singularities. However, we are taking for granted here that the observed momenta $\vec{p}$, $\vec{q}$ are chosen to avoid the shell on which these contribute. Thus for our purposes T_{reg} is L^2 for all momenta of interest.

[9] Notice that the factor $R_\alpha/v''_\perp$ is just the time spent by the intermediate particle β in the beam α.

[10] The region II as defined by (4.15) is not *exactly* the same as region I except for the special case of equal masses, $\mu = 1$. In particular region II has a hole in it for all values of the external parameters m_β/m_α and q_0/p_0 (except on a set of measure zero).

[11] The components $v''_{\perp\alpha}$ and $v''_{\perp\beta}$ can both vanish only if the incident beams are parallel; but one would never do experiments in this configuration since it would lead to an overwhelming contribution from the disconnected process discussed in Ref. 1. (See, in particular, footnote 19 of Ref. 1.)

[12] We can improve the parallel between the different disconnected processes if we generalize our experi-

ment to include several target particles. For example, if we have N_{tar} targets in a volume $\mathcal{V}_{tar}$ (entirely inside both beams) then the "α-scattering" contribution is proportional to $\rho_\alpha \rho_{tar} \mathcal{V}_{tar}$, which clearly shows its relation to the α-β contribution, which is proportional to $\rho_\alpha \rho_\beta \mathcal{V}_{\alpha\beta}$.

[13]Notice that the double-scattering region always includes the two disconnected shells. Thus if $\vec{p}$ is not in the former it is certainly not on the latter.