Diffusion of fast electrons in the presence of an electric field*

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The diffusion of high-energy electrons ($\geq$ 10 keV) in the presence of an electric field is discussed. Solutions are obtained by extending the Bethe age theory to include effects of the applied electric field. Since we restrict ourselves to the P_1 approximation, the results are only valid for diffuse beams, i.e., beams whose distribution function can be adequately described by a current and number density. The solution predicts a range enhancement in the presence of an accelerating electric field. A simple exponential decay is obtained when the electric field is large enough so that the electrons gain as much energy in the electric field as they lose by inelastic collisions. The validity and accuracy of the solutions and predictions are discussed.

I. INTRODUCTION

In the last four years high-energy electron beams have been increasingly used both in direct laser excitation, as in molecular rare gases[1] and the rare-gas monohalides,[2] and as a source of ionization, as the E-beam sustained and controlled laser.[3,4] Recently Monte Carlo calculations, including effects of electric and magnetic fields, have been published.[5] However, these simulations do not provide physical insight or scaling laws. In this paper we treat the electron scattering in the gas by assuming that the fast electrons can be described by a current and number density. Such a simplification is not valid for a well-collimated beam. However, the high-energy electrons have to pass through a foil first. If the foil thickness is greater than $\frac{2}{5}$ of the transport mean free path, the fast electrons, after traversing the foil, are fairly diffuse.[6,7] For a diffuse beam, in the absence of an electric field, we get the age theory.[7,8] We will extend the age theory to include the effects of the discharge electric field.

The effect of the electric field is twofold: (i) it sustains the electron energy; and (ii) it tends to turn the electron trajectories in the direction of the electric field. The importance of the latter effect is easily estimated by the dimensionless parameter γ which is the energy gained per transport mean free path divided by the electron energy. The above effects will enhance the age and range of the fast electrons in the direction of the electric field. The modified diffusion theory is valid for weak electric fields, i.e., fields of such a magnitude that on the average the electron gains as much or less energy from the electric field than it loses by inelastic collisions. As we will see subsequently, most E-beam controlled lasers have "weak" electric fields.

In Sec. II, the general age diffusion equations are derived in the presence of an arbitrary electric field. These are obtained by assuming that the electrons can be adequately described by a number and current density. We then integrate the Boltzmann equation over the angular velocity variables only. The first two moments yield equations similar to the usual continuity and momentum equations.

In Sec. III, we derive the exponential solution. This solution is valid when the energy gained by the electrons in the electric field is equal to the energy lost via inelastic collisions. In this section we also derive the mean angular distribution of the electrons by integrating over all energies. The angular distribution is affected by both the applied electric field and the energy lost by inelastic collisions. One can define a second dimensionless parameter ξ that is the energy lost per transport mean free path along the electron trajectory divided by the electron energy. The angular distribution function is more strongly peaked when either γ or ξ increases. The angular distribution is important as it enables us to determine the validity of the P_1 approximation.

In Sec. IV, the age theory in the presence of an E-field is developed by a Fourier transform method. The inverse transform is obtained by a saddle-point integration. For the special case of zero electric field, the Bethe age theory is recovered. When the electric field reaches a critical value, an exponential solution similar to that derived in Sec. II is obtained. The validity and accuracy of the theory are discussed. Finally, in Sec. V an approximate half-space solution is derived and discussed.

II. TRANSPORT EQUATION INCLUDING AN ELECTRIC FIELD

The transport of fast electrons is adequately described by the Boltzmann equation

$$\hat{u}\cdot\nabla f+\frac{\vec{a}}{u}\cdot\nabla_p f=N_n\int\{f(\vec{r},\hat{u}',w)-f(\vec{r},\hat{u},w)\}\times\sigma(|\hat{u}-\hat{u}'|,w)d\hat{u}'+\left.\frac{\partial f}{\partial s}\right|_{\text{incl}},\qquad(1)$$

where the distribution function f is a function of the kinetic energy w, velocity angular coordinates ϕ and θ, and position $\vec{r}$. f is the number of electrons in a six-dimensional volume around the point p_x, p_y, p_z, x, y, and z. p_x, p_y, and p_z are the three-momentum coordinates while x, y, and z are the spatial coordinates. The unit vector $\hat{u}$ is in the direction of the velocity $\vec{u}$; u is the magnitude of $\vec{u}$. The first term on the right-hand side (RHS) of (1) is just the collision integral for elastic scattering. N_n is the neutral number density and σ is the scattering cross section per unit solid angle. We account for inelastic deflections[9] approximately by replacing Z^2 in the screened Rutherford cross section σ by $Z(Z+1)$ (see Ref. 10). Z is, of course, the atomic number. The second term on the RHS of (1) describes the change in f resulting from inelastic collisions along its trajectory s. The force $\vec{a}$ results from an electric field $\vec{E}$:

$$\vec{a}=e\vec{E},\qquad(2)$$

where e is the charge on the electron.

Following Landau[11] the inelastic collision integral can be written explicitly as

$$\left.\frac{\partial f}{\partial s}\right|_{\text{incl}}=\int_0^\infty\{P(w+\Delta,\Delta)f(w+\Delta)-P(w,\Delta)f(w)\}\,d\Delta,\qquad(3)$$

where $P(w,\Delta)$ is the probability that an electron having an energy w will lose an energy Δ. The upper limit ∞ is valid because $P(w,\Delta)=0$ for $\Delta>w$; as $\Delta\geqslant 0$ the lower limit is set to zero. Typically $\Delta/w\sim 10^{-4}$ so we can write

$$P(w+\Delta,\Delta)f(w+\Delta)=P(w,\Delta)f(w)+\Delta\frac{\partial}{\partial w}\{P(w,\Delta)f(w)\}+\cdots.\qquad(4)$$

Substituting (4) into (3) we find

$$\left.\frac{\partial f}{\partial s}\right|_{\text{incl}}=\frac{\partial}{\partial w}(\chi f)\qquad(5)$$

where χ is the Bethe stopping power which is the energy lost by an electron per unit path length along its trajectory and is defined by[12]

$$\chi(w)=\int_0^w P(w,\Delta)\Delta\,d\Delta.\qquad(6)$$

In writing (6) we have assumed that the electrons lose energy continuously and effects due to straggling have been neglected.

Assuming that f is diffuse enough, we can write

$$f\approx(1/4\pi)(N+3\vec{J}\cdot\hat{u})\qquad(7)$$

where

$$N=\int_0^{2\pi}d\phi\int_{-1}^{+1}d\mu f,\qquad(8)$$

$$\vec{J}=\int_0^{2\pi}d\phi\int_{-1}^{+1}d\mu\,\hat{u}f,\qquad(9)$$

$$\mu=\cos\theta.$$

If N and $\vec{J}$ are integrated over all energies we will obtain the electron number density and current, respectively.

Integrating (1) over all directions we obtain[7]

$$\nabla\cdot\vec{J}-\frac{\partial}{\partial w}(\chi N)+e\vec{E}\cdot\frac{\partial\vec{J}}{\partial w}+\frac{e\vec{E}\cdot\vec{J}}{pu}=0.\qquad(10)$$

The result of multiplying (1) by $\hat{u}$ and integrating over all directions is

$$\frac{\nabla N}{3}-\frac{\partial(\chi\vec{J})}{\partial w}+\frac{e\vec{E}}{3}\frac{\partial N}{\partial w}+\frac{\vec{J}}{\lambda}=0.\qquad(11)$$

The transport mean free path λ is defined by[13]

$$\lambda=\left(2\pi N_n\int_{-1}^{+1}d\mu\;\sigma(\mu)(1-\mu)\right)^{-1}.$$

In the usual age theory it is assumed that $\partial(\chi\vec{J})/\partial w\ll\vec{J}/\lambda$.[7] This assumption is valid for $Z>30$.[14] The region of validity can be extended to $Z\approx 13$ by the modified age theory.[8] The third term in Eq. (11) is of opposite sign to $\partial(\chi\vec{J})/\partial w$. So we would expect the age theory to improve as the electric field is increased. This is because the effective stopping power decreases in the presence of an applied electric field. So neglecting $\partial(\chi\vec{J})/\partial w$ and $(e\vec{E}/3)\partial N/\partial w$, Eq. (11) reduces to

$$\tfrac{1}{3}\nabla N+\vec{J}/\lambda=0.\qquad(12)$$

Equations (10) and (12) are the lowest-order diffusion equations and are sometimes referred to as the P_1 approximation. They are valid provided that the electron distribution function f is diffuse enough to be adequately represented by (7) and the effective stopping power is sufficiently weak. Such will be the case if the foil separating the lasing mixture and the high vacuum chamber is equal to or thicker than $2\lambda/5$.[6,7] For example, this criterion is satisfied for 150-kV electrons impinging on a 1-mil aluminum foil, or 300-keV electrons incident on a 1-mil stainless foil.

III. THE EXPONENTIAL SOLUTION

A simple solution to the electron diffusion in the presence of an electric field can be obtained by integrating the Boltzmann equation over energy. In one dimension, Eq. (1) becomes

$$2\mu\lambda\frac{\partial f}{\partial x}-2\lambda\frac{\partial[\epsilon(w)f]}{dw}+\gamma(1-\mu^2)\frac{\partial f}{\partial\mu}=\frac{\partial}{\partial\mu}(1-\mu^2)\frac{\partial f}{\partial\mu}\,. \tag{13}$$

Parameters in this equation will be explained presently. The parameter γ, in the nonrelativistic limit, is the ratio of the energy gained per transport mean free path to the electron energy. For relativistic electrons γ is given by

$$\gamma=\frac{2eE\lambda}{pu}=\frac{eE\lambda}{w}\,\frac{1+T}{1+T/2}\,. \tag{14}$$

The electric field is assumed to be along the x direction and $T=w/m_0c^2$. γ is a familiar parameter in plasma physics. $\epsilon(w)$ is the net energy lost by an electron per unit length along its trajectory.

$$\epsilon(w)=\chi-eE\mu\,. \tag{15}$$

In the above equation the collision term has been approximated by the Fokker-Planck expansion.[7] The transport mean free path may be evaluated and is given by[13]

$$\frac{1}{\lambda}=\frac{2\pi N_n Z(Z+1)e^4}{p^2u^2}\left[\ln\left(1+\frac{1}{\eta}\right)-\frac{1}{1+\eta}\right], \tag{16}$$

where $\alpha=Z/137$ and the screening angle η is given by

$$\eta=\frac{1}{2}\left(\frac{\hbar}{p}\,\frac{1.12Z^{1/3}}{0.855a_0}\right)^2$$

where a_0 is the Bohr radius.

We now multiply (13) by $w(1+T/2)/\lambda$ and integrate over all energy; then we encounter such terms as

$$\int_0^\infty dw\, fw(1+T/2)=F(\mu,x)\,, \tag{17}$$

$$\int_0^\infty dw\, fw(1+T/2)/\lambda=F/\lambda_0\,, \tag{17a}$$

$$\int_0^\infty dw\,\gamma fw(1+T/2)/\lambda=F\gamma_0/\lambda_0\,. \tag{17b}$$

We assume that f is spread only over a moderate range of energy, an assumption which will turn out to be well fulfilled for the case to which we wish to apply the solution of the present section. Then γ_0, λ_0, and χ_0 in (17a) and (17b) refer to some average value w_0 in the energy interval over which f extends.

The integral of the second term in (13), after multiplication by $w(1+T/2)/\lambda$ is

$$-2\int_0^\infty dw\,w(1+T/2)\frac{\partial}{\partial w}[\epsilon(w)f]=2\int_0^\infty dw\,(1+T)\epsilon f=\frac{2(\chi_0-eE\mu)(1+T_0)F}{w_0(1+T_0/2)}\,. \tag{18}$$

Multiplying by λ_0 again and dropping the subscript 0,

$$2\mu\lambda\frac{\partial F}{\partial x}+2\xi F-2\gamma\mu F+\gamma(1-\mu^2)\frac{\partial F}{\partial\mu}=\frac{\partial}{\partial\mu}(1-\mu^2)\frac{\partial F}{\partial\mu}\,, \tag{19}$$

where

$$\xi=\frac{2\lambda\chi}{pu}=\frac{\lambda\chi}{w}\,\frac{1+T}{1+T/2}\,. \tag{19a}$$

The dimensionless parameter ξ, in the limit of $T\to0$, is the ratio of the energy lost per transport mean free path to the electron energy.

Integrating (19) over all μ we find

$$\lambda\frac{dj}{dx}+\frac{\xi j}{\langle\mu\rangle}=0\,, \tag{20}$$

where

$$j=\int\mu F\,d\mu\,, \tag{20a}$$

$$\langle\mu\rangle=j\Big/\int F\,d\mu\,. \tag{20b}$$

Notice when $\xi=0$ the current j is conserved. Equation (20) has a simple exponential solution

$$j=j_0\exp\left(-\int^x dx'\frac{\xi}{\langle\mu\rangle}\,\frac{1}{\lambda}\right). \tag{21}$$

Assuming that

$$F\sim g(\mu)\exp\left(-\int^x dx'\,\frac{\xi}{\langle\mu\rangle}\,\frac{1}{\lambda}\right), \tag{22}$$

Eq. (19) reduces to an ordinary differential equation for $g(\mu)$,

$$-\frac{2\mu\xi g}{\langle\mu\rangle}+2\xi g-2\gamma\mu g+\gamma(1-\mu^2)\frac{dg}{d\mu}=\frac{d}{d\mu}(1-\mu^2)\frac{dg}{d\mu}\,. \tag{23}$$

In the limit of $\xi=0$, if we are to have no singularities at $\mu=\pm1$, it is easy to show that

$$g=ce^{\gamma\mu}\,. \tag{24}$$

We see from Eq. (24) that the electron distribution function is peaked in the direction of the electric field. The degree of the peaking is given by γ. In the nonrelativistic limit γ is the energy gained per transport mean free path divided by the electron energy. Physically, this peaking occurs because of a balance between the elastic collisions tending to drive the distribution function isotropic and focusing resulting from the electric field. The exponential decay in x arises because of backscatter that takes place continuously. The inelastic collisions are described by ξ. As ξ increases from 0, g will become more sharply peaked in angle. This is because particles with trajectories along the direction of the field will lose energy more slowly than the rest. For a nonzero ξ, Eq. (23) may easily be solved by expanding g in Legendre polynomials.

$$g=\sum_{l=0} A_l P_l(\mu)\,. \tag{25}$$

Substituting (25) into (23) results in an infinite set of simultaneous equations. The nth equation is

$$A_{n+1}\left(\frac{n+1}{2n+3}\right)\left[\gamma n-\frac{2\xi}{\langle\mu\rangle}\right]+A_n[n(n+1)+2\xi]-A_{n-1}\left(\frac{n}{2n-1}\right)\left[(n+1)\gamma+\frac{2\xi}{\langle\mu\rangle}\right]=0\,. \tag{26}$$

A solution can be obtained by truncating the series given by (25) at $l=L$. An iterative procedure is necessary as $\langle\mu\rangle$ is not known *a priori*. For $\gamma \approx 1$, $\xi\approx\frac{1}{2}$ (which will turn out to be a limiting case). The series converges rapidly. In fact, the final value for $\langle\mu\rangle$ differs by only 5% from that obtained by truncating at $L=1$. Using this "P_1 approximation" we can write

$$\langle\mu\rangle_{\pm}=\frac{\gamma\pm[\gamma^2+12(\xi)(\xi+1)]^{1/2}}{6(1+\xi)}\,. \tag{27}$$

For a bounded solution as $x\to\infty$, we must choose $\langle\mu\rangle_+$.

In fact, using Eq. (26) and truncating the expansion given by Eq. (25) at some larger n ($n\approx 25$), we can obtain a solution to any desired accuracy provided g is not singular. In Fig. 1 we show the plot of $g(\mu)$ for $\gamma=1$ and $\xi=\frac{1}{2}$. We will show subsequently that $\gamma=1$ for a CO_2 laser mixture corresponds to an electric field of 6.5 kV/cm atm for 100-keV electrons. The applied electric field in a CO_2 laser is typically[3] 4–6 kV/cm atm. The value of ξ for this mixture is about 0.46. So $g(\mu)$ shown in Fig. 1 will be the most sharply peaked distribution function considered. For $\gamma>1$ and $\xi>0.5$ the P_1 approximation is no longer valid. From a solution (25) and (26) we can estimate the accuracy of the P_1 (lowest-order diffusion) approximation (for our extreme case of $\gamma=1$, $\xi=\frac{1}{2}$) by evaluating the following ratio:

$$\sum_{n=2} A_n^2 \Big/ \sum_{n=0} A_n^2$$

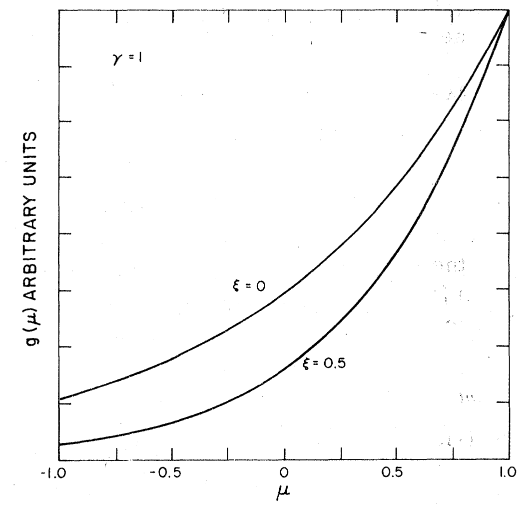

FIG. 1. The electron distribution function $g(\mu)$ for the special cases of $\gamma=1$ and $\xi=0$ and 0.5.

which for our limiting case is 15%. For comparison in Fig. 1, we have also plotted the exponential distribution function ($\xi=0$) given by (24). It should be pointed out that if the elastic scattering were isotropic and not Fokker-Planck, the lowest-order P_1 approximation could no longer be valid when $\gamma=1$. This point is discussed further in Appendix A.

Now if we define an electric field E_b such that on average the electron energy is constant, this electric field will be given by

$$eE_b=\chi/\langle\mu\rangle_b \tag{28}$$

or

$$\langle\mu\rangle_b=\xi/\gamma_b \tag{29}$$

where γ_b is the value of γ when $E=E_b$. Combining

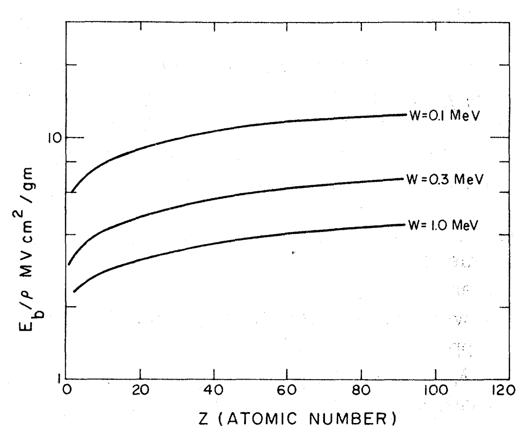

FIG. 2. E_b, the electric field required to balance the energy lost by electrons due to inelastic collisions, as a function of atomic number.

(29) and (27), γ_b and $\langle\mu\rangle_b$ can be evaluated simply as

$$\gamma_b=[\tfrac{3}{2}\xi(1+\xi)]^{1/2},\quad \langle\mu\rangle_b=[\tfrac{5}{3}\,\xi/(\xi+1)]^{1/2}\ .$$

For our limiting case of $\gamma\approx 1$, $\xi\approx\frac{1}{2}$, $\langle\mu\rangle_b\approx 0.47$. Then from (21) we find that the electron current has a simple exponential decay

$$j=j_0\exp[-(eE_b x/w)]\ . \tag{30}$$

Since the energy is constant the energy deposition will also have this simple exponential decay.

Figure 2 shows a plot of E_b/ρ as a function of atomic number and for various electron energies. It should be noted that E_b in Fig. 2 is normalized to the density ρ in the medium of interest. In the case of gases, the density is proportional to the molecular weight. Putting in numbers one finds that in xenon, E_b for 100-keV electrons is 66 kV/cm atm as opposed to 6.5 kV/cm atm in a CO_2 laser having a mixture $He/N_2/CO_2$ in a volumetric ratio 3/2/1. For 1-MeV electrons in Xe, E_b drops to 23 kV/cm atm. The dc breakdown electric field in xenon is about 7 kV/cm atm, so it is clear that in Xe discharges the effect of the electric field on the fast electrons is negligible.

Finally, we come back to the assumption of moderate energy spread of the electrons which we made below Eq. (17). Multiplying Eq. (13) by w or w^2 and performing the same operations as in Eq. (17) to (20), we can show (see Appendix B) that the mean-square spread of the energy increases exponentially, thus

$$\langle\delta w^2\rangle(x)=\langle\delta w^2\rangle(0)e^{2x/S} \tag{30a}$$

where

$$S=R\langle\mu\rangle/(1+\alpha)\ . \tag{30b}$$

Here R is the range of an electron in the absence of an electric field,

$$R=w_0/\chi_0 \tag{30c}$$

and α measured the correlation between the average μ for a given energy, $\langle\mu(w)\rangle$, and the energy. A separate calculation gave $\alpha=\frac{1}{9}$. Since S is reasonably large, the energy spread will remain moderate if it is small to begin with, i.e., if $\langle\delta w^2\rangle(0)\ll w_0^2$. The latter condition is satisfied for electrons coming from a source through a thin foil into the gas.

IV. AGE DIFFUSION IN THE PRESENCE OF AN ELECTRIC FIELD

Going back to Eq. (13) and expanding the electron distribution function f as

$$f=(1/4\pi)(N+3J\mu)\ , \tag{31}$$

we obtain

$$\frac{\partial J}{\partial x}-\frac{\partial}{\partial w}(\chi N)+eE\frac{\partial J}{\partial w}+\frac{\gamma J}{\lambda}=0\ , \tag{32}$$

$$\frac{1}{3}\frac{\partial N}{\partial x}-\frac{\partial}{\partial w}(\chi J)+\frac{eE}{3}\frac{\partial N}{\partial w}+\frac{J}{\lambda}=0\ . \tag{33}$$

If we neglect $\partial(\chi J)/\partial w$ and $\frac{1}{3}eE(\partial N/\partial w)$, Eq. (33) simplifies to

$$J\approx\frac{-\lambda}{3}\frac{\partial N}{\partial x}\ . \tag{34}$$

This is the diffusion approximation. Neglecting $\partial(\chi J)/\partial w$ in (33) results in an overestimate of the number of electrons at large distances from the source. This term ensures that electrons that traverse a distance x must lose an energy $x\chi/\langle\mu\rangle$. Thus in the absence of an electric field the age theory will overestimate the penetration distance of electrons into the medium. The term $eE\,\partial N/\partial w$ is of opposite sign, and at large electric fields ($E\approx E_b$) it will become important. It is for this reason that the critical electric field calculated using (32) and (34) is always larger than the balancing field E_b as given by (28).

Equations (32) and (34) can be Fourier transformed and combined to give

$$(ik+\gamma/\lambda)(ik\lambda/3)N+\frac{d}{dw}\{(ik\lambda eE/3+\chi)N\}=0\ . \tag{35}$$

Let

$$U=(ikeE\lambda/3+\chi)N\ . \tag{36}$$

So (35) becomes

$$\frac{dU}{dw}-\frac{\lambda}{3\chi}\,\frac{k(k-i\gamma/\lambda)}{1+ikeE\lambda/3\chi}\,U=0\ . \tag{37}$$

We set $a=eE\lambda/3\chi$, $b=\gamma/\lambda$, and $dt=(-\lambda/3\chi)dw$. Then treating a and b as constants we get

$$N(x,t)=\int_{-\infty}^{+\infty}\frac{dk\,e^y}{\chi(1+ika)} \tag{38}$$

where

$$y=ikx-\frac{k(k-ib)}{1+ika}\,t\ . \tag{39}$$

Equation (38) may be evaluated by a saddle-point integration. By letting $dy/dk=0$, the saddle point is given by

$$z=i\pm i\left(\frac{c\tau}{x+\tau}\right)^{1/2}\ , \tag{40}$$

where $z=ka$, $\tau=t/a$, and $c=1-ab$. The appropriate saddle point is given by the negative root. This is because the contour cannot be deformed to go beyond $k=i/a$ which is an essential singularity. Hence at the saddle point the value of k is given by

$$k_s=\frac{i}{a}\left[1-\left(\frac{c\tau}{x+\tau}\right)^{1/2}\right]\ . \tag{41}$$

Thus the contour will always remain below the essential singularity.

Substituting (40) into (39) we find that apart from a normalization constant

$$N(x,t)\approx\frac{(t+ax)^{-1/2}}{\chi}\left(\frac{c\tau}{\tau+x}\right)^{-1/4}$$

$$\times\exp\{-(1/a)[(x+\tau)^{1/2}-(c\tau)^{1/2}]^2\}. \quad (42)$$

Equation (42) is the full-space solution. In fact, the result is valid for $-\tau<x<\infty$. For $x<-\tau$ no solution exists because no electrons can reach this spatial region. Electrons traveling towards negative x are being retarded by the electric field and $x=-\tau$ is the point at which they will be reflected even if there was no medium. In the limit of the product $ab\to 0$, i.e., the limit of zero electric field, we get the usual age diffusion solution

$$N\approx\frac{1}{\chi}\frac{1}{\sqrt{t}}\exp\left(-\frac{x^2}{4t}\right). \quad (43)$$

When $ab=1$, (37) simplifies to

$$\frac{dU}{dw}-\frac{\lambda}{3\chi}\frac{k}{ia}U=0. \quad (43a)$$

Combining the solution of the above equation with (36) and taking the inverse transform, we get a simple exponential solution for N:

$$N=\frac{\pi}{\chi a}\exp\left(-\frac{eE_c}{w}(x+\tau)\frac{1+T}{1+T/2}\right), \quad (44)$$

where the critical electric field E_c is given by

$$eE_c=\left(\frac{3\chi w}{\lambda}\right)^{1/2}\left(\frac{1+T/2}{1+T}\right)^{1/2}. \quad (45)$$

It is interesting to note that the limit of Eq. (42) as $ab\to 1$ is different from Eq. (44) by an algebraic prefactor. The reason for this difference is that in the limit of $E\to E_c$ the saddle point approaches the essential singularity located $k=i/a$. Hence the saddle-point solution is no longer valid. Going back to differential Eq. (37), however, one finds that in the limit of $ab=1$, Eq. (43a), there is no longer an essential singularity at $k=i/a$. In fact $N(k)$ has a simple pole at $k=i/a$ which results in the solution given by Eq. (44). Comparing (45) and (28) it is easy to show that the ratio

$$E_b/E_c=[\tfrac{1}{2}(1+\xi)]^{1/2},$$

where ξ is defined by (19a). For a CO_2 laser mixture of interest $\xi\approx\frac{1}{2}$ and E_b is 13% less than E_c. So for electric fields equal to the critical field, age theory underestimates the penetration distance by 15% for CO_2 laser mixes. For larger-Z mixtures ξ will become smaller and the difference between E_b and E_c larger. Thus we can see that at large distances from the source and $E=0$ the age theory overestimates the number of electrons, while for electric field of magnitude E_c it underestimates the number of electrons. Hence the theory breaks down for distances that are much greater than the transport mean free path. However, the number of electrons that penetrate the medium to this range are exponentially small.

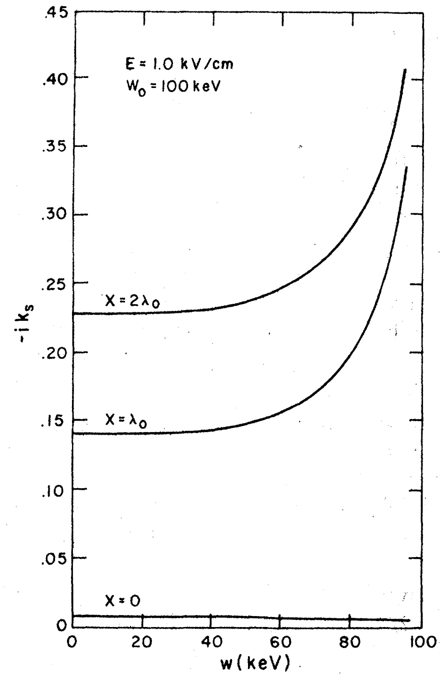

FIG. 3. Variation of the saddle point k_s as a function of electron energy for various values of x and an applied electric field of 1 kV/cm.

For nonconstant a and b the saddle point has to be evaluated numerically. Equation (39) may be written as

$$y=ikx-\int_w^{w_0}\frac{\lambda}{3\chi}k\frac{k-ib}{1+ika}dw \quad (46)$$

$$=ikx-\phi(k). \quad (47)$$

The saddle point is evaluated by equating

$$\frac{\partial y}{\partial k}=ix-\frac{\partial\phi(k)}{\partial k}=0. \quad (48)$$

Figures 3, 4, and 5 show the evaluation of the saddle point for electric fields of 1, 3, and 5 kV/cm and initial electron energies of 100 keV. The medium was assumed to be a CO_2 laser mixture containing $He/N_2/CO_2$ in a volumetric ratio of 3/2/1 at a total pressure of 760 torr. In the absence of an electric field ($a=b=0$), the saddle point may be evaluated from (41) to be

$$k_s=ix/2t.$$

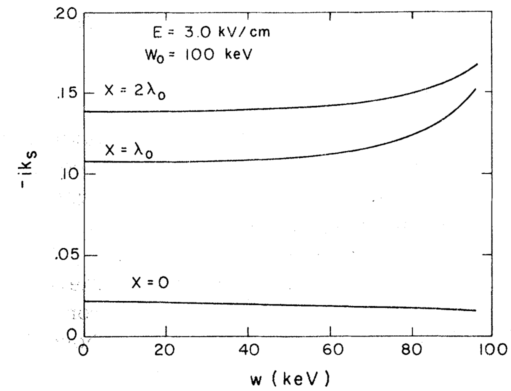

FIG. 4. Same as Fig. 3 except the electric field is 3 kV/cm.

Note that as x increases, the saddle point approaches the essential singularity asymptotically, similar to the predictions of (41). Also, the saddle point varies more slowly with x as the electric field is increased. This behavior is also apparent from (41). Finally, U may be evaluated numerically and is given by

$$U=[\pi/\phi''(k_s)]^{1/2}\exp[ik_s x-\phi(k_s)]\,. \tag{49}$$

We are, of course, interested in the energy deposited which is given by

$$\begin{aligned}\mathcal{E}_D&=\int_0^{w_0} N\chi\,dw\\&=\int_0^{w_0}\frac{\chi U}{\chi+ik_s eE/3}\,dw\,.\end{aligned} \tag{50}$$

Equation (50) will reduce to the age diffusion solution for zero electric fields.

In Fig. 6 we see the results of the age theory. For clarity the results when $E=0$ and $E=E_c$ (7.5 kV/cm) are shown. For electric fields less than E_c, the energy deposition as predicted by (50) decreases more rapidly in space. The initial electron

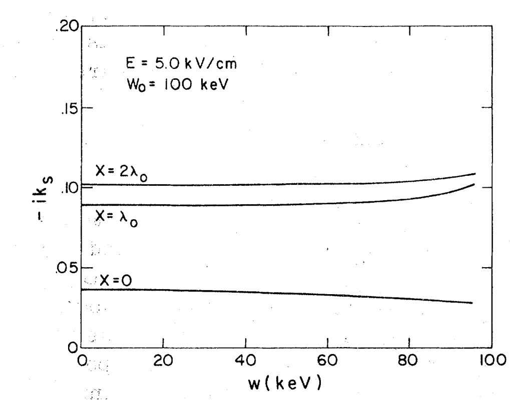

FIG. 5. Same as Fig. 3 and 4 except the electric field is 5 kV/cm.

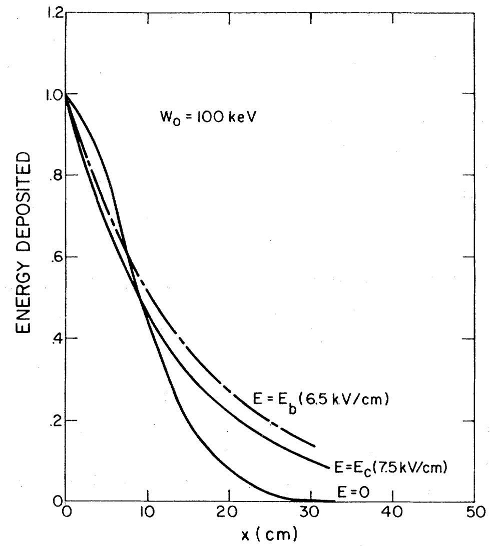

FIG. 6. The predicted energy deposited for zero electric field and the critical field of 7.5 kV/cm. Also shown is the exponential solution derived in Sec. III.

energy was taken to be 100 keV which is approximately the mean energy of a beam of 130-keV electrons having traversed a 1-mil Al foil. The medium was assumed to be a CO_2 laser mixture containing $He/N_2/CO_2$ in a volumetric mixture ratio of 3/2/1 at a total pressure of 760 torr. The mean atomic number was computed to be 5.6 and mean square of the atomic number 37.2. Also shown in Fig. 6 is the exponential solution derived in Sec. II [see Eq. (30)].

V. HALF-SPACE SOLUTION

We are actually interested in the half-space solution where the laser gas fills the space $x>0$. On the side $x<0$, there is usually a foil. We may describe this foil by an albedo α: If a certain current J_- of electrons comes out of the gas at $x>0$, flowing toward negative x, then the foil will reflect back into the gas the current

$$J_+=\alpha J_-\,; \tag{51}$$

the *net* current flowing into the gas is then

$$J_b=J_+-J_-=-(1-\alpha)J_-\,. \tag{52}$$

The electron density is proportional to J_++J_-; more accurate consideration of the angle averages gives

$$N=\tfrac{3}{2}(J_++J_-)\,; \tag{53}$$

hence

$$J_b=-\frac{2}{3}\frac{1-\alpha}{1+\alpha}N\equiv-\tfrac{2}{3}\zeta N\,. \tag{54}$$

[Note that N and J are defined to have the same dimension, as in (31).]

If we now use the diffusion approximation (34), we get

$$\frac{N}{dN/dx}\bigg|_{x=0} = \frac{1}{2}\,\frac{\lambda}{\zeta}\,. \tag{55}$$

This is the boundary condition on the electron density at the free surface. A more accurate theory[15] gives a factor 0.71 instead of $\frac{1}{2}$ on the right-hand side of (55). After this correction if the curvature of N is not too big, and ζ not too small, (55) is equivalent to the condition[16]

$$N=0 \quad \text{at } x=x_1 \equiv -0.71\lambda/\zeta\,. \tag{56}$$

This is known as the extrapolated end-point condition.

The simplest situation is $\alpha=0$, which means that the foil does not reflect *any* electrons; i.e., it is a perfect absorber. (We shall find below that this is not far from the truth.) In this case, $\zeta=1$ in (55). The electric field does not appreciably change the situation because in the foil it has negligible effect, compared with elastic and inelastic scattering; α is not affected by the field E.

The extrapolated end point (56) can easily be put into the solution (42) provided λ/ζ is constant (we have anyway assumed a and b in (37) to be consants).[17] The solution, which is valid for $\gamma\leq 0.3$, can then be given in terms of a mirror source. To construct this, we use Appendix C which shows that solution (42) holds for negative as well as positive x. We note that (42) is still valid if the electron source is at x_0, rather than $x=0$; the x in (42) must be replaced by $x-x_0$.

The "source" of electrons may be put about two-fifths of a transport mean free path inside the laser gas because this is about the distance at which the direction of the electrons becomes random,[6,7] so

$$x_0=0.4\lambda. \tag{57}$$

(The foil thickness should be subtracted.) Third, the condition (56) will be satisfied, for all τ, if we put a "mirror source" of opposite sign at

$$x_2=2x_1-x_0 \tag{58}$$

where, of course, $x_1<0$. If we now use Eq. (C1), we find that the correct solution for the half-space is

$$N_h(x,t)\approx N(x-x_0,t)-N(x-2x_1+x_0,t) \times\exp\{[-2(x_1-x_0)/a](1-\sqrt{c})\}\,, \tag{59}$$

where $N(x,t)$ is given by (42). It can easily be verified, using (C1), that

$$N_h(x_1,t)=0 \quad \text{for all } t\,. \tag{59a}$$

Inserting (57) and (56) with $\zeta=1$,

$$N_h(x,t)=N(x-0.4\lambda,t)-N(x+1.82\lambda,t) \times\exp[-(1.82\lambda/a)(1-\sqrt{c})]\,. \tag{60}$$

This is easy to evaluate.

The solution given by (60) is approximate because as the electron energy decreases λ, or better λ/ϵ, decreases rapidly with w. Then the extrapolated end point (56) changes with the age t, and (59) is no longer a solution. Fortunately, however, just the rapid decrease of λ with w makes a numerical solution fairly easy. Relatively few electrons return from the gas to the surface $x=0$, and those that do have much reduced energy. If they are reflected by the foil at all, this will entail a further large energy decrease. Hence the existence of a surface at $x=0$, and especially the albedo of the foil, will only affect the electrons of greatly reduced energy.

A possible solution procedure is again numerical: (1) Calculate the electron distribution for an infinite gaseous medium; (2) calculate the number and energy distribution of the electron flowing out through $x=0$, consider these as a negative source and calculate its effect on the electron distribution; (3) taking the outflowing electrons as a source, calculate the electrons reflected by the foil, and consider these as a positive source in the gas, etc. This series should converge very rapidly.

A further help comes from the case without electric field. Then, if Z is the same in the foil and in the gas, and if the foil is thick enough, foil and gas may be considered one medium, and there is no surface effect. It is only necessary to measure x in terms of the mean free path λ, which means a change of scale at the surface $x=0$. If the foil has a smaller atomic number than the gas, the result will be between no surface and a perfectly absorbing surface. With an electric field and $Z_{\text{gas}}=Z_{\text{foil}}$, the same will be the case: In the foil, the electrons are not experiencing an electric field, while, if the foil were replaced by gas for $x<0$, such a field would exist and would tend to bring the electrons back to $x=0$ and $x>0$. Therefore, fewer electrons come from a full space of gas. We believe, however, that all these effects are small, and have neglected them.

VI. CONCLUSION

The age theory for electron scattering has been extended to include effects of an applied electric field. When the electric field is less than a critical field E_c a solution which is valid for moderate distances is obtained by a saddle-point integration. For an electric field equal to E_c an exponential dependence on distance is obtained. A simple exponential solution is obtained when the energy gained in the electric field E_b is equal to the energy lost by inelastic collisions. However, care should be taken to insure that the angular distribution is diffuse enough to allow the use of the P_1 approximation. We have verified in Sec. III that for our limiting case ($\gamma \approx 1$; $\xi \approx \frac{1}{2}$) the error in neglecting the higher-order polynomials is about 15%.

APPENDIX A: ISOTROPIC VS FOKKER-PLANCK SCATTERING

One reason that P_1 approximation is valid for the age theory is that small-angle Coulomb scattering [approximated in (13) by the Fokker-Planck expansion] tends to smear the electron distribution. For isotropic scattering Eq. (13) may be rewritten as

$$2\mu\lambda\frac{\partial f}{\partial x} - 2\lambda\frac{\partial}{\partial w}[\epsilon(w)f] + \gamma(1-\mu^2)\frac{\partial f}{\partial \mu} = -2f + \int_{-1}^{+1} f(\mu')\,d\mu' . \quad \text{(A1)}$$

We can then integrate over energy to obtain an equation in F [see Eq. (17)]. Assuming that F has the form given by (22), we can write

$$\frac{-2\mu\xi g}{\langle\mu\rangle} + 2\xi g - 2\gamma\mu g + \gamma(1-\mu^2)\frac{dg}{d\mu} = -2 + \int_{-1}^{+1} g(\mu')\,d\mu' . \quad \text{(A2)}$$

Note when $\xi = 0$, g doesn't have the simple exponential solution given by (24). Using the Legendre polynomial expansion as given by (25), (A2) can be represented by infinite set of simultaneous equations. The nth equation is

$$A_{n+1}\frac{n+1}{2n+3}\left(\gamma n - \frac{2\xi}{\langle\mu\rangle}\right) + A_n[2(1-\delta_{0,n}) + 2\xi] - A_{n-1}\frac{n}{2n-1}\left(\frac{2\xi}{\langle\mu\rangle} + (n+1)\gamma\right) = 0 , \quad \text{(A3)}$$

where $\delta_{0,n} = 1$ when $n = 0$, otherwise $\delta_{0,n} = 0$.

The P_1 approximation for isotropic scattering gives the identical result as the Fokker-Planck. However, going to higher-order approximations makes it clear that the solution converges slowly and the P_1 approximation is inadequate for the limiting case of $\gamma = 1$, $\xi = \frac{1}{2}$. In fact, even when $\xi = 0$ and $\gamma = 1$ the P_1 approximation is a poor representation. Figure 7 shows a plot of $g(\mu)$ for these cases using 25 polynomials. The oscillations in $g(\mu)$ are indicative of the fact that even 25 polynomials are not adequate to represent $g(\mu)$. In Fig. 8 we see the plots for $g(\mu)$ when $\gamma = \frac{1}{2}$ and $\xi = \frac{1}{2}$ and 0. For isotropic scattering the P_1 approximation is valid only for $\gamma = \frac{1}{2}$, $\xi = 0$. Figure 9 shows $g(\mu)$ when $\gamma = 0$ (i.e., $E = 0$), and $\xi = 0.5$ and 0.25. By evaluating the ratio

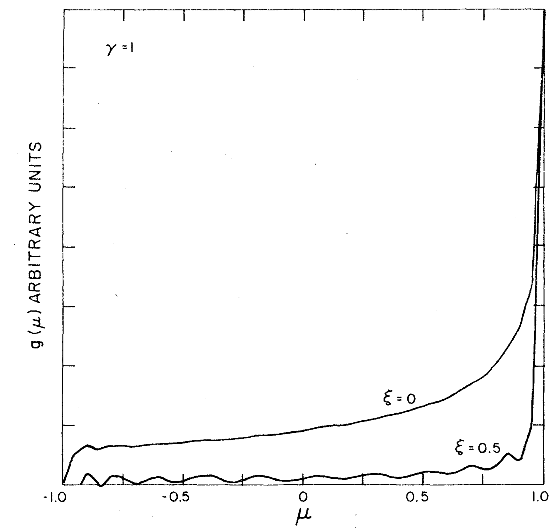

FIG. 7. The electron distribution function $g(\mu)$, assuming isotropic scattering for the special cases of $\gamma = 1$ and $\xi = 0$ and 0.5.

$$\sum_{n=2} A_n^2 \Big/ \sum_{n=0} A_n^2$$

the error in using the P_1 approximation can be estimated. When $\gamma = 0$ and $\xi = 0.5$, which corresponds to $Z \simeq 6$, the error in using the P_1 approximation is 30% and decreases to 15% for $\xi = 0.25$ (i.e., $Z \approx 12$).

It is interesting to note that for the special case of $\gamma = 0$ the distribution function $g(\mu)$ can be solved exactly and in closed form.[18] Integrating (A1) over energies for this special case, we find

$$\mu\lambda\frac{dF}{dx} + \xi F = \frac{1}{2}\int_{-1}^{+1} F\,d\mu - F . \quad \text{(A4)}$$

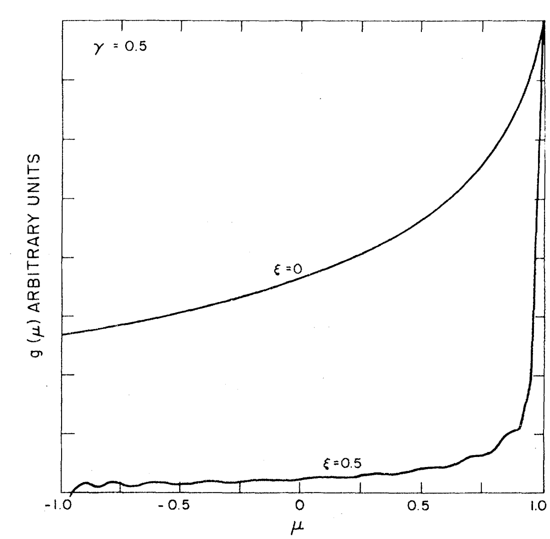

FIG. 8. Same as Fig. 7 except $\gamma = 0.5$.

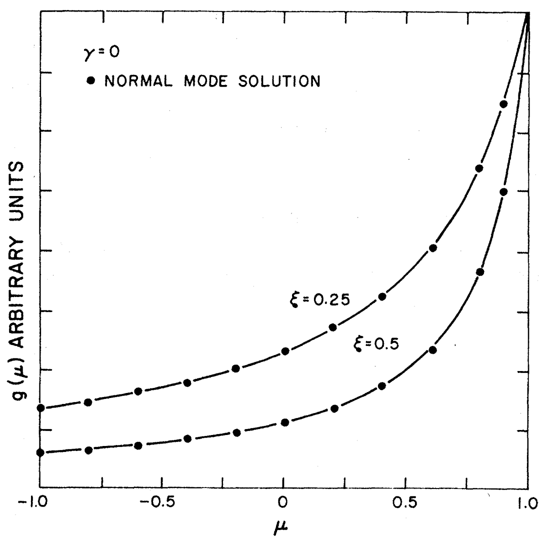

FIG. 9. Same as Figs. 7 and 8 except $\gamma=0$, i.e., zero electric field. The points correspond to the eigenvalue solution [Eq. (A6)].

Making the substitutions

$$c=1/(1+\xi) \quad \text{and} \quad \eta=x/c\lambda$$

(A4) may be rewritten as

$$\mu\frac{dF}{d\eta}+F=\frac{c}{2}\int_{-1}^{+1}F\,d\mu. \tag{A5}$$

Equation (A5) is the one-speed neutron transport equation. c is the average number of secondary neutrons produced per collision. The solution for g is[18]

$$g=q/(\nu-\mu) \tag{A6}$$

where q is a constant and ν is the eigenvalue satisfying the equation

$$\nu c\tanh^{-1}(1/\nu)=1. \tag{A7}$$

For $\xi=0.5$, $\nu\approx1.165$ and for $\xi=0.25$, $\nu\approx1.41$. Solutions given by (A6) are shown in Fig. 9 for comparison to the P_{25} result.

APPENDIX B: ENERGY SPREAD OF ELECTRONS

Starting from Eq. (13) in Sec. III, and assuming $T\ll 1$, we obtain, after multiplication by $w/2\lambda$ and integrating over w:

$$\mu\frac{\partial}{\partial x}\int wf\,dw-\int w\frac{\partial}{\partial w}(\epsilon f)\,dw+\frac{1-\mu^2}{2}\int\frac{\gamma}{\lambda}\frac{\partial f}{\partial\mu}w\,dw=\frac{1}{2}\frac{\partial}{\partial\mu}(1-\mu^2)\int\frac{\partial f}{\partial\mu}\frac{w}{\lambda}dw. \tag{B1}$$

The second term can be integrated by parts, giving

$$\int\epsilon f\,dw=\int\chi f\,dw-eE\mu\int f\,dw. \tag{B2}$$

Set

$$\chi=\chi_0 w_0/w \tag{B3}$$

and introduce

$$F_n(\mu,x)=\int w^{1+n}f\,dw; \tag{B4}$$

then

$$\int\epsilon f\,dw=\chi_0 w_0 F_{-2}-eE\mu F_{-1}. \tag{B5}$$

The third term of (B1) can be written as

$$\tfrac{1}{2}(1-\mu^2)eE\,\partial F_{-1}/\partial\mu. \tag{B6}$$

The term on the right of (B1) is not important for the following, but it may be brought into the same form by setting

$$\lambda=\lambda_0 w^2/w_0^2. \tag{B7}$$

The entire Eq. (B1) then becomes

$$\mu\frac{\partial F_0}{\partial x}+\chi_0 w_0 F_{-2}-eE\mu F_{-1}+\tfrac{1}{2}eE(1-\mu^2)\frac{\partial F_{-1}}{\partial\mu}=\frac{1}{2}\frac{w_0^2}{\lambda_0}\frac{\partial}{\partial\mu}(1-\mu^2)\frac{\partial F_{-2}}{\partial\mu}. \tag{B8}$$

We now integrate over μ. Then the right-hand side vanishes. We introduce the notations

$$\int\mu\,d\mu\,F_n=j_n,\qquad \int d\mu\,F_n=\rho_n. \tag{B9}$$

The fourth term in (B8) after partial integration gives

$$eE\int\mu F_{-1}\,d\mu=eEj_{-1} \tag{B10}$$

which exactly cancels the integral of the third term, as it should. Thus the electric field drops out, and we get

$$\partial j_0/\partial x+\chi_0 w_0\rho_{-2}=0. \tag{B11}$$

This is exactly analogous to Eq. (20) in Sec. III, and gives the solution (21) if the energy spread is small.

Now we repeat the same procedure, multiplying Eq. (B1) by w^n under the integral signs. (B2) is now replaced by

$$(n+1)\int\epsilon fw^n\,dw=\left(\int\chi fw^n\,dw-eE\mu\int fw^n\,dw\right)(n+1). \tag{B12}$$

Accordingly, (B8) is replaced by

$$\mu \frac{\partial F_n}{\partial x} + \chi_0 w_0 (n+1) F_{n-2} - (n+1) eE\mu F_{n-1} + \tfrac{1}{2} eE(1-\mu^2)\frac{\partial F_{n-1}}{\partial \mu} = \frac{1}{2}\frac{w_0^2}{\lambda_0}\frac{\partial}{\partial \mu}(1-\mu^2)\frac{\partial F_{n-2}}{\partial \mu}. \tag{B13}$$

Integrating over μ, the right-hand side again vanishes. The last term on the left cancels one unit of the third term, leaving n units. Using the notation (B9) we get

$$\frac{\partial j_n}{\partial x} + (n+1)\chi_0 w_0 \rho_{n-2} - neEj_{n-1} = 0. \tag{B14}$$

Consider first the average energy,

$$W = j_1/j_0. \tag{B15}$$

Using (B14) with $n=1$ together with (B11), we get

$$-\frac{\partial W}{\partial x} = \chi_0 w_0 \frac{2\rho_{-1} - W\rho_{-2}}{j_0} - eE. \tag{B16}$$

This then is the exact condition for the balancing field:

$$eE_b = \chi_0 w_0 \frac{2\rho_{-1} - w_0\rho_{-2}}{j_0} \approx \frac{\chi_0}{\langle\mu\rangle} \tag{B17}$$

which is the same as Eq. (28) in Sec. III.

Now the energy spread,

$$\langle w^2\rangle - \langle w\rangle^2 = (j_2 j_0 - j_1^2)/j_0^2. \tag{B18}$$

Let us first consider

$$\frac{d}{dx}(j_2 j_0 - j_1^2) = -\chi_0 w_0 (j_2\rho_{-2} + 3j_0\rho_0 - 4j_1\rho_{-1}). \tag{B19}$$

The terms with eE cancel between $j_0 j_2'$ and $2j_1 j_1'$, and the electric field has no effect. Now *assume* that

$$j_n/\rho_n = \langle\mu\rangle \tag{B20}$$

independent of n, then (B19) becomes

$$(\chi_0 w_0 j_0^2/\langle\mu\rangle)[4\langle w\rangle\langle w^{-1}\rangle - 3 - \langle w^2\rangle\langle w^{-2}\rangle]. \tag{B21}$$

Now let $w_0 = \langle w\rangle$ and $w = w_0 + \delta w$. Then, by definition, $\langle\delta w\rangle = 0$, and

$$\langle w^n\rangle = w_0^n\left(1 + \frac{n(n-1)}{2}\frac{\langle\delta w^2\rangle}{w_0^2}\right). \tag{B22}$$

On substituting (B22) into (B21) we obtain zero. Hence the numerator in (B18) does not change, up to order $\langle\delta w^2\rangle$. Concerning the denominator, we have

$$-\frac{1}{j_0}\frac{dj_0}{dx} = \frac{\chi_0 w_0 \rho_{-2}}{j_0} \approx \frac{\chi_0}{w_0\langle\mu\rangle}. \tag{B23}$$

Hence

$$\frac{d}{dx}(\langle w^2\rangle - \langle w^2\rangle) = \frac{2\chi_0}{w_0\langle\mu\rangle}\langle\delta w^2\rangle. \tag{B24}$$

So the $\langle\delta w^2\rangle$ will increase exponentially; an increase by a factor e^2 will take place in a distance

$$(w_0/\chi_0)\langle\mu\rangle = R\langle\mu\rangle, \tag{B25}$$

where R is the *range* of the electron, disregarding the electric field. In the distance (B25), $(\langle\delta w^2\rangle)^{1/2}$ increases by a factor e.

Unfortunately, assumption (B21) is unjustified. For larger n, faster electrons are emphasized, and for these $\langle\mu\rangle$ will be larger. This correlation can be accounted for by assuming

$$\mu_n \equiv j_n/\rho_n = \mu_0 + n\delta\mu \tag{B26}$$

and

$$\delta\mu = \mu_0 \alpha\langle\delta w^2\rangle/w_0^2; \tag{B27}$$

then the term in the square brackets in (B21) becomes

$$2\frac{\delta\mu}{\mu_0} = \frac{2\alpha}{w_0^2}\langle\delta w^2\rangle. \tag{B28}$$

The rate of increase of $\langle\delta w^2\rangle$ will now be faster than (B24). The root mean square of δw will increase by a factor e in the distance

$$[R/(1+\alpha)]\langle\mu\rangle. \tag{B29}$$

APPENDIX C: AGE THEORY FOR NEGATIVE x

The solution (38), (39) is valid for any x and t, whether x is positive or negative. However, the saddle-point integration is only valid if t is large enough. The saddle-point result, Eq. (42), presents some difficulty if $x < -\tau$. However (42) may be certainly used for small negative x.

We compare the results for positive and negative x. Assuming $|x| \ll \tau$, we may expand (42) in powers of x and find

$$N(x,t) \approx \frac{c^{1/4}t^{-1/2}}{\chi} \times \exp\left[-\left(\frac{\tau}{a}(1-\sqrt{c})^2 + \frac{x}{a}(1-\sqrt{c}) + \frac{x^2}{4t}\right)\right]. \tag{C1}$$

The characteristic diffusion result (43) is part of this answer, and (C1) reduces to (43) for $c=1$. It is interesting that for given $|x|$, (C1) is smaller for positive than for negative x. This result seems paradoxical: the electrons go preferentially against rather than with the electric field. Physically, this occurs because the electric field accelerates the electrons in the direction of $x > 0$. Hence there exists a positive current at $x = 0$.

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