PHYSICAL REVIEW A VOLUME 16, NUMBER 5 NOVEMBER 1977

Similarity analysis of magnetohydrodynamic flows with viscous stress relaxation

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(Received 1 March 1977)

A novel similarity solution in terms of a hyperelliptic integral is given for a magnetohydrodynamic flow across an azimuthal magnetic field in a diverging duct, under consideration of viscous stress relaxation. Velocity profiles and the critical duct angle for flow separation are calculated as a function of the Reynolds number and the Hartmann number. It is shown that viscous stress relaxation modifies the velocity distribution and reduces considerably the critical duct angle for flow separation at low Reynolds numbers. At large Reynolds numbers, viscous stress relaxation is less important, and the results approach asymptotically those of ordinary magnetofluiddynamics, which is based on a static relation between viscous stresses and the velocity component gradients.

I. INTRODUCTION

In classical fluid mechanics[1] and magnetohydrodynamics,[2] it is assumed that inhomogeneities $\nabla_i v_j$ in the velocity components v_j produce instantaneously viscous stresses Π_{ij}. Mathematically, this is expressed through a phenomenological "flux"-"force" relation, given for incompressible fluids or subsonic flows by[1,2]

$$\Pi_{ij} = -\mu(\nabla_i v_j + \nabla_j v_i).$$

In a real continuum, velocity inhomogeneities do not switch on viscous stresses instantaneously but rather in accordance with a relaxation process of characteristic time τ. Indeed, the moment

$$\Pi_{ij} = m \iiint (c_i c_j - \tfrac{1}{3}c^2\delta_{ij}) f(\vec{c}, \vec{r}, t) d^3\vec{c}$$

of the Boltzmann equation yields, for an incompressible viscous fluid ($\nabla \cdot \vec{v} = 0$), the relaxation equation (discussed in connection with the 13-moment approximation in the Appendix):

$$\partial_t \Pi_{ij} + v_k \nabla_k \Pi_{ij} = -\tau^{-1}\Pi_{ij} - p(\nabla_i v_j + \nabla_j v_i),$$

if thermal forces and terms of higher order in the field derivatives are neglected. The viscosity μ, the fluid pressure p, and the viscous stress relaxation time τ are interrelated by $\mu = p\tau$. This equation satisfies the basic requirements of a (classical) $\vec{r}, t$-dependent field equation since it (i) contains space and time derivatives, and (ii) is invariant against Galilei transformation. The static Navier-Stokes relation is not in accordance with either requirement.

The third equation is in the typical form of an inhomogeneous relaxation equation, with a forcing term, $-p(\nabla_i v_j + \nabla_j v_i)$. It is seen that it reduces to the first equation if the temporal ($\partial_t \Pi_{ij}$) and convective ($v_k \nabla_k \Pi_{ij}$) relaxation terms are disregarded.

According to the first equation, viscous stresses would propagate in accordance with a (parabolic) diffusion equation (continuous "signals" and infinite speed of propagation). According to the third equation, viscous stresses would propagate in accordance with a (hyperbolic) wave equation (discontinuous "signal" and finite speed of propagation). This is readily shown, e.g., by combining the equation of motion for the viscous fluid with the first and third stress transport equations, respectively, for the case of a small one-dimensional velocity perturbation. Thus, these equations give rise to a qualitatively significant different behavior in viscous stress transport. Quantitatively, the term $\partial_t \Pi_{ij}$ is of importance for short processes with a duration time $t \lesssim \tau = \mu/p$. A criterion for the quantitative significance of the term $v_k \nabla_k \Pi_{ij}$ is not as easily establishable, since $\nabla_k \Pi_{ij}$ may be quasisingular at certain points of the fluid. Similarly, the rigorous theory of heat transport has to be based on a (hyperbolic) wave equation.[3]

We consider herein subsonic flows of dense, ionized gases across an external azimuthal magnetic field $\vec{B}_0$ in a duct with inclined walls (so-called diffuser, Fig. 1). The analysis is based on the magnetohydrodynamic equations with viscous-stress relaxation, i.e., we disregard effects of "magnetic" viscosity (which occur in highly rarefied plasma flows) assuming that $\omega_i \tau_i \ll 1$, where $\omega_i = e_i B_0/m_i$ and τ_i are the gyration frequency and collision time of the ions, respectively. By means of an exact (nonlinear) similarity solution, we demonstrate that convective-stress relaxation affects the onset of flow separation, i.e., the first occurrence of wall back flows which, in general, are unstable and result in a turbulent boundary layer. Flow separation is commonly observed if for given Reynolds (R) and Hartmann ($\mathcal{H}$) numbers, the duct angle θ_0 is increased beyond a critical value θ_0^s. The calculated velocity distributions are qualitatively in agreement with velocity profiles observed in diffusers.[4]

Magnetohydrodynamic diffusers with transverse magnetic fields and nonvanishing electric load are frequently used to study the transformation of kinetic flow energy (due to thermal expansion) into electric energy. For the experimental realization, it is suitable to install the diffuser in a similar larger diverging duct through which the working fluid is pumped at a constant rate in order to minimize three-dimensional entrance effects. The development of the boundary-layer with entrance effect is a complex problem which has been analyzed only for magnetohydrodynamic flows between noninclined walls by means of Goertler series expansions.[5]

II. NONLINEAR BOUNDARY-VALUE PROBLEM

Let cylindrical coordinates (r, θ, z) be introduced for the description of the magnetohydrodynamic flow model (Fig. 1). The conducting fluid is bounded in the surfaces $(\theta=+\theta_0, r_1 \leq r \leq r_2)$ and $(\theta=-\theta_0, r_1 \leq r \leq r_2)$ by insulating walls, and in the surfaces $(z=+z_\infty)$ and $(z=-z_\infty)$ by electrodes, which are connected through an ideal circuit $(R=0)$. The conducting fluid is injected through the inner cylinder section $(r=r_1, -\theta_0 \leq \theta \leq +\theta_0, -z_\infty \leq z \leq +z_\infty)$ and removed downstream through the outer cylinder section $(r=r_2, -\theta_0 \leq \theta \leq +\theta_0, -z_\infty \leq \theta \leq +z_\infty)$. The boundary layers at the electrodes are disregarded compared with those at the insulating walls by assuming that the interelectrode spacing is large, $z_\infty \gg \frac{1}{2}(r_1+r_2)\theta_0$. The magnetic field has its sources in an electric current I flowing through a conducting rod $(0 \leq r \leq r_0, -\infty \leq z \leq +\infty, r_0 < r_1)$. In accordance with Stokes's law, $\oint \vec{B}\cdot d\vec{s} = \mu_0 I$, the magnetic field is azimuthal (μ_0 is the permeability of vacuum) and has the induction

$$\vec{B} = (\mu_0/2\pi)(I/r)\vec{e}_\theta, \quad r_0 \leq r < \infty.$$

The radial flow $\vec{v} = u\vec{e}_r$ of the conducting fluid across the magnetic field $\vec{B}$ induces axial electric (E_z) and current density (j_z) fields, presuming

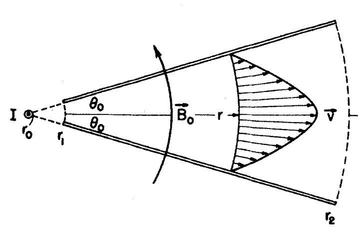

FIG. 1. Geometry of diverging duct with radial velocity field $\vec{v}$ and azimuthal magnetic field $\vec{B}_0$.

that the Hall effect is negligible $(\omega_e \tau_e \ll 1)$,

$$\vec{j} = \sigma(E_z + uB_\theta)\vec{e}_z.$$

The resulting Lorentz force density is a purely radial field which opposes the inducting flow,

$$\vec{j}\times\vec{B} = -\sigma(E_z + uB_\theta)B_\theta\vec{e}_r.$$

Because of $\nabla\times\vec{E} = \vec{0}$ and $\nabla\cdot\vec{j} = \sigma(\nabla\cdot\vec{E} + \vec{B}\cdot\nabla\times\vec{v} - \vec{v}\cdot\nabla\times\vec{B}) = \sigma\nabla\cdot\vec{E} = 0$, the axial electric field is inhomogeneous, and vanishes,

$$\vec{E} = E_z\vec{e}_z = \vec{0}, \quad R = 0,$$

since the load of the external circuit is zero. In this case, the current in the external circuit assumes the maximum value

$$J = \sigma \int_{r_1}^{r_2}\int_{-\theta_0}^{+\theta_0} uB_\theta r\, dr\, d\theta.$$

These equations are based on the assumption that the induced magnetic field is small compared with the external magnetic field, which implies small magnetic Reynolds numbers,[5]

$$R_B = \mu_0\sigma u(r,0)r \ll 1.$$

In this elementary radial-flow model, fluid dynamic and electric end effects at $r=r_1$ and $r=r_2(|\theta| \leq \theta_0)$ are disregarded.

The magnetohydrodynamic diffuser flow under consideration is described by the nonlinear boundary-value problem for the radial velocity $[u = u(r,\theta)]$, stress $[\Pi_{ij} = \Pi_{ij}(r,\theta)]$, and pressure $[p = p(r,\theta)]$ fields:

$$\rho u \frac{\partial u}{\partial r} = -\frac{\partial p}{\partial r} - \left(\frac{1}{r}\frac{\partial}{\partial r}(r\Pi_{rr}) + \frac{1}{r}\frac{\partial \Pi_{\theta r}}{\partial\theta} - \frac{\Pi_{\theta\theta}}{r}\right) - \sigma B_\theta^2(r_0)\left(\frac{r_0}{r}\right)^2 u, \tag{1}$$

$$0 = -\frac{1}{r}\frac{\partial p}{\partial\theta} - \left(\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\Pi_{\theta r}) + \frac{1}{r}\frac{\partial\Pi_{\theta\theta}}{\partial\theta}\right), \tag{2}$$

$$\frac{\partial u}{\partial r} + \frac{u}{r} = 0, \tag{3}$$

where

$$\mu\frac{u}{p}\frac{\partial\Pi_{rr}}{\partial r} + \Pi_{rr} = -2\mu\frac{\partial u}{\partial r}, \tag{4}$$

$$\mu\frac{u}{p}\frac{\partial\Pi_{\theta r}}{\partial r} + \Pi_{\theta r} = -\mu\frac{1}{r}\frac{\partial u}{\partial\theta}, \tag{5}$$

$$\mu\frac{u}{p}\frac{\partial\Pi_{\theta\theta}}{\partial r} + \Pi_{\theta\theta} = -2\mu\frac{u}{r}, \tag{6}$$

and

$$u(r, \theta = \pm\theta_0) = 0, \tag{7}$$

$$\rho\int_{-\theta_0}^{+\theta_0} u(r,\theta) r\, d\theta = Q. \tag{8}$$

Equation (8) specifies the flow rate Q through the diverging duct. For similarity reasons, Eq. (8) is equivalent to inlet $(r=r_1)$ and outlet $(r=r_2)$ boundary conditions. Instead of Eq. (8), it is more convenient to assume the Reynolds number $R(0)$ of the central streamline to be given,[6]

$$\rho u(r, \theta=0) r/\mu = R(0). \tag{9}$$

Equations (1), (2) are the r and θ components of the equation of motion of the conducting fluid in the azimuthal magnetic field $[B_\theta = B_\theta(r_0) r_0/r]$. Equation (3) represents the continuity equation for incompressible radial flow, and Eqs. (4)–(6) describe the convective stress relaxation with a viscosity $\mu = p\tau$. It is noted that for pure radial flow

$$\vec{v}\cdot\nabla\overline{\overline{\Pi}} = u\partial\,\overline{\overline{\Pi}}/\partial r.$$

III. SIMILARITY TRANSFORMATION

The dimensions of the flow fields and the fluid constants are interrelated by

$$(u) = (\mu/\rho r), \quad (p) = (\Pi_{ij}) = (\mu^2/\rho r^2).$$

Accordingly, we try to reduce the partial Eqs. (1)–(6) into ordinary differential equations by means of the similarity transformation:

$$u(r,\theta) = (\mu/\rho) r^{-1} f(\theta), \tag{10}$$

$$p(r,\theta) = (2\mu^2/\rho) r^{-2} P(\theta), \tag{11}$$

$$\Pi_{rr}(r,\theta) = (2\mu^2/\rho) r^{-2} g_{rr}(\theta), \tag{12}$$

$$\Pi_{\theta r}(r,\theta) = (2\mu^2/\rho) r^{-2} g_{\theta r}(\theta), \tag{13}$$

$$\Pi_{\theta\theta}(r,\theta) = (2\mu^2/\rho) r^{-2} g_{\theta\theta}(\theta). \tag{14}$$

The functions $f(\theta)$, $P(\theta)$, and $g_{ij}(\theta)$ are nondimensional. Upon substitution of Eqs. (10)–(14), we obtain from Eqs. (1)–(9) the ordinary boundary-value problem:

$$f^2 = -4P + 2g'_{\theta r} + \mathcal{H}^2 f, \tag{15}$$

$$P = P_0 - g_{\theta\theta}, \tag{16}$$

$$[(P-f)/P]\, g_{rr} = f, \tag{17}$$

$$[(P-f)/P]\, g_{\theta r} = -\tfrac{1}{2} f', \tag{18}$$

$$[(P-f)/P]\, g_{\theta\theta} = -f, \tag{19}$$

where

$$f(\theta = \pm\theta_0) = 0, \tag{20}$$

$$\int_{-\theta_0}^{+\theta_0} f(\theta)\, d\theta = Q/\mu, \tag{21}$$

and

$$P_0 = P(\theta=\pm\theta_0), \quad f(0) = R(0), \tag{22}$$

$$\mathcal{H}^2 = \left(\frac{\sigma}{\mu}\right) B_\theta^2(r) r^2 = (\sigma/\mu)(\mu_0 I/2\pi)^2. \tag{23}$$

From the stress-relaxation Eqs. (17)–(18) one obtains the conventional static stress relations for $P \gg |f|$: $g_{rr} \to f$, $g_{\theta r} \to -\frac{1}{2} f'$, $g_{\theta\theta} \to -f$; $P \gg |f|$.

IV. CLOSED-FORM SOLUTION

Substitution of Eq. (16) into Eq. (19) gives the pressure function $P(\theta)$ in terms of the velocity function $f(\theta)$,

$$P = \tfrac{1}{2}[2f + P_0 + (4f^2 + P_0^2)^{1/2}]. \tag{24}$$

The minus sign of the square root is not applicable as one verifies by means of Eqs. (20) and (22). By eliminating $P(\theta)$ by means of Eq. (16) and $g'_{\theta r}$ by means of Eq. (18), we find from Eqs. (15) and (20) the nonlinear boundary-value problem for the velocity function $f(\theta)$,

$$\begin{aligned} f''[2f + P_0 + (4f^2 + P_0^2)^{1/2}] &+ 2P_0(4f^2 + P_0^2)^{-1/2} f'^2 \\ + [P_0 + (4f^2 + P_0^2)^{1/2}]\{f^2 &+ (4 - \mathcal{H}^2) f \\ &+ 2[P_0 + (4f^2 + P_0^2)^{1/2}]\} = 0, \end{aligned} \tag{25}$$

where

$$f(\theta = \pm\theta_0) = 0. \tag{26}$$

The differential equation for the corresponding flow without stress relaxation[7] is obtained from Eq. (15) as

$$f'' + f^2 + (4 - \mathcal{H}^2) f + 4P_0 = 0 \text{ for } P \gg |f|.$$

The substitution,

$$\frac{df}{d\theta} = \psi, \quad \frac{d^2 f}{d\theta^2} = \frac{1}{2}\frac{d\psi^2}{df}, \tag{27}$$

transforms Eq. (25) into a nonlinear differential equation of first order for $\psi = \psi(f)$,

$$\frac{d}{df}\psi^2 + F(f)\psi^2 + G(f) = 0, \tag{28}$$

where

$$\begin{aligned} F(f) &\equiv 4P_0\{[2f + P_0 + (4f^2 + P_0^2)^{1/2}](4f^2 + P_0^2)^{1/2}\}^{-1} \\ &> 0, \end{aligned} \tag{29}$$

$$\begin{aligned} G(f) \equiv\; & 2[P_0 + (4f^2 + P_0^2)^{1/2}] \\ & \times \{f^2 + (4 - \mathcal{H}^2) f + 2[P_0 + (4f^2 + P_0^2)^{1/2}]\} \\ & \times [2f + P_0 + (4f^2 + P_0^2)^{1/2}]^{-1} \\ & \gtrless 0. \end{aligned} \tag{30}$$

The general solution of Eq. (28) is found by the method of variation of the integration constant of the solution of the associated homogeneous equation as

$$\psi^2(f) = \exp\left(-\int_{f_0}^{f} F(f)\,df\right) \times\left[\psi_0^2 - \int_{f_0}^{f}\exp\left(+\int_{f_0}^{f}F(f)\,df\right)G(f)\,df\right], \quad (31)$$

$f_0 \equiv f(0)$,

where

$$\psi_0 \equiv \psi(f = f(0)) = \left(\frac{df}{d\theta}\right)_{\theta=0} = 0 \quad (32)$$

for symmetrical flows. Combining of Eqs. (27) and (31) yields an integral solution for $\theta=\theta(f)$ from which one obtains the analytical solution $f=f(\theta)$ by inversion

$$\pm\theta = \int_{f_0}^{f(\theta)} H^{-1}(f)\,df, \quad f_0 \equiv f(0), \quad (33)$$

where

$$H(f) = \left\{\exp\left(-\int_{f_0}^{f}F(f)\,df\right) \times\left[\psi_0^2 - \int_{f_0}^{f}\exp\left(+\int_{f_0}^{f}F(f)\,df\right)G(f)\,df\right]\right\}^{1/2}. \quad (34)$$

Equations (33) and (34) represent a closed-form solution for the magnetohydrodynamic diffuser flow with viscous stress relaxation in terms of a hyperelliptic integral, since by Eq. (34),

$$\begin{aligned} H^2(f) = \frac{[P_0 + (4f^2+P_0^2)^{1/2}]}{[2f + (4f^2+P_0^2)^{1/2}]}\Bigg(&\frac{[2f_0 + (4f_0^2+P_0^2)^{1/2}]}{[P_0 + (4f_0^2+P_0^2)^{1/2}]}\psi_0^2 - \tfrac{1}{2}(\tfrac{2}{3}(f^3 - f_0^3) + \tfrac{1}{2}(16 - 2\mathcal{H}^2 - P_0)(f^2 - f_0^2) + P_0\mathcal{H}^2(f - f_0) \\ &+ \tfrac{1}{12}[(4f^2+P_0^2)^{3/2} - (4f_0^2+P_0^2)^{3/2}] \\ &+ \tfrac{1}{2}(8 - \mathcal{H}^2)[f(4f^2+P_0^2)^{1/2} - f_0(4f_0^2+P_0^2)^{1/2}] \\ &+ \tfrac{1}{4}(8 - \mathcal{H}^2)P_0^2\{\ln[2f + (4f^2+P_0^2)^{1/2}] - \ln[2f_0 + (4f_0^2+P_0^2)^{1/2}]\})\Bigg). \end{aligned} \quad (35)$$

In Eq. (33), the ± sign has to be used depending on whether $df/d\theta \gtreqless 0$, or $\theta \lesseqgtr 0$ in case of pure outflows, $f(\theta) > 0$. The integration constant ψ_0 is determined in Eq. (32) for symmetrical flows with an extremum at $\theta = 0$, which are of main practical interest. The remaining integration constant P_0 contained in the solution of Eq. (33) is determined by the boundary condition in Eq. (26), which gives

$$\theta_0 = \int_0^{f_0} H^{-1}(f)\,df. \quad (36)$$

Based on Eqs. (33)–(36), velocity distributions $f(\theta) \geq 0$ of pure outflows have been computed for the typical duct angle $\theta_0 = 5^0$ and given Reynolds numbers $R \equiv R(0) = f(0)$ of the central stream line $\theta = 0$, with the Hartmann number $\mathcal{H}$ as parameter, $\mathcal{H}^2 \geq 0.7R > \frac{2}{3}R$. In the presence of viscous stress relaxation, the onset of flow separation, as will be shown, is inhibited at large Reynolds numbers R for Hartmann numbers

$$\mathcal{H} > \mathcal{H}_{cr}; \; \mathcal{H}_{cr}^2 \cong \tfrac{2}{3}R \text{ for } R \gg 1.$$

The velocity distributions in Figs. 2, 3, and 4 represent net outflows without backflow regions since $\mathcal{H} > \mathcal{H}_{cr}$. Figure 2 shows $f(\theta)$ for the relatively small Reynolds number $R = 10^3$ and $\mathcal{H}^2 = 0.7R - 10\,R(\theta = 5^0)$. The velocity distributions become flatter and the velocity gradients at the walls $\theta = \pm\,\theta_0$ increase in magnitude as $\mathcal{H}$ increases.

Figure 3 shows $f(\theta)$ for the moderate Reynolds number $R = 10^4$ and $\mathcal{H}^2 = 0.7R - 5R(\theta_0 = 5^0)$. In this case, $|df(\theta = \pm\theta_0)/d\theta|$ decreases with decreasing $\mathcal{H}$ so that a well developed flow exists only in the central region for small $\mathcal{H}^2 > \frac{2}{3}R$. Figure 4 shows $f(\theta)$ for the relatively large Reynolds number $R = 10^5$ and $\mathcal{H}^2 = 0.7R - 5R$ $(\theta_0 = 5^0)$. For small Hartmann numbers, $\mathcal{H}^2 > \frac{2}{3}R$, the flow is considerably depressed in the extended regions adjacent to the walls so that Reynolds numbers $R(\theta) \cong 10^5$ are realized only in the limited central section $|\theta| < \frac{1}{10}\theta$ of the duct. The $f(\theta)$ curves in Fig. 4 show clearly the transition to the limiting velocity distribution, for which $|df(\theta = \pm\theta_0)/d\theta|$ assumes the smallest realizable value, as $\mathcal{H} \to \frac{2}{3}R$. It is con-

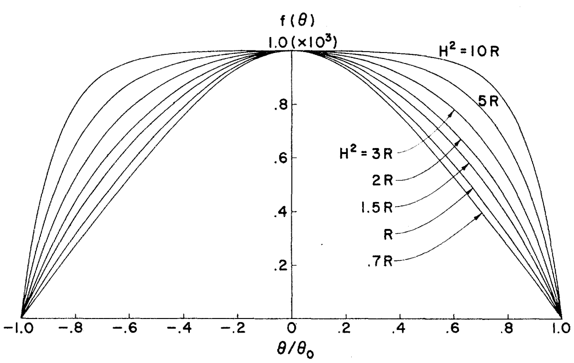

FIG. 2. $f(\theta)$ vs θ for $R = 10^3$ and various $\mathcal{H}$ $(\theta_0 = 5°)$.

cluded that well developed velocity distributions exist for sufficiently large Hartmann numbers $\mathcal{H}=\mathcal{H}(R;\theta_0)$, e.g., $\mathcal{H}^2>\frac{2}{3}R$ for $R=10^3$ (Fig. 2) and $\mathcal{H}^2\gtrsim 2R$ for $R=10^4$ (Fig. 3) and $R=10^5$ (Fig. 4).

Velocity distributions of the typical form shown in Figs. 3 and 4 have been observed in magnetohydrodynamic diffuser-flow experiments.[4] A qualitative comparison is, however, not possible, since a conical diffuser was used in the experiments.[4] Velocity distributions without stress relaxation have been calculated previously.[8]

V. MAGNETOHYDRODYNAMIC FLOW SEPARATION

The integral solution derived in Eqs. (33), (35), and (36) describes physical flows as long as $P_0 \equiv P(\theta=\pm\theta_0)\geq 0$ for the given parameters R, $\mathcal{H}$, and θ_0, since the pressure field has to be positive everywhere, $P(\theta)\geq 0$, $|\theta|\leq\theta_0$. The boundary-value problem in Eqs. (25)–(26) becomes, in the limiting case $P_0=0$,

$$\tilde{f}''+\tfrac{1}{2}(8-\mathcal{H}^2)\tilde{f}+\tfrac{1}{2}R\tilde{f}^2=0, \tag{37}$$

$$\tilde{f}(\theta=\pm\theta_0)=0, \tag{38}$$

where

$$\tilde{f}(\theta)\equiv\frac{f(\theta)}{f(0)}=\frac{f(\theta)}{R};\quad \tilde{f}(\theta=0)\equiv 1;\quad R\equiv R(0). \tag{39}$$

Since $P_0=0$ in Eq. (37), it cannot be reduced to the relaxation-free case $(P, P_0\gg f)$. The solution for the limiting flow with vanishing wall pressure P_0 is by Eqs. (37)–(38),

$$\pm\theta=(3/R)^{1/2}\int_1^{\tilde{f}(\theta)}\frac{d\tilde{f}}{[-Q(\tilde{f})]^{1/2}}, \tag{40}$$

where

$$Q(\tilde{f})=\tilde{f}^3+\frac{3}{2R}(8-\mathcal{H}^2)\tilde{f}^2-\frac{3}{2R}(8-\mathcal{H}^2)-1 \tag{41}$$

is a trinomial in $\tilde{f}$ which has one real and two complex conjugate roots since, in general, $2R>8-\mathcal{H}^2$:

$$\tilde{f}_1=1,$$

$$\tilde{f}_{2,3}=\frac{3}{4R}(8-\mathcal{H}^2+\tfrac{2}{3}R)\left[-1\pm i\left(-1+\frac{(\frac{8}{3}R)}{8-\mathcal{H}^2+\frac{2}{3}R}\right)^{1/2}\right]. \tag{42}$$

Equation (40) contains an elliptic integral which is resolved by means of the substitution

$$\tilde{f}(\theta)=1-\lambda^2\left(\frac{1-\cos\phi}{1+\cos\phi}\right),\quad 0\leq\phi\leq\pi \tag{43}$$

which gives

$$\pm\theta=-(3/R)^{1/2}\lambda^{-1}F(\phi,k) \tag{44}$$

where

$$\lambda=\{3[1+(8-\mathcal{H}^2)R^{-1}]\}^{1/4}, \tag{45}$$

$$k^2=\tfrac{1}{2}\{1+\tfrac{3}{4}\lambda^{-2}[2+(8-\mathcal{H}^2)R^{-1}]\}. \tag{46}$$

Inversion of Eq. (44) and substitution of ϕ in Eq. (43) yields the explicit solution for the case $P_0=0$:

$$\tilde{f}(\theta)=1-\lambda^2\,\frac{1-\mathrm{cn}[(\frac{1}{3}R)^{1/2}\lambda\theta,k]}{1+\mathrm{cn}[(\frac{1}{3}R)^{1/2}\lambda\theta,k]}. \tag{47}$$

According to Eq. (44), the critical duct angle $\theta_0(P_0=0)$ at which the wall pressure is zero is given by the boundary condition $\tilde{f}(\theta=\pm\theta_0)=0$ as,

$$\theta_0(P_0=0)=(3/R)^{1/2}\lambda^{-1}F(\phi_0,k), \tag{48}$$

$$\phi_0=\arccos[(\lambda^2-1)/(\lambda^2+1)], \tag{49}$$

in terms of the characteristic flow numbers R and $\mathcal{H}$ [$\lambda=\lambda(R,\mathcal{H})$, $k=k(R,\mathcal{H})$ by Eqs. (45)–(46)]. Since $f_{2,3}=0$ for $\mathcal{H}^2-8=\frac{2}{3}R$ by Eq. (42), the integral in Eq. (40) diverges for $\tilde{f}(\theta=\pm\theta_0)=0$. Similarly, Eq. (47) diverges in this case since $k\to 1$ for $\mathcal{H}^2-8\to\frac{2}{3}R$ by Eq. (46). It is recognized that

$$0\leq\theta_0\leq\theta_0(P_0=0)=\pi\ \text{for}\ \mathcal{H}^2\geq 8+\tfrac{2}{3}R, \tag{50}$$

$$0<\theta_0\leq\theta_0(P_0=0)<\pi\ \text{for}\ \mathcal{H}^2<8+\tfrac{2}{3}R. \tag{51}$$

Accordingly, physical flow solutions with $P(\theta)$

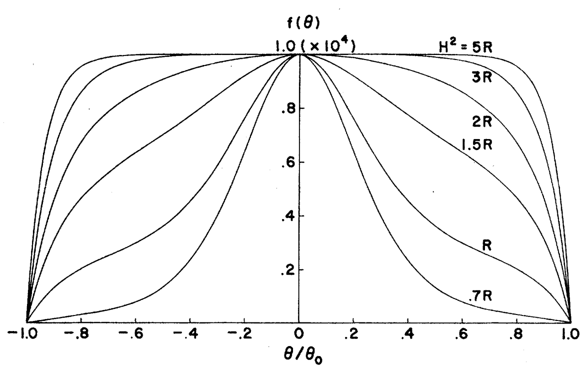

FIG. 3. $f(\theta)$ vs θ for $R=10^4$ and various $\mathcal{H}(\theta_0=5°)$.

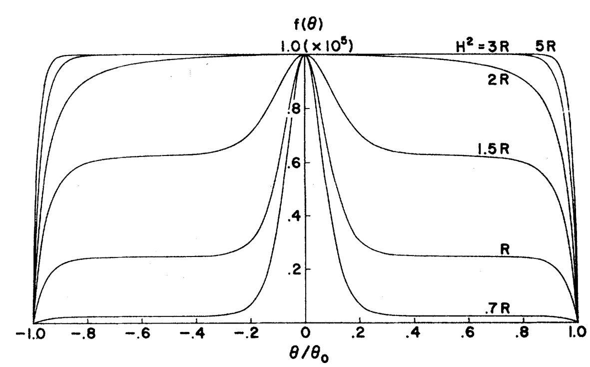

FIG. 4. $f(\theta)$ vs θ for $R=10^5$ and various $\mathcal{H}(\theta_0=5°)$.

≥ 0, $|\theta| \leq \theta_0$, exist for all duct angles $0 < \theta_0 \leq \pi$ if the Hartmann number is

$$\mathcal{H} \geq \mathcal{H}_{cr}, \quad \mathcal{H}_{cr}^2 = 8 + \tfrac{2}{3}R. \tag{52}$$

On the other hand, if $\mathcal{H} < \mathcal{H}_{cr}$, physical flow solutions with $P(\theta) \geq 0$, $|\theta| \leq \theta_0$, exist only for duct angles $\theta_0 \leq \theta_0(P_0 = 0) < \pi$.

In Fig. 5, the critical duct angle $\theta_0^s \equiv \theta_0(P_0 = 0)$ for vanishing wall pressure is plotted versus $R \equiv R(0)$ with $\mathcal{H}$ as a parameter. It is seen that θ_0^s decreases with increasing R, but increases with increasing $\mathcal{H}$. The stabilizing effect of the magnetic field at sufficiently large Hartmann numbers $\mathcal{H}$ is apparent, in particular, in the regions $\mathcal{H} > \mathcal{H}_{cr}(R)$.

The critical duct angle $\theta_0^s = \theta_0(P_0 = 0)$ is also obtained from the condition $d\bar{f}(\theta = \pm\theta_0)/d\theta = 0$, if viscous-stress relaxation is not taken into consideration.[7] The corresponding curves $\theta_0^s = \theta_0(R, \mathcal{H})$ are shown dashed in Fig. 5. Comparison indicates that for the same R and $\mathcal{H}$, θ_0^s "without stress relaxation" is considerably larger than θ_0^s "with stress relaxation" for relatively small Reynolds numbers, $R < 10^2$. Accordingly, viscous-stress relaxation has a destabilizing effect on the flow, which, however, is completely negligible for large Reynolds numbers, $R \gg 10^2$. As the wall pressure drops to zero, the laminar flow solution can no longer be realized, and flow separation sets in for duct angles $\theta_0 > \theta_0^s(R, \mathcal{H}) < \pi$.

In the classical similarity theory for incompressible viscous flow between inclined walls,[6] solutions with a (positive) homogeneous overpressure p_0 exist so that one does not have to be concerned about negative pressures in back-flow regions at the onset of separation. The similarity analysis of the corresponding compressible flow[8] no longer permits solutions with over pressure, and the limiting flow solution with $df(\theta = \pm\theta_0)/d\theta = 0$ exhibits a negative wall pressure, $P_0 < 0$. For this reason, the onset of separation was determined from the condition $P(\theta = \pm\theta_0) \equiv 0$.[8] Similarly, here we have associated the onset of separation in flows with stress relaxation with the vanishing of the wall pressure, $P_0 = 0$. The observed stabilizing effect of the magnetic field is due to the increase in the wall pressure (Fig. 6) with increasing Hartmann number $\mathcal{H}$. In conventional incompressible fluid dynamics without stress relaxation, the conditions $df(\theta = \pm\theta_0)/d\theta = 0$ and $P(\theta = \pm\theta_0) = 0$ lead to the same separation criterion.

Interest in this theoretical problem arose in connection with experiments on boundary-layer separation in incompressible liquid metal flow and subsonic magnetogasdynamic flow in nonuniform magnetic fields and ducts.[4,9] If the Hartmann number is set to zero, the closed-form solutions presented reduce to those for the flow of electrically nonconducting ordinary fluids with viscous-stress relaxation in diverging ducts.

ACKNOWLEDGMENT

This work was supported in part by the U. S. Office of Naval Research.

APPENDIX: VISCOUS STRESS TRANSPORT EQUATION

Incompressible magnetohydrodynamics is applicable to conducting liquids such as liquid metals, and also, as an approximation, to collision-dominated ionized gases and plasmas at subsonic flow speeds. In each case, the viscous momentum transport is due to the heavy atomic particles while the electrons affect only the stress relaxation time τ through cross-collisions. For this reason, conducting fluids can be described by one-fluid magnetohydrodynamic equations.

The stress-relaxation equation for incompres-

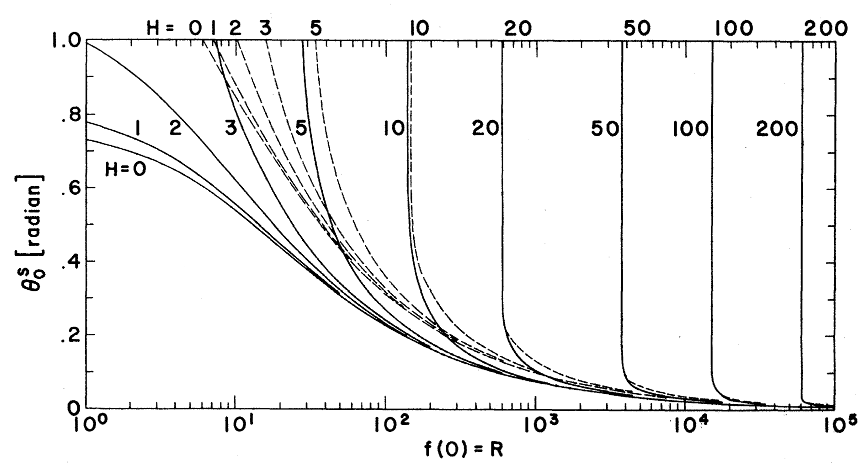

FIG. 5. Critical duct angle θ_0^s for separation vs R for various $\mathcal{H}$, with (——) and without (---) viscous-stress relaxation.

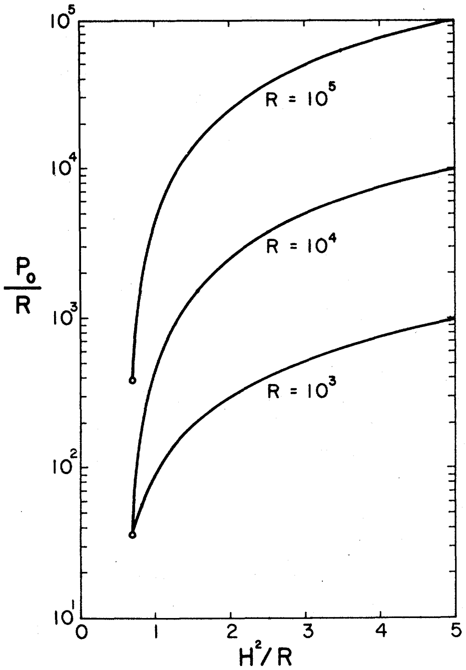

FIG. 6. Wall pressure P_0 vs $\mathcal{H}$ for various R ($\theta_0 = 5^0$).

sible ($\nabla \cdot \vec{v} = 0$) flows is derived as a special case of the stress-relaxation equation for compressible ($\nabla \cdot \vec{v} \neq 0$) flows based on the Boltzmann equation presuming that the fluids are sufficiently dense and collision dominated. For liquids a similar stress relaxation equation can be derived from the Born-Green kinetic equation which differs from the Boltzmann equation through the collision integral for many-body collisions. The most elegant method of solution for the Boltzmann equation is the 13-moment approximation due to Grad,[10-12] which is mathematically also more rigorous than the Chapman-Enskog and Hilbert methods.[13] The closure of the 13-moment approximation is forced by truncating the third-order heat-flux tensor Q_{ijk} in terms of the heat-flux vector $\vec{q}$

$$Q_{ijk} \equiv \iiint_{-\infty}^{+\infty} mc_i c_j c_k f(\vec{c}, \vec{r}, t) d^3\vec{c}$$

$$= \tfrac{2}{5}(q_i \delta_{jk} + q_j \delta_{ki} + q_k \delta_{ij}), \qquad \text{(A1)}$$

where $f(\vec{c}, \vec{r}, t)$ is the distribution of thermal velocities $\vec{c}$. This truncation affects mainly the heat-flow dynamics, and is, therefore, an excellent approximation for quasi-isothermal magnetohydrodynamic flows ($\vec{q} = \vec{0}$).

Following the original deduction of Grad for a neutral one-component gas,[10] we multiply the Boltzmann equation for particles of mass m_r and charge e_r in an electromagnetic field $\vec{E} - \vec{B}$ by $\frac{1}{2} m_r [\vec{c}_r \vec{c}_r - (\frac{1}{3}) c_r^2 \overleftrightarrow{\delta}]$ and integrate it over the entire space $\int\int\int d^3\vec{c}_r$ of the thermal velocities $\vec{c}_r$ of the r particles. Thus, the following moment-conservation equation is obtained for the nonhydrostatic stress tensor $\overleftrightarrow{\Pi}_r = \overleftrightarrow{P}_r - p_r \overleftrightarrow{\delta}$ of the r component (summation s over the remaining components s of the fluid):

$$\frac{\partial}{\partial t}\overleftrightarrow{\Pi}_r + \overleftrightarrow{\Pi}_r \times \vec{\omega}_r + [\overleftrightarrow{\Pi}_r \times \vec{\omega}_r]^{-1} + \vec{v}_r \cdot \nabla \overleftrightarrow{\Pi}_r + \overleftrightarrow{\Pi}_r \nabla \cdot \vec{v}_r + p_r(\nabla \vec{v}_r + [\nabla \vec{v}_r]^{-1} - \tfrac{2}{3}\nabla \cdot \vec{v}_r \overleftrightarrow{\delta}) + \tfrac{2}{5}(\nabla \vec{q}_r + [\nabla \vec{q}_r]^{-1} - \tfrac{2}{3}\nabla \cdot \vec{q}_r \overleftrightarrow{\delta})$$
$$+ (\overleftrightarrow{\Pi}_r \cdot \nabla \vec{v}_r + [\overleftrightarrow{\Pi}_r \cdot \nabla \vec{v}_r]^{-1} - \tfrac{2}{3}\overleftrightarrow{\Pi}_r : \nabla \vec{v}_r \overleftrightarrow{\delta}) = \sum_s \iiint_{-\infty}^{+\infty} m_r [\vec{c}_r \vec{c}_r - \tfrac{1}{3}\vec{c}_r^2 \overleftrightarrow{\delta}] C_{rs} d\vec{c}_r, \qquad \text{(A2)}$$

where

$$C_{rs} = \int \cdots \int (f_r^* f_s^* - f_r f_s) \sigma_{rs}(g_{rs}, \Omega) g_{rs} d\Omega \, d\vec{c}_s \qquad \text{(A3)}$$

is the binary collision integral,[10] and $\vec{\omega}_r = -e_r \vec{B}/m_r$ is the gyration frequency ($[\;]^{-1}$ designates the inverse tensor). In the 13-moment approximation, the distribution functions $f_r(\vec{c}_r, \vec{r}, t)$ and $f_s(\vec{c}_s, \vec{r}, t)$ are expanded in Hermite tensorial polynomials, the expansion coefficients being the first (scalar) 13 moments of the distribution function.[10] By means of these expansions and Eq. (A3) it can be shown that Eq. (A2) is of the form

$$\frac{\partial \overleftrightarrow{\Pi}_r}{\partial t} + \overleftrightarrow{B}_r = -\tau_r^{-1} \overleftrightarrow{\Pi}_r - \sum_{s \neq r} \alpha_{rs} \tau_{rs}^{-1} \overleftrightarrow{\Pi}_s, \qquad \text{(A4)}$$

where τ_r is the viscous-stress relaxation time[14] and τ_{rs} is the relaxation time describing the linear momentum exchange[14] between the components r and $s \neq r$. $\overleftrightarrow{B}_r$ is an abbreviation for the remaining tensor terms on the left side of Eq. (A2), and α_{rs} are numerical coefficients. For representative times $t \gg \tau_r$, the term $\partial \overleftrightarrow{\Pi}_r / \partial t$ in Eq. (A4) is negligible, and one obtains the quasi-equilibrium, $\overleftrightarrow{\Pi}_r$ proportional to the sum of the various driving force tensors.[10] The tensor $\overleftrightarrow{B}_r$ reduces not always to the velocity gradients $\nabla \vec{v}_r$ of the Navier-Stokes relation,[10] and for representative times $t \lesssim \tau_r$, the 13-moment approximation does not approximate the phenomenological Navier-Stokes relation and gives better results.[10]

For magnetohydrodynamic applications, a simple-stress transport equation for the electrically conducting, incompressible fluid as a whole can be deduced from Eq. (A2) by neglecting those terms which are small compared with the leading terms. The contribution of the electrons (e) to the fluid of ions (i) and atoms (a) as a whole is insignificant since $m_e \ll m_{i,a}$ and[14] $\alpha_{ei}, \alpha_{ea} \ll 1$. The magnetic anisotropy terms are negligible since $\omega_i \tau_i \ll 1$ for the heavy ions, and vanish for the neutral atoms ($\omega_a = 0$). The $\nabla \cdot \vec{v}_r$ terms can be disregarded for incompressible fluids and subsonic (compressible) flows. The $\vec{q}_r$ terms are negligible for quasi-isothermal flows, and the terms $\overleftrightarrow{\Pi}_r \cdot \nabla \vec{v}_r$ are of the order of magnitude of quadratic terms in $\nabla \vec{v}_r$, and therefore small compared to the linear ones. The stresses in the electron gas have no effect on the stress distribution of the fluid as a whole ($m_e \ll m_{i,a}$). Thus, one obtains from Eq. (A2) as stress transport equation for incompressible, quasi-isothermal fluid as a whole:

$$\partial \overleftrightarrow{\Pi}/\partial t + \vec{v} \cdot \nabla \overleftrightarrow{\Pi} = -\tau^{-1} \overleftrightarrow{\Pi} \; - p(\nabla \vec{v} + [\nabla \vec{v}]^{-1}), \qquad \text{(A5)}$$

where $\vec{v} = \sum n_r m_r \vec{v}_r / \sum n_r m_r$ is the mean-mass velocity of the fluid. The relaxation frequency τ^{-1} of the total stress tensor $\overleftrightarrow{\Pi}$ is a linear combination of the inverse relaxation times $\tau_i^{-1}, \tau_a^{-1}, \tau_{rs}^{-1}$. The Reynolds number in the preceding similarity analysis and τ are related by $R(0) = [\rho u(0) r / p(0)] \tau^{-1}$.

Equation (A5) can also be derived directly from elementary physical arguments. The Navier-Stokes driving force $-p(\nabla \vec{v} + [\nabla \vec{v}]^{-1})$ of $\overleftrightarrow{\Pi}$ follows from the symmetry argument of Einstein. The term $\partial \overleftrightarrow{\Pi}/\partial t$ results from the fact that the Navier-Stokes quasi-equilibrium $\overleftrightarrow{\Pi}/\tau = -p(\nabla \vec{v} + [\nabla \vec{v}]^{-1})$ develops within a time of the order of the "collision" time τ. Finally, the convective term $\vec{v} \cdot \nabla \overleftrightarrow{\Pi}$ has to be added in order to make Eq. (A5) invariant against Galilei transformations ($\vec{r}' = \vec{r} - \vec{w}t, t' = t$).

We have based the deduction of the stress transport equation on Grad's 13-moment theory, which gives all driving forces (for the viscous stresses) which have a simple physical meaning, i.e., not only the Navier-Stokes forces $\sim \nabla \vec{v}$ and $\sim [\nabla \vec{v}]^{-1}$. It is seen that the 13-moment theory is more comprehensive and more rigorous than the Navier-Stokes theory, and "refuses to predict results which may be inaccurate."[10] On the other hand, the failures of the Navier-Stokes stress equation may assume catastrophic proportions, e.g., it "predicts smooth solutions for shock strengths of infinite magnitude (with a transition from negative to a positive density)."[10]

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