Comment on the use of the low-energy theorem in determining the bremsstrahlung spectrum*

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Much understanding of the bremsstrahlung spectrum can be obtained from the behavior at its two end points. The low-energy theorem can be used to obtain the soft-photon end point from elastic scattering; this may be combined with data from the hard-photon portion of the spectrum to interpolate through the soft-photon region, which is difficult to calculate directly because partial-wave series converge slowly. The low-energy theorem may also be used to show that, contrary to statements in the literature, the ratio of the Born approximation to the exact cross section approaches unity at the soft-photon end point for the Coulomb potential but not for a screened potential.

We have recently presented[1] a discussion of properties of the electron bremsstrahlung spectrum, based on direct numerical calculations in partial-wave series for the process described as a single-electron transition in a screened central potential. However, we did not sufficiently emphasize that much understanding of the spectrum can be obtained from the behavior at its two end points: the tip $k/T_1=1$, where all the incident-electron kinetic energy T_1 is radiated in the photon energy k, and the soft-photon limit $k/T_1=0$. We had previously discussed[2] the tip region, which can be related to atomic photoeffect. The soft-photon region of the spectrum can be related through the low-energy theorem[3] to elastic electron scattering. What we wish to point out is that this is of practical importance: Direct numerical calculation of the spectrum is difficult in this region because the partial-wave series converge slowly; we shall demonstrate that an interpolation can be based on the soft-photon end-point value. We shall also use the low-energy theorem to show that, contrary to statements found in the literature,[4] the ratio of Born approximation to the exact cross section approached unity at the soft-photon end point for the Coulomb potential but not for a screened potential. We emphasize that our concern here is with the usefulness of the low-energy theorem for the end point, not with the attempts to extend the theorem. (We shall elsewhere apply the same ideas in discussions of the angular distribution of emitted radiation and of partially ionized atoms.)

The low-energy theorem states that, for fixed incident energy T_1, in the soft-photon limit the totally differential bremsstrahlung cross section (summed over final and averaged over initial electron spins) may be written in terms of the unpolarized elastic electron-scattering cross section as

$$\lim_{k\to 0} k\frac{d^3\sigma}{d\Omega_k\, d\Omega_2\, dk}=\frac{\alpha}{4\pi^2}\left(\frac{d\alpha}{d\Omega_2}\right)_{\rm el}\left(\frac{\epsilon\cdot p_1}{\bar{k}\cdot p_1}-\frac{\epsilon\cdot p_2}{\bar{k}\cdot p_2}\right)^2 . \quad (1)$$

Here p_1 and p_2 are the respective four-momenta of incident and scattered electrons, ϵ the polarization of the photon, and $\bar{k}$ the four-vector $(1,\hat{k})$ with photon direction $\hat{k}$. There is also a low-energy theorem for *approximate* cross sections, for example relating the Born approximation for bremsstrahlung to the Born approximation for elastic scattering.

Summing over photon polarization states, integrating over photon angles, and finally integrating over electron angles, the end point of the bremsstrahlung spectrum is given by

$$\lim_{k\to 0} k\frac{d\sigma}{dk}=4\alpha\int_0^\pi d\theta\,\sin\theta\left(\frac{d\sigma}{d\Omega}\right)_{\rm el}\times[A(A^2-B^2)^{-1/2}\cosh^{-1}(A/B)-1], \quad (2)$$

where $A=1-\beta_1^2\cos\theta$, $B=1-\beta_1^2$, $\beta_1=|p_1|/E_1$, and θ is the electron scattering angle, and so for a screened potential the end point may be calculated from the elastic-scattering cross section.[5] For the Coulomb potential, $(d\sigma/d\Omega)_{\rm el}$ diverges as $\sim\theta^{-4}$ in the forward direction, so that the integral diverges, corresponding to the well-known fact that $k\,d\sigma/dk$ diverges as $\ln k$ in this case. To demonstrate this, one can subtract the low-energy theorem for the Born-approximation cross section from Eq. (1) and integrate, obtaining instead of Eq. (2) the *difference* between the exact and the Born-approximation end points in terms of an integral over the *difference* between the exact and the Born-approximation elastic-scattering cross sections. The difference between these elastic-scattering cross sections diverges no faster than θ^{-3} in the forward direction,[6] the bracketed term of Eq. (2) goes as θ^2, and so the integral exists. This says that the difference between the exact and the Born-approximation end points is finite, but since the Born-approximation end point,[7] $k\to 0$ for $k d\sigma/dk$, diverges as $[16\alpha Z^2/(3\beta_1^2)]r_0^2\ln k^{-1}$ this means that the exact end point diverges in the same way. Thus

TABLE I. Numerical data for the bremsstrahlung energy spectrum. The column headings are as follows: Coul: for totally ionized atoms; Screen: for neutral atoms; B: Born approximation; EC: exact calculation. EH: Elwert-Haug approximation; ES: exact calculation. BF: Born approximation modified by the form factor; EHF: Elwert-Haug approximation modified by the form factor.

		$\beta^2 \frac{k}{Z^2}\frac{d\sigma}{dk}$ (Coul)					$\beta^2 \frac{k}{Z^2}\frac{d\sigma}{dk}$ (Screen)				
T_1 (keV)	k/T_1	B (mb)	B/EC	EH (mb)	EH/EC	EC (mb)	BF (mb)	BF/ES	EHF (mb)	EHF/ES	ES (mb)
						$Z=13$					
5	0.80	2.93	0.454	6.36	0.986	6.45	2.53	0.444	5.56	0.975	5.70
	0.60	4.55	0.650	6.89	0.984	7.00	3.81	0.637	5.92	0.989	5.98
	0.40	6.32	0.809	7.71	0.987	7.81	4.93	0.787	6.32	1.01	6.27
	0.20	8.87	0.947	9.26	0.989	9.36	5.79	0.894	6.63	1.02	6.50
	0.0						6.28	0.974	6.28	0.947	6.63
10	0.99	0.60	0.103	5.65	0.972	5.82	0.56	0.101	5.26	0.951	5.53
	0.80	2.89	0.466	6.05	0.975	6.21	2.63	0.453	5.56	0.958	5.81
	0.60						4.00	0.642	6.00	0.964	6.23
	0.40	6.26	0.806	7.61	0.979	7.77	5.33	0.784	6.61	0.973	6.80
	0.0						7.50	0.962	7.50	0.962	7.80
50	0.80	2.61	0.572	4.39	0.963	4.56	2.55	0.569	4.32	0.965	4.48
	0.60	4.14	0.745	5.36	0.966	5.55	4.00	0.738	5.19	0.958	5.42
	0.40	5.88	0.857	6.68	0.974	6.86	5.52	0.838	6.30	0.956	6.59
	0.20	8.44	0.929	8.84	0.974	9.08	7.51	0.906	7.89	0.952	8.29
	0.0						9.60	0.932	9.60	0.932	10.3
500	0.90	1.02	0.688	1.38	0.931	1.48	1.02	0.693	1.38	0.938	1.47
	0.75	1.93	0.824	2.19	0.937	2.34	1.92	0.825	2.18	0.937	2.33
	0.50	3.71	0.908	3.88	0.949	4.09	3.66	0.904	3.83	0.946	4.05
	0.0						10.9	0.916	10.9	0.916	11.9
						$Z=79$					
5	0.80	2.93	0.469	5.82	0.933	6.24	2.13	0.643	4.73	1.43	3.31
	0.60	4.55	0.704	6.05	0.936	6.46	3.09	0.993	4.93	1.58	3.11
	0.40						3.82	1.34	5.12	1.80	2.85
	0.20	8.87	1.18	7.17	0.956	7.50	4.33	1.74	5.36	2.16	2.48
	0.0						4.57	2.05	4.57	2.05	2.23
10	0.99										
	0.80	2.89	0.445	5.74	0.883	6.50	2.33	0.547	4.99	1.17	4.26
	0.60	4.50	0.669	6.03	0.896	6.73	3.45	0.830	5.22	1.26	4.16
	0.40	6.26	0.878	6.49	0.910	7.13	4.40	1.11	5.49	1.38	3.98
	0.20	8.81	1.11	7.42	0.930	7.95	5.15	1.40	5.88	1.60	3.68
	0.0						5.52	1.61	5.52	1.61	3.42
50	0.80	2.61	0.385	4.96	0.732	6.78	2.40	0.415	4.65	0.803	5.79
	0.60	4.14	0.570	5.48	0.754	7.26	3.71	0.612	5.11	0.843	6.06
	0.40	5.88	0.740	6.27	0.789	7.95	5.03	0.791	5.72	0.901	6.35
	0.20	8.44	0.906	7.70	0.828	9.30	6.50	0.957	6.64	0.976	6.80
	0.05	12.8	1.06	11.1	0.902	12.3					
	0.0						7.66	1.07	7.66	1.07	7.18
500	0.90	1.02	0.266	1.85	0.483	3.83	0.99	0.273	1.80	0.497	3.62
	0.75						1.86	0.436	2.33	0.546	4.27
	0.50	3.71	0.611	3.74	0.616	6.07	3.51	0.620	3.58	0.632	5.66
	0.0						9.67	0.843	9.67	0.843	11.5

TABLE II. Comparison between the interpolated and the exact numerical bremsstrahlung energy spectrum, $\beta^2(k/Z^2)(d\sigma/dk)$ (mb).

		Al				Au			
		Coulombic		Screened		Coulombic		Screened	
T_1 (keV)	k/T_1	Interp.	Exact	Interp.	Exact	Interp.	Exact	Interp.	Exact
5	0.2	9.33	9.36	6.57	6.50	7.45	7.50	2.57	2.48
50	0.2	8.99	9.08	8.31	8.29	8.93	9.30	6.83	6.80

in the Coulomb case, unlike the neutral-atom case, the ratio to the Born approximation approaches 1 at the soft-photon end of the spectrum, whereas the difference approaches a constant.

To see the feasibility of interpolation, we show in Table I our exact numerical data for Coulomb and screened potentials, for $Z=13$ and 79, $T_1=5$, 10, 50, and 500 keV, together with the predictions of possible interpolating forms, based on the Born-approximation (Bethe-Heitler) formula[7] or the Sommerfeld-Maue wave functions (Elwert-Haug calculations[4]), with screening described via a form factor. It should be understood that there is no *theoretical* justification for combining the Elwert-Haug formula with a Born-approximation form factor, but the Table shows that it provides a convenient numerical interpolating form. These methods are described in more detail in our previous paper.[1] The exact numerical data are obtained from our bremsstrahlung code except for the soft-photon end point $k/T=0$, where it is obtained from a calculation of elastic scattering via the low-energy theorem. It will be noted that in the neutral-atom case the Born approximation is not exact at the end point. In the point-Coulomb case the Born approximation does become exact, but the Elwert-Haug formula provides a smoother interpolating form in the soft-photon region.

We show in Table II some predictions of the bremsstrahlung intensities for $k/T_1=0.2$ obtained by interpolation from the hard-photon data at $k/T_1=0.4$, 0.6, and 0.8 (more easily calculated in partial-wave series) and the soft-photon $k/T_1=0$ end-point value, using ratios to the analytic results noted above. The table compares the predictions by interpolation with our exact numerical results for $k/T=0.2$; an accuracy well within 5% was achieved. In addition, interpolation from the same data gives 11.6 at 50 keV for Au at $k/T_1=0.05$ in the Coulomb case, compared with 12.3 from an exact numerical calculation; utilization of the $k/T_1=0.2$ data would improve the prediction to 12.0. It should be emphasized that it requires a tremendous amount of computation time to calculate the bremsstrahlung intensity at such small k/T_1: In this last case about 100 h were required, while with interpolation the result is quickly obtained within a few percent.

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[5]In the nonrelativistic (small β_1) case the right-hand side of Eq. (2) reduces to $(\frac{16}{3})\alpha\beta_1^2\int_0^\pi(d\sigma/d\Omega)_{\rm el}d(\sin^4\frac{1}{2}\theta)$. At very low energies for which one has isotropic elastic scattering $(d\sigma/d\Omega)_{\rm el}=4\pi R^2$, with R the s-wave scattering length, the soft-photon end point of $k\,d\sigma/dk$ is simply $(\frac{64}{3})\pi\alpha\beta_1^2R^2$.

[6]See p. 909 of J. W. Motz, H. Olsen, and H. W. Koch, Rev. Mod. Phys. 36, 881 (1964); B. Nagel, Trans. R. Inst. Technol. Stockh. No. 157 (1960).

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