Solitary wave solutions in coherent two-photon pulse propagation

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Solitary wave solutions of an electric power envelope for two-photon pulse propagation in a resonant medium are given under limited conditions. In addition, we show new types of periodic solutions without Jacobian elliptic functions and the well-known Lorentzian-shape solutions for a coherent two-photon amplifier.

There is considerable current interest in solitary wave solutions for various fields,[1] especially for coherent pulse propagation at the one-photon resonance[2] and in Raman transitions.[3-5] In this paper, we present solitary wave solutions for coupling the equations of motion of the two-photon Feynman vector[6-8] $\vec{r}$ to the reduced Maxwell equations. The representation of $\vec{r}$ is very useful for the adiabatic following advanced by Grishkowsky *et al.*[7] and the two-photon precession decay.[9] In addition, the coupling to the Maxwell equations has led to two-photon self-induced transparency (TPSIT).[6,10,11] In order to find some stationary-state solutions, including solitary wave solutions, we make several assumptions: no inhomogeneous broadening, equal pulse velocity, and equal perturbation energy at two different frequency fields

$$E_i = \mathcal{E}_i \cos(\omega_i t + \phi_i - k_i z), \quad i = 1, 2. \tag{1}$$

The atom is assumed to be irradiated by two light beams with frequencies ω_1 and ω_2 and propagation vectors k_1 and k_2, where $k_i = \omega_i / c$.

The two-photon Bloch equations[6-11] with no relaxation terms are described as follows:

$$\frac{\partial \vec{r}}{\partial t} = \vec{\gamma} \times \vec{r}, \tag{2}$$

with components

$$\frac{\partial r_1}{\partial t} = -[\Delta\omega + (\Delta E_1 - \Delta E_2)/\hbar] r_2, \tag{3a}$$

$$\frac{\partial r_2}{\partial t} = [\Delta\omega + (\Delta E_1 - \Delta E_2)/\hbar] r_1 + \kappa \mathcal{E}_1 \mathcal{E}_2 r_3, \tag{3b}$$

$$\frac{\partial r_3}{\partial t} = -\kappa \mathcal{E}_1 \mathcal{E}_2 r_2, \tag{3c}$$

$$\Delta\omega = \Omega_{12} - (\omega_1 + \omega_2) - (\dot\phi_1 + \dot\phi_2),$$

where $\vec{r}(r_1, r_2, r_3)$ is the relevant vector $\vec{r}$ and $\vec{\gamma}$ $[\Delta\omega + (\Delta E_1 - \Delta E_2)/\hbar, 0, -\kappa\mathcal{E}_1\mathcal{E}_2]$ is the torque vector; $\Delta\omega$ is the off-resonant frequency to the eigenenergy separation $\hbar\Omega_{12}$ involving the phase shift $\dot\phi_1 + \dot\phi_2$. The two-photon gyroelectric ratio κ is defined as

$$\kappa = \frac{1}{2\hbar^2} \left| \sum_n p_{1n} p_{n2} \left(\frac{1}{\Omega_{n2} - \omega_1} + \frac{1}{\Omega_{n2} - \omega_2} \right) \right| ,$$

where p_{in} is the matrix element of the electric dipole moment. The optical Stark effect is denoted by $\Delta E_1 - \Delta E_2$, which may be canceled under the assumption of the equal perturbation energy at two fields and given combinations of two-photon frequencies.

We take into account an exact resonance case with no instantaneous phase shift (namely, $\Delta\omega = 0$) and no additional effect such as a parametric coupling. This choice simplifies the problem considerably. The second-order induced polarizations for the two-photon resonance expressed by the r components result in driving forces for the Maxwell equations under a slowly varying amplitude approximation,

$$\frac{\partial \mathcal{E}_i}{\partial z} + \frac{1}{c}\frac{\partial \mathcal{E}_i}{\partial t} = -\frac{N_0 \hbar \omega_i \kappa}{2\epsilon_0} \mathcal{E}_j r_2, \tag{4}$$

where N_0 is the atomic number density and ϵ_0 is the dielectric constant.

We now proceed to calculate the right-hand side of Eq. (4) using the above-mentioned assumptions. We can easily obtain solutions for r_2 and r_3 under adiabatic-following approximation to the intermediate states from Eqs. (3). The solutions are $r_2 = \sin\varphi$ and $r_3 = -\cos\varphi$ with

$$\varphi(t) = \kappa \int_{-\infty}^{t} \mathcal{E}_1(t', z)\, \mathcal{E}_2(t', z)\, dt. \tag{5}$$

After substituting the solution into Eqs. (4), the propagating equations become

$$\frac{\partial \mathcal{E}_1}{\partial z} + \frac{1}{c}\frac{\partial \mathcal{E}_1}{\partial t} = -\tfrac{1}{2}\beta_1 \mathcal{E}_2 \sin\varphi, \tag{6a}$$

$$\frac{\partial \mathcal{E}_2}{\partial z} + \frac{1}{c}\frac{\partial \mathcal{E}_2}{\partial t} = -\tfrac{1}{2}\beta_2 \mathcal{E}_1 \sin\varphi, \tag{6b}$$

where the characteristic propagation constant $\beta_i = N_0 \hbar \omega_i \kappa / \epsilon_0$.

Our aim is to find stationary solutions including solitary wave forms on the moving frame $\zeta = t - z/$

V (V being the equal pulse velocity at each field). The first integral on the moving frame for $\mathcal{E}_i^2$ from Eqs. (6) multiplied by $2\mathcal{E}_i$ yields $\beta_2'\mathcal{E}_1^2 - \beta_1'\mathcal{E}_2^2 = c_1$ [c_1 being the constant number of integral and $\beta_i' = \beta_i c V/(c-V)$] which may be called two-photon conservation. Here, we consider an equal photon number at each frequency contributed to the two-photon transition. Since the assumption $\Delta E_1 = \Delta E_2$ has been made, we can set c_1 to be zero, although $\Delta E_1 = \Delta E_2$ and $\beta_2\mathcal{E}_1^2 = \beta_1\mathcal{E}_2^2$ force ω_1 and ω_2 to have one of at most several discrete values because ΔE_1 and ΔE_2 are functions of ω_1 and ω_2. For a three-level system, these values are $\omega_1 = \Omega_{12}/(1+|p_{1n}/p_{2n}|^2)$ and $\omega_2 = \Omega_{12}/(1+|p_{2n}/p_{1n}|^2)$. Other several distinct values exist for a multilevel system.

From Eqs. (5) and (6), we transform

$$\frac{\partial^2}{\partial\zeta^2}\varphi = \beta' \frac{\partial\varphi}{\partial\zeta}\sin\varphi , \tag{7}$$

where $\beta' = \beta_1'^{1/2}\beta_2'^{1/2}$. Equation (7) seems to be a sine-Gordon equation modified by $\partial\varphi/\partial t$, while it reduces to

$$\frac{\partial}{\partial\zeta}\varphi = -\beta'(\cos\varphi + C_0) . \tag{8}$$

It is convenient to classify the solutions depending on different values of the integral constant C_0. Since $\partial\varphi/\partial\zeta = 0$ implies $C_0 + \cos\varphi = 0$ in a sense of solitary wave propagation, we may immediately conclude that (i) if $|C_0| > 1$, no shape-preserving pulse solutions are possible. However, we may find periodic solutions as a result of a step-function input, although the area is not preserved. (ii) If $|C_0| = 1$, any shape-preserving pulse solutions can be found to be of area $2n\pi$, $n = 0,1,2,$ etc. (iii) If $0 < |C_0| < 1$, any shape-preserving solution has area $2n\pi$, or $2n\pi \pm 2\cos^{-1}(-C_0)$. (iv) If $C_0 = 0$, shape-preserving solutions having area $n\pi$ are possible.

(i) $|C_0| > 1$. This restriction leads to a new type of periodic traveling wave solution, with no solitary pulse shape preserving,

$$\varphi_\pm(\zeta) = \cos^{-1}\left(\frac{C_0\cos[\beta'(C_0^2-1)^{1/2}(\zeta-\zeta_0)] \mp 1}{C_0 \mp \cos[\beta'(C_0^2-1)^{1/2}(\zeta-\zeta_0)]}\right), \tag{9}$$

and

$$\mathcal{E}_{i\pm}^2(\zeta) = \frac{\beta_i'}{\kappa}\frac{C_0^2-1}{C_0 \mp \cos[\beta'(C_0^2-1)^{1/2}(\zeta-\zeta_0)]} , \tag{10}$$

where ζ_0 is the arbitrary constant on the moving frame. Equations (10) are physically identical, because a change of ζ_0 makes $\mathcal{E}_{i+}^2$ to $\mathcal{E}_{i-}^2$.

(ii) $|C_0| = 1$. The choice $C_0 = -1$ implies, physically, the two-photon resonant medium is an absorber. This solution has been already obtained[10,11] and the electric power envelope has a Lorentzian shape in the moving frame with the two-photon area

$$\varphi(\zeta) = 2\cot^{-1}[-\beta'(\zeta-\zeta_0)] . \tag{11}$$

Thus, each power varies as

$$\mathcal{E}_i^2(\zeta) = \frac{\beta_i'/\kappa}{1+\beta'^2(\zeta-\zeta_0)^2} . \tag{12}$$

The branch solution of Eq. (11), the area $\varphi(\zeta)$ vs ζ, implies that the area is attenuated toward $2n\pi$ with increasing ζ from $\zeta = -\infty$. It is noticed that the full integral of the area Eq. (5) gives $\varphi(\infty) = 2\pi$ indicating TPSIT.

The other case, $C_0 = +1$, may be identical mathematically to $C_0 = -1$. However, this choice involves a complete population inversion at an earlier time, and it corresponds to coherent two-photon amplification.[12,13] The solutions are

$$\varphi(\zeta) = 2\tan^{-1}[-\beta'(\zeta-\zeta_0)] , \tag{13}$$

and

$$\mathcal{E}_i^2(\zeta) = -\frac{\beta_i'}{\kappa}\frac{1}{1+\beta'^2(\zeta-\zeta_0)^2} . \tag{14}$$

In this case, we clearly see from the branch solution of Eq. (13) that the area is amplified toward $(2n+1)\pi$ from $(2n-1)\pi$ with increasing ζ. The area integral yields also $\varphi(+\infty) - \varphi(-\infty) = 2\pi$, namely the leading edge is amplified, while the trailing edge is attenuated. It is not in contradiction with relativity because information exceeds $\zeta = -\infty$ on the pulse leading edge.

(iii) $0 < |C_0| < 1$. This limitation yields single-pulse solutions with the areas

$$\varphi_\pm(\zeta) = \sin^{-1}\left((1-C_0^2)^{1/2}\frac{(C_0 e^{-\beta'(\zeta-\zeta_0)} - 1) \pm (e^{-\beta'(\zeta-\zeta_0)} - C_0)e^{-\beta'(\zeta-\zeta_0)}}{1-C_0^2 + (e^{-\beta'(\zeta-\zeta_0)} - C_0)^2}\right). \tag{15}$$

Here, taking the minus sign of $\varphi_\pm(\zeta)$, we obtain $\varphi_-(\zeta) = \sin^{-1}[-(1-C_0^2)]^{1/2}$, which gives an unstable growing-up solution. Taking $\varphi_+(\zeta)$, we can easily obtain

$$\mathcal{E}_i^2(\zeta) = \frac{\beta_i'}{\kappa}\frac{(1-C_0^2)^{1/2}}{\cosh[\beta'(\zeta-\zeta_0)] - C_0} . \tag{16}$$

(iv) $C_0 = 0$. The solution is given by

$$\varphi_+(\zeta) = -\tanh\beta'(\zeta-\zeta_0) , \tag{17}$$

and the electric power envelope becomes

$$\mathcal{E}_i^2(\zeta) = \frac{\beta_i'}{\kappa}\operatorname{sech}[\beta'(\zeta-\zeta_0)] . \tag{18}$$

The envelopes represent solitary wave solutions analogous to the soliton solution obtained in the sine-Gordon equation.

The cases $|C_0|<1$, i.e., (iii) and (iv), represent not only nonzero population in the excited state, but also a coherent superposition of states preserved over a long time. Especially, the situation of $C_0=0$ may imply an equal population density and complete mixed states both in the upper state and the lower ground state of the two-photon allowed transition. Therefore, the experimental observation of these solutions seems to be very difficult at present, but perhaps an ingenious idea will be forthcoming.

In summary, we found the stationary solutions for coherent two-photon pulse propagation. In the special case $C_0=0$, the solitary wave solutions were obtained. The general solutions exhibiting transient pulse propagation[10,11] for any input wave form could be obtained in a similar way. One of the future problems is to find transient solutions[14] for experimental observation, removing the assumptions for simplicity.

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