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Quantization of the linearly damped harmonic oscillator

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A previously developed novel theory for the formal canonical quantization of *classically dissipating systems* will be the starting point for a detailed discussion of the quantum statistical aspects of the simple linearly damped harmonic oscillator. The formalism essentially involves *complex* classical canonical coordinates and momenta, and thus quite naturally leads to the possibility of creation and annihilation operators. Furthermore, the occurrence of quantal *noise* operators appears to be of principal importance for the conservation of the fundamental commutator in the course of time, as will be expressed in a simple *fluctuation-dissipation relation*. Making a canonical transformation back to the real, Cartesian Hermitian position and momentum an "effective" Hermitian Hamiltonian will be derived, with which a transformation is made from the Heisenberg frame to the Schrödinger frame where the density operator equation will be computed. This will make it obvious that no proper Schrödinger equation exists for the dissipative subsystem on its own, thus reflecting an *incomplete knowledge*. The master equation will then be translated into its Wigner representation. The intimate connection between the diffusion coefficients in the resulting Fokker-Planck equation and the *uncertainty relation* will be demonstrated in a clear fashion.

I. INTRODUCTION

The quantization of dissipative systems appears to have presented a tantalizing problem, as may be concluded from the abundance of papers on the subject over the last decades. For a rather extensive listing of them we refer to Refs. 1–3. One of the most powerful methods for quantization may be found in Dirac's canonical procedure. However, since the usual classical Hamiltonian formulation cannot cope with frictional phenomena,[4,5] the solution of the quantization problem is not obvious in that case.

Several historical solutions to the problem have been proposed, of which we only briefly mention the following. Kanai[6] introduced an explicitly time-dependent Hamiltonian, which however has been shown to violate the fundamental commutator of position and momentum, and hence violates the uncertainty principle. Kostin[7] invented a nonlinear Schrödinger equation. Apart from the fact that Kostin's Hamiltonian thus obscures the usual Hilbert space formulation of quantum mechanics,[8] in that it violates the superposition principle and does not produce the frictional frequency shift, we mention the paradoxical existence of a complete set of stationary energy states. The latter result also applies to Hasse's[2] nonlinear frictional potential.

A recent novel generalization of Hamiltonian classical mechanics to complex canonical variables lifted the restriction to real Hamiltonians.[1] The intimate relation between the purely imaginary part of the Hamiltonian and the dissipation has been shown. The quantization procedure now leads to a non-Hermitian Hamiltonian operator and the possible occurrence of quantal "noise" sources.

The noise sources reflect the incomplete knowledge we have about a quantum system when going from known deterministic dissipative dynamics to its quantized version, without having detailed information concerning the underlying microscopic quantal interactions. That is, one can always add to quantum equations operator functions whose expectation values vanish. In the quantum mechanics of dissipative systems these "hidden" operators appear to be fundamentally associated with the damping. These findings are in line with the results from, for example, Senitzky's[9] fully quantal reservoir-coupling approach.

In the present paper we report a detailed investigation of the consequences of the above described Hamiltonian quantization procedure. For this purpose we have considered one of the simplest conceivable dissipative models: the linearly damped harmonic oscillator. In Sec. II we transform from the classical, real, Cartesian coordinates to the complex canonical variables and apply the formal quantization. Having transformed back to the original dynamical variables we discuss in Sec. III some relevant properties of the appearing noise operators and introduce a Hermitian effective Hamiltonian. In Sec. IV we then transform from the Heisenberg frame to the Schrödinger frame and derive the master equation, which will be displayed in Wigner form in Sec. V. Then, in Sec. VI, the result can be conveniently discussed in relation to Heisenberg's uncertainty principle. Some final remarks will be made in Sec. VII.

II. THE FORMAL QUANTIZATION

The classical deterministic equation of motion for a linearly damped harmonic oscillator in one dimension reads

$$\ddot{x}+2\lambda\dot{x}+\Omega^2 x=0. \quad (1)$$

In the analysis we suppose the system to be undercritically damped[10] and introduce the complex coordinate

$$q=\omega^{-1/2}[p+(\lambda-i\omega)x], \quad (2)$$

where $\omega=(\Omega^2-\lambda^2)^{1/2}$ and $p=\dot{x}$. By means of (1) the equation of motion for q is easily found to be

$$\dot{q}+i\omega q+\lambda q=0. \quad (3)$$

We briefly recall from the general theory[1] that (3) may be obtained as the Euler-Lagrange equation from the complex Lagrangian

$$\mathcal{L}=\tfrac{1}{2}i(q^*\dot{q}-q\dot{q}^*)-(\omega-i\lambda)q^*q \quad (4)$$

by variation of q^*. The complex conjugate of (3) results from $\mathcal{L}^*$ by variation of q. From (4) the canonical momentum conjugate to q has been defined as

$$\pi=\frac{\partial\mathcal{L}}{\partial\dot{q}}=\frac{i}{2}q^*. \quad (5)$$

Now it has been shown to be possible to formulate a complex Hamiltonian $\tilde{\mathcal{H}}$ which generates the dynamics and which in the present case may be given as[1]

$$\tilde{\mathcal{H}}=\tilde{H}+i\tilde{\Gamma} \quad (6)$$

$$\tilde{H}=-i\omega\pi q;\quad \tilde{\Gamma}=i\lambda\pi q. \quad (7)$$

It is immediately verified that the generalized Hamiltonian equations

$$\dot{q}=\frac{\partial\tilde{\mathcal{H}}}{\partial\pi};\quad \dot{\pi}=-\frac{\partial\tilde{\mathcal{H}}^*}{\partial q}, \quad (8)$$

indeed lead to the correct equation of motion (3) and its conjugate. The general formalism as presented in Ref. 1 now states that the above classical mechanics leads to a quantum mechanics in the Heisenberg frame with the Hamiltonian operator $\hat{\mathcal{H}}$

$$\hat{\mathcal{H}}=\hat{H}+i\hat{\Gamma}, \quad (9)$$

$$\hat{H}=-i\omega\hat{\pi}\hat{q};\quad \hat{\Gamma}=-\tfrac{1}{2}\hbar\lambda, \quad (10)$$

and where the equations of motion read

$$\dot{\hat{q}}=-(i/\hbar)[\hat{q},\hat{H}]_-+(1/\hbar)[\hat{q},\hat{\Gamma}]_++\hat{\mathfrak{N}}_q(t), \quad (11)$$

$$\dot{\hat{\pi}}=-(i/\hbar)[\hat{\pi},\hat{H}]_-+(1/\hbar)[\hat{\pi},\hat{\Gamma}]_++\hat{\mathfrak{N}}_\pi(t), \quad (12)$$

wherein $[\ ]_-$ and $[\ ]_+$ represent commutators and anticommutators, respectively. The $\hat{\mathfrak{N}}_q$ and $\hat{\mathfrak{N}}_\pi$ have been identified as noise operators, having the property

$$\langle\hat{\mathfrak{N}}_q(t)\rangle_N=\langle\hat{\mathfrak{N}}_\pi(t)\rangle_N=0,$$

where the formal averaging may be taken with respect to the noise only. In Ref. 1 the quantal noise was shown to be intimately connected with the conservation of the fundamental commutator

$$[\hat{\pi},\hat{q}]_-=-i\hbar \quad (13)$$

in the course of time. We will touch in more detail on this in the next section.

It is noted in passing that one can presently introduce the operators $\hat{a}=(2\hbar)^{-1/2}q$ and $\hat{a}^\dagger=-i(\hbar/2)^{-1/2}\hat{\pi}$, which thus have the commutator $[\hat{a},\hat{a}^\dagger]_-=1$ and therefore exhibit the properties of annihilation and creation operators. The oscillation part of the Hamiltonian (9) then leads to the familiar expression $\hat{H}=\hbar\omega\hat{a}^\dagger\hat{a}$.

Having performed the quantization based on the essentially complex classical coordinates and momenta q, π we now wish to return in the quantal formulation to our real physical Cartesian variables $\hat{x}$, $\hat{p}$. In view of (2) and (5) we have the transformations

$$\hat{q}=\omega^{-1/2}[\hat{p}+(\lambda-i\omega)\hat{x}], \quad (14)$$

$$\hat{\pi}=\tfrac{1}{2}i\omega^{-1/2}[\hat{p}+(\lambda+i\omega)\hat{x}]. \quad (15)$$

These relations are easily inverted to yield

$$\hat{x}=\omega^{-1/2}(\tfrac{1}{2}i\hat{q}-\hat{\pi}), \quad (16)$$

$$\hat{p}=\omega^{-1/2}[(\lambda-i\omega)\hat{\pi}-\tfrac{1}{2}i(\lambda+i\omega)\hat{q}]. \quad (17)$$

Since the transformation is linear the general form of the equations of motion (11) and (12) is preserved. Furthermore, simple algebra shows that (13) transforms into

$$[\hat{p},\hat{x}]_-=-i\hbar, \quad (18)$$

so that the transformation is canonical. Then, inserting (14) and (15) into (10) one obtains

$$\hat{H}=\tfrac{1}{2}\hat{p}^2+\tfrac{1}{2}\lambda[\hat{p},\hat{x}]_++\tfrac{1}{2}\Omega^2\hat{x}^2;\quad \hat{\Gamma}=-\tfrac{1}{2}\hbar\lambda, \quad (19)$$

where we have used (18). The result (19) has been communicated before.[8,11,12] It is to be emphasized that our Hamiltonian $\hat{\mathcal{H}}=\hat{H}+i\hat{\Gamma}$ in the case of vanishing friction $\lambda\to 0+$ properly reduces to the conventional expression for the free harmonic oscillator and thus leads to the well-known Hermite polynomials for the stationary energy states of Schrödinger's equation. If $\lambda\neq 0$ the Hamiltonian itself cannot be identified with the energy, of course, and merely plays the role of generator of motion in the sense of (11) and (12).

III. THE EFFECTIVE HAMILTONIAN

In terms of the Hamiltonian $\hat{\mathcal{H}}$ the equation of motion for $\hat{x}$ may be written as

$$\dot{\hat{x}}=-(i/\hbar)(\hat{x}\hat{\mathcal{H}}-\hat{\mathcal{H}}^\dagger\hat{x})+\hat{\mathfrak{N}}_x(t), \quad (20)$$

which by means of (18) and (19) yields, as it should,

$$\dot{\hat{x}} = \hat{p} + \hat{\mathfrak{N}}_x(t). \tag{21}$$

Similarly, the equation for $\hat{p}$ explicitly reads

$$\dot{\hat{p}} = -2\lambda\hat{p} - \Omega^2\hat{x} + \hat{\mathfrak{N}}_p(t). \tag{22}$$

In view of the vanishing expectation values of the noise operators, (21) and (22) properly fulfil Ehrenfest's theorem regarding our starting equation (1).

We next consider some relevant properties of the noise operators using standard techniques from fluctuation theory (see Refs. 1, 13, 14). We write down from the fluctuation equation (20) its coarse-grained short-time solution,[15,16]

$$\hat{x}(\Delta t) = \hat{x}(0) - (i/\hbar)[\hat{x}(0)\hat{\mathcal{H}}(0) - \hat{\mathcal{H}}^\dagger(0)\hat{x}(0)]\Delta t + \int_0^{\Delta t} \hat{\mathfrak{N}}_x(\tau)\,d\tau, \tag{23}$$

where $\hat{x}(0) = \hat{x}(t)$ and $\hat{x}(\Delta t) = \hat{x}(t+\Delta t)$. Invoking the corresponding solution for $\hat{p}$ we form the product

$$\begin{aligned}\hat{p}(\Delta t)\hat{x}(\Delta t) = {} & \hat{p}(0)\hat{x}(0) - \frac{i}{\hbar}[\hat{p}(0)\hat{x}(0)\hat{\mathcal{H}}(0) - \hat{p}(0)\hat{\mathcal{H}}^\dagger(0)\hat{x}(0) + \hat{p}(0)\hat{\mathcal{H}}(0)\hat{x}(0) - \hat{\mathcal{H}}^\dagger(0)\hat{p}(0)\hat{x}(0)]\Delta t \\ & + \int_0^{\Delta t} \hat{p}(0)\hat{\mathfrak{N}}_x(\tau)\,d\tau - \frac{i}{\hbar}\Delta t \int_0^{\Delta t} [\hat{p}(0)\hat{\mathcal{H}}(0) - \hat{\mathcal{H}}^\dagger(0)\hat{p}(0)]\hat{\mathfrak{N}}_x(\tau)\,d\tau \\ & + \int_0^{\Delta t} \hat{\mathfrak{N}}_p(\tau)\hat{x}(0)\,d\tau - \frac{i}{\hbar}\Delta t \int_0^{\Delta t} \hat{\mathfrak{N}}_p(\tau)[\hat{x}(0)\hat{\mathcal{H}}(0) - \hat{\mathcal{H}}^\dagger(0)\hat{x}(0)]\,d\tau \\ & + \int_0^{\Delta t}\int_0^{\Delta t} d\tau\,d\tau'\,\hat{\mathfrak{N}}_p(\tau)\hat{\mathfrak{N}}_x(\tau'), \end{aligned} \tag{24}$$

where a term of second order in Δt has already been omitted. One may also form the product $\hat{x}(\Delta t)\hat{p}(\Delta t)$, subtract the two products, divide by Δt and average over the noise variables, taking account of the usual Markov property[1,3,15] which states that any oscillator operator at some time t is uncorrelated with the noise operators at any later time t'. Now assuming the commutator (18) to be conserved when going from t to $t+\Delta t$, and letting Δt tend to zero, one may present the result as

$$\langle[\hat{\mathfrak{N}}_p(\tau), \hat{\mathfrak{N}}_x(\tau')]_-\rangle_N = (2/\hbar)\langle[\hat{x}, \hat{p}\hat{\Gamma}]_- + [\hat{x}\hat{\Gamma}, \hat{p}]_-\rangle_N\delta(\tau-\tau'). \tag{25}$$

If we define, also for later use, for example[17]

$$\langle\hat{\mathfrak{N}}_p(\tau)\hat{\mathfrak{N}}_x(\tau')\rangle_N = 2D_{px}\delta(\tau-\tau'), \tag{26}$$

$$\Delta_{px} = D_{px} - D_{xp}, \tag{27}$$

and insert $\hat{\Gamma}$ from (19) for the special case of the linearly damped process into (25), the result reads[18]

$$\Delta_{px} = -i\hbar\lambda. \tag{28}$$

We note first of all that Δ_{px} being nonzero is seen to be an essentially quantum mechanical feature (in view of $\hbar \neq 0$). Furthermore, we remark that (28) clearly demonstrates the intimate relation of the operator noise with both the fundamental commutator (regarding $-i\hbar$) and the dissipation (regarding λ). Equation (28) represents a distinct example of a quantal fluctuation-dissipation relation. It will be seen to be of vital importance in the following.

We also require some knowledge concerning the correlations of system operators and noise at the same instant, about which the Markovian property itself does not yield any information. We therefore multiply, by way of example, (23) from the left with $\hat{\mathfrak{N}}_p(\Delta t)$, average and invoke (26). Since $\int_0^\infty \delta(\xi)\,d\xi = \frac{1}{2}$ one obtains the result

$$\langle\hat{\mathfrak{N}}_p(t)\hat{x}(t)\rangle_N = D_{px}. \tag{29}$$

Other possible correlations will be obvious by now.

Our next step will be to incorporate the noise sources into the Hamiltonian. To this end we return to (23), where in view of the conserved fundamental commutator we may rewrite the noise term as

$$\begin{aligned}\int_0^{\Delta t} \hat{\mathfrak{N}}_x(\tau)\,d\tau &= -\frac{i}{\hbar}\int_0^{\Delta t} \hat{\mathfrak{N}}_x(\tau)[\hat{x}(0)\hat{p}(0) - \hat{p}(0)\hat{x}(0)]\,d\tau \\ &= -\frac{i}{2\hbar}\left\{(1+\epsilon)\int_0^{\Delta t} [\hat{x}(0)\hat{\mathfrak{N}}_x(\tau)\hat{p}(0) - \hat{p}(0)\hat{\mathfrak{N}}_x(\tau)\hat{x}(0)]\,d\tau \right. \\ &\qquad \left. + (1-\epsilon)\int_0^{\Delta t} [\hat{x}(0)\hat{p}(0)\hat{\mathfrak{N}}_x(\tau) - \hat{\mathfrak{N}}_x(\tau)\hat{p}(0)\hat{x}(0)]\,d\tau\right\}, \end{aligned} \tag{30}$$

where in the second step we have used the Markov property, and where ϵ will be taken to be real but otherwise arbitrary. Inserting the above expression again into (23) it is easily seen that we may obtain a "noisy" Hamiltonian regarding $\hat{x}$ by just adding to $\mathcal{H}$ a contribution

$$\hat{\mathfrak{h}}_x = \tfrac{1}{2}(1+\epsilon)\hat{\mathfrak{N}}_x \hat{p} + \tfrac{1}{2}(1-\epsilon)\hat{p}\hat{\mathfrak{N}}_x . \tag{31}$$

In the same manner one computes, δ being arbitrary,

$$\hat{\mathfrak{h}}_p = -\tfrac{1}{2}(1+\delta)\hat{\mathfrak{N}}_p \hat{x} - \tfrac{1}{2}(1-\delta)\hat{x}\hat{\mathfrak{N}}_p . \tag{32}$$

Combining these noise terms with (19) we may write our total non-Hermitian noisy Hamiltonian in a single expression as follows

$$\mathcal{H}_N = \hat{H} + i\hat{\Gamma} + \hat{\mathfrak{h}}_x + \hat{\mathfrak{h}}_p . \tag{33}$$

Since it will be our goal to derive the density-operator equation for the damped oscillator while the quantization has been performed in the Heisenberg frame, we have to transform to the Schrödinger frame. Such a transformation can only be done properly by means of a unitary operator; that is, by using a Hermitian Hamiltonian. In the following we show the possibility to construct an effective Hermitian Hamiltonian by a proper choice of the parameters ϵ and δ. For that purpose we require $\langle \mathcal{H}_N - \mathcal{H}_N^\dagger \rangle_N$ to be zero. By means of (33) one obtains

$$2i\langle \hat{\Gamma}\rangle_N + \epsilon\langle \hat{\mathfrak{N}}_x \hat{p} - \hat{p}\hat{\mathfrak{N}}_x\rangle_N - \delta\langle \hat{\mathfrak{N}}_p \hat{x} - \hat{x}\hat{\mathfrak{N}}_p\rangle_N = 0 . \tag{34}$$

We now invoke the previously calculated correlations of which (29) is a typical example. This leads to

$$(\epsilon + \delta)\Delta_{px} = 2i\langle \hat{\Gamma}\rangle_N , \tag{35}$$

where we have used the definition (27). By virtue of the fluctuation-dissipation relation (28) and the expression (19) for $\hat{\Gamma}$, we conclude that any choice of ϵ and δ constrained by the relation $\epsilon + \delta = 1$ appears to be appropriate in order to effectively "Hermitize" the Hamiltonian. Their explicit specification is simply circumvented by the notion that, effectively, the terms with ϵ and δ in $\hat{\mathfrak{h}}_x$ and $\hat{\mathfrak{h}}_p$ cancel the anti-Hermitian part of the original $\mathcal{H}$. Therefore, putting $\epsilon = \delta = 0$ entails putting $\hat{\Gamma} = 0$. Thus we are left with the following manifestly Hermitian noisy effective Hamiltonian.

$$\hat{H}_N = \tfrac{1}{2}\hat{p}^2 + \tfrac{1}{2}\lambda[\hat{p}, \hat{x}]_+ + \tfrac{1}{2}\Omega^2\hat{x}^2 + \tfrac{1}{2}[\hat{\mathfrak{N}}_x, \hat{p}]_+ - \tfrac{1}{2}[\hat{\mathfrak{N}}_p, \hat{x}]_+ . \tag{36}$$

Before proceeding with this Hamiltonian, we will verify in detail whether it correctly reproduces the equations of motion (21) and (22) for $\langle\hat{x}\rangle_N$ and $\langle\hat{p}\rangle_N$. Because $\hat{H}_N$ is Hermitian it serves as a proper time-translator, so that the equation of motion for $\hat{x}$ now reduces to its usual form

$$\dot{\hat{x}} = -(i/\hbar)[\hat{x}, \hat{H}_N]_- . \tag{37}$$

Inserting (36), and averaging over the noise, one finds

$$\langle \dot{\hat{x}}\rangle_N = \langle\hat{p}\rangle_N + \lambda\langle\hat{x}\rangle_N - (i/2\hbar)\langle \hat{x}(\hat{\mathfrak{N}}_x\hat{p} + \hat{p}\hat{\mathfrak{N}}_x - \hat{\mathfrak{N}}_p\hat{x} - \hat{x}\hat{\mathfrak{N}}_p)\rangle_N + (i/2\hbar)\langle(\hat{\mathfrak{N}}_x\hat{p} + \hat{p}\hat{\mathfrak{N}}_x - \hat{\mathfrak{N}}_p\hat{x} - \hat{x}\hat{\mathfrak{N}}_p)\hat{x}\rangle_N . \tag{38}$$

In order to simplify the computation of the correlations we shall presently assume the well-known Gaussian property to hold.[14,19] We have not verified in detail to what extent the final result may be influenced by this assumption. In fact, this is not thought to be relevant within the framework of the present approach. Since, as has been emphasized before, we have started from classical deterministic equations which already contain the dissipation, no detailed knowledge has been available regarding the underlying microscopic interactions,[1,9] and the noise sources had in fact to be introduced by necessity only to prevent the killing of a fundamental quantum mechanical requisite, namely Planck's constant $\hbar$. This lack of knowledge will be seen, first of all, to lead to a description of the damped oscillator in terms of a density operator rather than an explicit Schrödinger-state vector and, moreover, to some not completely specified but essentially nonzero coefficients such as D_{px} in the master equation. So, regarding our starting point and the observed partial knowledge all we can do is to find the simplest system which does not violate quantum mechanics in the course of its evolution, and the assumption of the Gaussian property does not seem to be very restrictive.

Invoking the correlations like (29) one now easily evaluates (38). As a typical example one may consider

$$\begin{aligned}\langle \hat{x}\hat{p}\hat{\mathfrak{N}}_x\rangle_N &= \langle\hat{x}\rangle_N\langle\hat{p}\hat{\mathfrak{N}}_x\rangle_N + \langle\hat{x}\langle\hat{p}\rangle_N\hat{\mathfrak{N}}_x\rangle_N + \langle\hat{x}\hat{p}\rangle_N\langle\hat{\mathfrak{N}}_x\rangle_N \\ &= \langle\hat{x}\rangle_N D_{px} + \langle\hat{p}\rangle_N D_{xx} + i\hbar\langle\hat{\mathfrak{N}}_x\rangle_N \\ &= \langle\hat{x}\rangle_N D_{px} + \langle\hat{p}\rangle_N D_{xx} .\end{aligned} \tag{39}$$

In this way (38) leads to[17]

$$\langle\dot{\hat{x}}\rangle_N = \langle\hat{p}\rangle_N + [\lambda - (i/\hbar)\Delta_{px}]\langle\hat{x}\rangle_N = \langle\hat{p}\rangle_N , \tag{40}$$

where in the last step we have once more used the fundamental fluctuation-dissipation relation (28). A similar calculation leads to

$$\langle\dot{\hat{p}}\rangle_N = -2\lambda\langle\hat{p}\rangle_N - \Omega^2\langle\hat{x}\rangle_N , \tag{41}$$

which completes the proof that (36) indeed generates in a proper manner the correct effective, mean value dynamics of $\hat{x}$ and $\hat{p}$.

IV. THE MASTER EQUATION

Having now the Hermitian Hamiltonian $\hat{H}_N$ in (36) at our disposal we are in the position to transfer from the Heisenberg frame to Schrödinger's frame. For the density operator $\hat{\Re}$ one thus obtains

$$\dot{\hat{\Re}}(t) = -(i/\hbar)[\hat{H}_N(t), \hat{\Re}(t)]_- , \tag{42}$$

where the explicit time-dependence of $\hat{H}_N$ now arises from the noise only. In order to obtain the relevant reduced density operator for the oscillator we must take the trace over the noise variables:

$$\hat{\rho}(t) = \mathrm{tr}_N \hat{\Re}(t) . \tag{43}$$

This can be done, as in the Heisenberg frame, by coarse-graining. The general solution of (42) reads[20,21]:

$$\hat{\Re}(t) = \hat{\Re}(t-\tau) + \sum_{n=1}^{\infty} \left(-\frac{i}{\hbar}\right)^n \int_{t-\tau}^{t} dt_1 \int_{t-\tau}^{t_1} dt_2 \cdots \int_{t-\tau}^{t_{n-1}} dt_n \, [\hat{H}_N(t_1), [\hat{H}_N(t_2), \ldots [\hat{H}_N(t_n), \hat{\Re}(t-\tau)]_- \cdots]_-]_- , \tag{44}$$

where in order to decorrelate the oscillator and the noise, τ may be taken as the correlation time. In view of the assumed Markov property we presently consider $\tau = \Delta t \to 0+$. Detailed analysis shows that, in this case, confinement to the term with $n=1$ in (44) indeed represents a consistent first-order approximation in the following calculation. Thus we obtain

$$\hat{\Re}(t) = \hat{\Re}(0) - \frac{i}{\hbar} \int_0^{\Delta t} [\hat{H}_N(t')\hat{\Re}(0) - \hat{\Re}(0)\hat{H}_N(t')]\, dt' , \tag{45}$$

where on the right-hand side we have set for convenience, $t - \Delta t = 0$. In view of the definition of Δt one now has

$$\hat{\Re}(0) = \hat{\rho}\hat{\rho}_N , \tag{46}$$

where $\hat{\rho}_N$ represents the density operator for the noise sources only, while $\hat{\rho}$ solely concerns the oscillator variables. The equation of motion of the oscillator $\hat{\rho}$ can now be derived from (43) by means of (42), (45), and (46), while $\hat{H}_N$ has been given in (36). Let us only consider a typical example of terms involved:

$$[\dot{\hat{\rho}}]_1 = \hbar^{-2} \mathrm{tr}_N \int_0^{\Delta t} \hat{H}_N(t)\hat{\Re}(0)\hat{H}_N(t')dt' . \tag{47}$$

For ease of survey we set

$$\hat{H}_N(t) = \hat{H}_0 + \hat{\mathfrak{h}}(t) , \tag{48}$$

where, regarding (36), $\hat{H}_0$ may be identified with $\hat{H}$ in (19) so that $\hat{\mathfrak{h}}$ contains the noisy remainder of $\hat{H}_N$. In the Schrödinger frame $\hat{p} = \hat{p}(0)$ and $\hat{x} = \hat{x}(0)$, so that $\hat{p}$ and $\hat{\mathfrak{N}}_x(t)$ as well as $\hat{x}$ and $\hat{\mathfrak{N}}_p(t)$ for $t > 0$ always commute here, whence we may simplify $\hat{\mathfrak{h}}$ as

$$\hat{\mathfrak{h}}(t) = \hat{\mathfrak{N}}_x(t)\hat{p} - \hat{\mathfrak{N}}_p(t)\hat{x} . \tag{49}$$

Inserting (46) and (48) into (47) one obtains

$$[\dot{\hat{\rho}}]_1 = \hbar^{-2} \int_0^{\Delta t} \mathrm{tr}_N \{\hat{H}_0\hat{\rho}\hat{\rho}_N\hat{H}_0 + \hat{H}_0\hat{\rho}\hat{\rho}_N\hat{\mathfrak{h}}(t') + \hat{\mathfrak{h}}(t)\hat{\rho}\hat{\rho}_N\hat{H}_0 + \hat{\mathfrak{h}}(t)\hat{\rho}\hat{\rho}_N\hat{\mathfrak{h}}(t')\}\, dt' . \tag{50}$$

The first term on the right hand side of (50) will be of order Δt and hence vanishes as $\Delta t \to 0+$. Using (49) we may do the "tracing procedure" in the other terms in (50) as usual. The second and third term in (50) then are found to vanish because all noise operators have zero mean. Thus the only nonzero contributions in (50) arise from the last term. Accounting for such definitions as (26), and again noticing that $\int_0^\infty \delta(\zeta)\, d\zeta = \frac{1}{2}$, one finds[17]

$$[\dot{\hat{\rho}}]_1 = \hbar^{-2}\{\hat{p}\hat{\rho}\hat{p}D_{xx} - \hat{p}\hat{\rho}\hat{x}D_{px} - \hat{x}\hat{\rho}\hat{p}_{xp}D_{xp} + \hat{x}\hat{\rho}\hat{x}D_{pp}\} . \tag{51}$$

In view of the structure of our master equation the above contribution occurs twice. The evaluation of the other contributions will be clear now, so we give only the respective results:

$$[\dot{\hat{\rho}}]_2 = -\frac{i}{\hbar}\mathrm{tr}_N[\hat{H}_N(t), \hat{\Re}(0)]_- = -\frac{i}{\hbar}[\hat{H}_0, \hat{\rho}]_- , \tag{52}$$

$$\begin{aligned}[\dot{\hat{\rho}}]_3 &= -\hbar^{-2}\mathrm{tr}_N \int_0^{\Delta t} \hat{H}_N(t)\hat{H}_N(t')\hat{\Re}(0)\, dt' \\ &= -\hbar^{-2}\{\hat{p}^2\hat{\rho}D_{xx} - \hat{p}\hat{x}\hat{\rho}D_{xp} - \hat{x}\hat{p}\hat{\rho}D_{px} + \hat{x}^2\hat{\rho}D_{pp}\} ,\end{aligned} \tag{53}$$

$$\begin{aligned}[\dot{\hat{\rho}}]_4 &= -\hbar^{-2}\mathrm{tr}_N \int_0^{\Delta t} \hat{\Re}(0)\hat{H}_N(t')\hat{H}_N(t)\, dt' \\ &= -\hbar^{-2}\{\hat{\rho}\hat{p}^2D_{xx} - \hat{\rho}\hat{x}\hat{p}D_{px} - \hat{\rho}\hat{p}\hat{x}D_{xp} + \hat{\rho}\hat{x}^2D_{pp}\} .\end{aligned} \tag{54}$$

It is noted in passing that $[\dot{\hat{\rho}}]_4 = [\dot{\hat{\rho}}]_3^\dagger$ and that $D_{xp} = D_{px}^*$.[17] We finally obtain our reduced master equation by adding twice (51) to (52)–(54). The ultimate result may be written as

$$\begin{aligned}\dot{\hat{\rho}} = &-(i/\hbar)[\hat{H}_0, \hat{\rho}]_- - \hbar^{-2}D_{xx}[\hat{p}, [\hat{p}, \hat{\rho}]_-]_- \\ &+ \hbar^{-2}D_{px}[\hat{x}, [\hat{p}, \hat{\rho}]_-]_- \\ &+ \hbar^{-2}D_{xp}[\hat{p}, [\hat{x}, \hat{\rho}]_-]_- - \hbar^{-2}D_{pp}[\hat{x}, [\hat{x}, \hat{\rho}]_-]_- \\ &+ \hbar^{-2}\Delta_{px}([\hat{x}, \hat{\rho}\hat{p}]_- - [\hat{p}, \hat{\rho}\hat{x}]_-) .\end{aligned} \tag{55}$$

We remark that (55), being cast in pure commutator form, evidently shows that $\hat{\rho}$ obeys local continuity. However, it should also be obvious now that in constrast to $\hat{\Re}$ the oscillator operator $\hat{\rho}$ does not obey such a simple commutator equation as (42). The master equation (55) in fact cannot be obtained from a Schrödinger equation for a single-state vector related to $\hat{\rho}$ in the usual manner. This possibility is being spoiled by the dissipation-fluctuation mechanism, entering in (55) through the diffusion coefficients $D_{\mu\nu}$.

V. THE WIGNER DISTRIBUTION

It will be convenient to represent (55) in terms of a pseudoprobability distribution.[13,19–22] In the presently considered coordinate representation the well-known Wigner-distribution[23] may be presented in the form[13]

$$W(p,x,t)=\frac{1}{2\pi\hbar}\int_{-\infty}^{\infty}e^{ipy/\hbar}\langle x-\tfrac{1}{2}y|\hat{\rho}|x+\tfrac{1}{2}y\rangle\,dy\,. \quad (56)$$

If, for a pure state, we put $\hat{\rho}=|\ \rangle\langle\ |$ and note that $\langle r|\ \rangle=\psi(r)$ represents the wave function, we recover from (56) the more conventional definition of Wigner's distribution. The equation of motion for W for the damped oscillator may now be obtained from (56) with the aid of (55), where we would like to recall the following fundamental relations[24]:

$$\int dr\,|r\rangle\langle r|=1\,, \quad (57)$$

$$\langle r'|\hat{r}|r\rangle=r'\delta(r'-r)\,, \quad (58)$$

$$\langle r'|\hat{p}|r\rangle=-i\hbar\frac{\partial}{\partial r'}\delta(r'-r)\,. \quad (59)$$

Moreover, it is remembered that $\hat{H}_0$ in (55) stands for $\hat{H}$ as given in (19), taken at $t=0$. We thus have to evaluate a number of contributions of which we consider the first by way of example in some detail. The kinetic energy term $\frac{1}{2}\hat{p}^2$ from $\hat{H}_0$ leads to

$$\left[\frac{\partial W}{\partial t}\right]_1=-\frac{i}{2\hbar}\frac{1}{2\pi\hbar}\times\int_{-\infty}^{\infty}e^{ipy/\hbar}\langle x-\tfrac{1}{2}y|\hat{p}^2\hat{\rho}-\hat{\rho}\hat{p}^2|x+\tfrac{1}{2}y\rangle\,dy\,. \quad (60)$$

Inserting now the completeness relation (57) one may obtain

$$\left[\frac{\partial W}{\partial t}\right]_1=-\frac{i}{2\hbar}\frac{1}{2\pi\hbar}\int_{-\infty}^{\infty}dy\,e^{ipy/\hbar}\int_{-\infty}^{\infty}d\xi\{\langle x-\tfrac{1}{2}y|\hat{p}^2|\xi\rangle\langle\xi|\hat{\rho}|x+\tfrac{1}{2}y\rangle-\langle x-\tfrac{1}{2}y|\hat{\rho}|\xi\rangle\langle\xi|\hat{p}^2|x+\tfrac{1}{2}y\rangle\}. \quad (61)$$

In view of (59) one has

$$\langle\xi|\hat{p}^2|x\pm\tfrac{1}{2}y\rangle=-\hbar^2\frac{\partial^2}{\partial\xi^2}\delta(\xi-x\mp\tfrac{1}{2}y)\,. \quad (62)$$

Inserting (62) into (61) and performing in each term two partial integrations, one readily finds

$$\left[\frac{\partial W}{\partial t}\right]_1=\frac{i\hbar}{2}\frac{1}{2\pi\hbar}\int_{-\infty}^{\infty}dy\,e^{ipy/\hbar}\int_{-\infty}^{\infty}d\xi\left\{\delta(\xi-x+\tfrac{1}{2}y)\frac{\partial^2}{\partial\xi^2}\langle\xi|\hat{\rho}|x+\tfrac{1}{2}y\rangle-\delta(\xi-x-\tfrac{1}{2}y)\frac{\partial^2}{\partial\xi^2}\langle x-\tfrac{1}{2}y|\hat{\rho}|\xi\rangle\right\}. \quad (63)$$

We may further evaluate this expression as follows:

$$\left[\frac{\partial W}{\partial t}\right]_1=\frac{i\hbar}{2}\frac{1}{2\pi\hbar}\int_{-\infty}^{\infty}dy\,e^{ipy/\hbar}\frac{\partial}{\partial x}\int_{-\infty}^{\infty}d\xi\left\{\delta(\xi-x+\tfrac{1}{2}y)\frac{\partial}{\partial\xi}\langle\xi|\hat{\rho}|x+\tfrac{1}{2}y\rangle-\delta(\xi-x-\tfrac{1}{2}y)\frac{\partial}{\partial\xi}\langle x-\tfrac{1}{2}y|\hat{\rho}|\xi\rangle\right\}, \quad (64)$$

$$\left[\frac{\partial W}{\partial t}\right]_1=-i\hbar\frac{1}{2\pi\hbar}\frac{\partial}{\partial x}\int_{-\infty}^{\infty}dy\,e^{ipy/\hbar}\frac{\partial}{\partial y}\langle x-\tfrac{1}{2}y|\hat{\rho}|x+\tfrac{1}{2}y\rangle\,. \quad (65)$$

One final partial integration with respect to y yields

$$\left[\frac{\partial W}{\partial t}\right]_1=-\frac{\partial}{\partial x}pW\,. \quad (66)$$

Since the calculational procedure is expected to be intelligible now, we merely quote the subsequent results of the other terms from the master equation. The potential energy term $\frac{1}{2}\Omega^2\hat{x}^2$ from $\hat{H}_0$ leads to

$$\left[\frac{\partial W}{\partial t}\right]_2=\Omega^2\frac{\partial}{\partial p}xW\,, \quad (67)$$

while the remaining anticommutator in $\hat{H}_0$ gives

$$\left[\frac{\partial W}{\partial t}\right]_3=-\lambda\left(\frac{\partial}{\partial x}xW-\frac{\partial}{\partial p}pW\right). \quad (68)$$

This takes care of the contributions from $\hat{H}_0$. The next four terms in (55) are found to produce the following respective contributions to the Wigner equation, taken together as

$$\left[\frac{\partial W}{\partial t}\right]_4=D_{xx}\frac{\partial^2 W}{\partial x^2}+(D_{px}+D_{xp})\frac{\partial^2 W}{\partial x\partial p}+D_{pp}\frac{\partial^2 W}{\partial p^2}\,, \quad (69)$$

whereas the remaining part of the master equation gives rise to

$$\left[\frac{\partial W}{\partial t}\right]_5 = \frac{i}{\hbar}\Delta_{px}\left(\frac{\partial}{\partial x}xW + \frac{\partial}{\partial p}pW\right). \qquad (70)$$

Adding up (66)–(70), and noticing that in view of the fundamental fluctuation-dissipation relation (28) the first contributions in (68) and (70) cancel each other, one obtains

$$\frac{\partial W}{\partial t} = -\frac{\partial}{\partial x}pW + \Omega^2\frac{\partial}{\partial p}xW + 2\lambda\frac{\partial}{\partial p}pW + D_{xx}\frac{\partial^2 W}{\partial x^2} + (D_{px}+D_{xp})\frac{\partial^2 W}{\partial x\partial p} + D_{pp}\frac{\partial^2 W}{\partial p^2}. \qquad (71)$$

VI. DIFFUSION AND UNCERTAINTY

The foregoing calculations demonstrate several points in a clear fashion. One of the most important formulas appears to be the fluctuation-dissipation relation (28). First of all, it preserved the fundamental commutator in the Heisenberg frame in the course of time. Then it offered the possibility to find a proper Hermitian effective Hamiltonian, such that a transformation to the Schrödinger frame could be realized. And in the derivation in the preceding section it led to an appropriate equation of motion for the Wigner distribution. Appropriate, in the sense that (71) correctly reproduces the classical mean value equations of motion for $\langle p\rangle$ and $\langle x\rangle$.

But the entity Δ_{px} does not stand on its own. It has been clearly observed to be generated by the quantum noise operators, which also generate, for example, D_{pp}. In view of the underlying structure of the noise operators we do expect the diffusion coefficients to be nonzero in principle for dissipative systems. This notion agrees with the findings from Senitzky's approach,[9] where a commutator of noise operators prevents the killing of the fundamental commutator of $\hat{p}$ and $\hat{x}$, but where a symmetrized product is involved in the zero-point energy fluctuations.

We once more point out the incomplete knowledge concerning our damped oscillator. First, no dynamical description in terms of a state vector appears to be available, and second, no detailed expression of the diffusion coefficients in terms of known parameters has been found yet. Furthermore, it is noted in passing that the fact that (71) is of the simple Fokker-Planck type is a direct consequence of the structure of the noisy Hamiltonian (36), which also involves the Markovian property.[25] Otherwise, higher-order derivatives with respect to p and x might have appeared. But we note again that in view of the preceding it only seems to make sense in the present approach to search for the simplest possible quantal description.

From (71) one now easily computes the equations of motion for the second moments:

$$\frac{d}{dt}\sigma_{xx} = 2\sigma_{px} + 2D_{xx}, \qquad (72)$$

$$\frac{d}{dt}\sigma_{px} = \sigma_{pp} - 2\lambda\sigma_{px} - \Omega^2\sigma_{xx} + D_{px} + D_{xp}, \qquad (73)$$

$$\frac{d}{dt}\sigma_{pp} = -4\lambda\sigma_{pp} - 2\Omega^2\sigma_{px} + 2D_{pp}, \qquad (74)$$

where $\sigma_{xx} = \langle x^2\rangle - \langle x\rangle^2$, $\sigma_{pp} = \langle p^2\rangle - \langle p\rangle^2$ and $\sigma_{px} = \langle px\rangle - \langle p\rangle\langle x\rangle$.[26] Let us investigate the final steady-state fluctuations which emerge from these equations. Clearly, one has

$$\sigma_{px}(\infty) = -D_{xx}, \qquad (75)$$

$$\sigma_{pp}(\infty) = (2\lambda)^{-1}(D_{pp} + \Omega^2 D_{xx}), \qquad (76)$$

$$\sigma_{xx}(\infty) = (2\lambda\Omega^2)^{-1}[D_{pp} + (\Omega^2 + 4\lambda^2)D_{xx} + 2\lambda(D_{px} + D_{xp})]. \qquad (77)$$

From this we easily compute the final energy

$$E(\infty) = (2\lambda)^{-1}[D_{pp} + (\Omega^2 + 2\lambda^2)D_{xx} + \lambda(D_{px} + D_{xp})]. \qquad (78)$$

We set $E(\infty) = \hbar\nu$, where ν is undetermined yet. We next solve (78) for D_{pp}.

$$D_{pp} = 2\hbar\lambda\nu - (\Omega^2 + 2\lambda^2)D_{xx} - \lambda(D_{px} + D_{xp}). \qquad (79)$$

Having inserted this into (76) and (77) we form the uncertainty product

$$\sigma_{pp}(\infty)\sigma_{xx}(\infty) = \Omega^{-2}\{\hbar^2\nu^2 - [\lambda D_{xx} + \tfrac{1}{2}(D_{px} + D_{xp})]^2\}. \qquad (80)$$

In view of the conservation of the fundamental commutator by virtue of the fluctutation-dissipation relation (28), the expression (80) is expected to be $\geq \frac{1}{4}\hbar^2$. This leads to

$$[\lambda D_{xx} + \tfrac{1}{2}(D_{px} + D_{xp})]^2 \leq \hbar^2(\nu^2 - \tfrac{1}{4}\Omega^2). \qquad (81)$$

Since the left-hand side of (81) represents a pure quadratic form, we conclude that $\nu \geq \frac{1}{2}\Omega$, so that $E(\infty) \geq \frac{1}{2}\hbar\Omega$. That is, the quantal damped oscillator does not decay below its free-energy ground state.

If we presently assume the absence of fluctuations (such as thermally induced) other than the fundamental quantum ones, there is no means to excite the oscillator in the long run above its ground state. Therefore, let us set ν equal to its lower bound, i.e., $\nu = \frac{1}{2}\Omega$. In that case, from (81) we observe that

$$\tfrac{1}{2}(D_{px} + D_{xp}) = -\lambda D_{xx}, \qquad (82)$$

such that the uncertainty product (80) simplifies to

$$\sigma_{pp}(\infty)\sigma_{xx}(\infty) = \tfrac{1}{4}\hbar^2. \qquad (83)$$

As expected, this clearly indicates a minimum uncertainty state. For such a state, however, it is well-known that (Ref. 24, p. 301; also Ref. 26)

$$\sigma_{px} = \tfrac{1}{2}\langle (\hat{p} - \langle \hat{p}\rangle)(\hat{x} - \langle \hat{x}\rangle) + (\hat{x} - \langle \hat{x}\rangle)(\hat{p} - \langle \hat{p}\rangle)\rangle = 0 . \quad (84)$$

Regarding (75) this leads to

$$D_{xx} = 0 , \quad (85)$$

which upon insertion into (82) yields

$$D_{px} + D_{xp} = 0 . \quad (86)$$

Finally, by means of (79) one thus finds

$$D_{pp} = \hbar\lambda\Omega . \quad (87)$$

Herewith, the Wigner equation for the damped oscillator as presented in (71) reduces to

$$\frac{\partial W}{\partial t} = -\frac{\partial}{\partial x} pW + \Omega^2 \frac{\partial}{\partial p} xW + 2\lambda \frac{\partial}{\partial p} pW + \hbar\lambda\Omega \frac{\partial^2 W}{\partial p^2} . \quad (88)$$

We make the remarkable observation that thus the fluctuations here appear to be effectively coupled to the momentum. Actually, this seems to be the usual assumption in the Langevin approach of classical stochastic processes.[7,25,27]

VII. SOME CONCLUDING REMARKS

The treatment presented above of the quantum mechanical features of the damped oscillator seems to be a decisive step in the continuing debate concerning the quantization of dissipative systems. On the one hand, our procedure strongly parallels the fully quantum theoretic oscillator-coupled-to-reservoir approach. But, on the other hand, it essentially emerged from our generalized canonical quantization procedure for complex variables.

Our theory manifestly demonstrated why other quantization approaches to the damped oscillator have failed. Our master equation shows that no dynamical description in terms of a Schrödinger wave function can be expected to exist, the essential reason being the occurrence of fluctuations associated with the dissipation.

The noise sources could not be identified in detail, as a consequence of the absence of knowledge concerning the underlying microscopic-loss mechanisms in the classical deterministic damped oscillator equation on which our approach has been founded. We have only specified some of their correlation properties. They lead to a simple fluctuation-dissipation relation (28) which (i) prevented the killing of the fundamental commutator and (ii) allowed us to introduce a Hermitian effective Hamiltonian, so that (iii) we were able to derive an appropriate master equation. Moreover, the noise has been observed to guard the oscillator from decaying below its free-energy ground state. In other words, the quantal vacuum has not been turned off even by classical friction.

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[10] If $\lambda \geq \Omega$ the system will be aperiodically damped. Since ω then becomes purely imaginary its contribution must be included in the imaginary part of the Hamiltonian, as will become clear further on. But then we are not able to recover the appropriate Hamiltonian in terms of $\hat{p}$ and $\hat{x}$. However, the ultimate result of the analysis does not explicitly contain ω. Nevertheless, great care should be exercised in using it in the overcritically damped case.

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[15] By a coarse-grained short-time interval Δt we mean here $\tau_N \leq \Delta t \leq \lambda^{-1}$, where τ_N represents the correlation time of the noise. Since we expect the fundamental commutator to be conserved over any accessible time span we are more or less invited to assume the Markov property $\tau_N \to 0+$. See also Ref. 16.

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[17] From here on we shall assume the diffusion coefficients such as D_{px} to be scalars rather than operators. This assumption does not seem to be very restrictive in our approach, since the diffusion coefficients will be seen to be required in fact only for the survival of fundamental quantum principles.
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