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Aspects of exact dynamics for general solutions of the sine-Gordon equation with applications to domain walls*

J. F. Currie
Laboratory of Atomic and Solid State Physics, Cornell University, Ithaca, New York 14853
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A knowledge of the dynamics of nonlinear systems with multiple excitations is important for calculation of the continuum density of states and for one method of calculation of the classical statistical mechanics of the sine-Gordon Hamiltonian. Hirota's N-soliton construction for the sine-Gordon equation is extended by analytic continuation to incorporate usefully breather and continuum solutions as well. The significance of these classical solutions is illustrated in the context of the propagation and excitations of domain walls in a classical uniaxial ferromagnet of infinite extent. From these solutions, formulas for the relative phase shifts in the scattering of a soliton, breather, or continuum solution from any one of these, acting as scatterer, are derived. Following Hirota, it is shown that under certain assumptions about the asymptotic properties of the solution the relative phase shifts of solutions corresponding to many soliton, breather, and continuum states may be found by adding the derived pairwise phase shifts.

I. INTRODUCTION

The sine-Gordon partial differential wave equation has been widely studied by mathematicians and physicists because of not only its complete integrability and accompanying remarkable "soliton" properties, but also its ubiquity as a model of nonlinear physical phenomena.[1] Its pulselike solitary wave solution has been used to describe excitations in many areas of condensed matter physics: to name a few, domain walls in ferromagnets, dislocations in crystals, charge dislocations in one-dimensional Fröhlich charge-density-wave condensates, and flux quanta on Josephson-junction transmission lines.

From among these many applications, we choose here to discuss our general analysis in terms of a simple model of a classical uniaxial ferromagnet. Solitons and continuum solutions correspond to domain walls and plane linear spin excitations (not magnons). Breather solutions describe bound pairs of domain walls. Knowledge of the general analytic solutions yields information on how these various magnetic excitations interact. It will be seen that the classical solutions have interactions which cause phase shifts of the various excitations but do not permit them to be created or destroyed. Knowing the analytic expressions for these relative phase shifts allows one to calculate the continuum density of states in the presence of a given domain wall structure.

The classical statistical mechanics of systems modeled by the sine-Gordon equation and similar nonlinear partial differential equations represents an outstanding problem. There are two ways[2] to calculate partition functions. The first way, possible only for one-spatial-dimension models, is the functional integral method.[3-5] The second way forms a sum over configuration space of the appropriate Boltzmann weighting factor. The latter method requires a knowledge of the density of states for any infinitesimal volume in configuration space. The classical statistical-mechanics problem is similar to that of semiclassical quantization in field theory. In connection with the field-theoretic calculation, Dashen *et al.*[6] have outlined a method for computing some phase shifts. In this paper, I present the phase-shift calculation in detail, using an analytic continuation of Hirota's exact N-soliton solution, derive some new general phase shifts, and discuss their properties.

The outline of this paper is as follows: In Sec. II, Hirota's N-soliton construction[7] is defined in the context of a simple model ferromagnet, and its use and analytic properties are explored for the $N=2$ case. It is shown how Hirota's construction can be extended to include a breather and the trivial continuum solution ($k=0$) in addition to the pure soliton solutions. The connection between the classical solutions and the physical excitations of the ferromagnet is discussed. Section III defines the phase shift and shows how it may be used to deduce the continuum density of states in the presence of a collection of possibly propagating domain walls. Hirota's phase-shift theorem is stated, and an important corollary is noted. A heuristic proof of both the theorem and its corollary are presented. Section IV presents the calculation of the general phase shift for the continuum-soliton collision and includes a list of phase shifts for all general two-particle collisions.

II. N-SOLITON CONSTRUCTION OF HIROTA FOR THE SINE-GORDON EQUATION

In this section, we shall be discussing the main mathematical results. For concreteness, we shall adopt the language of one of the many possible applications. Consider a classical uniaxial ferromagnet of infinite extent whose magnetization density vector $\vec{M}$ prefers to align parallel or antiparallel to the z axis. As in Fig. 1, we define $\theta(\vec{r}, t)$ to be the polar angle and $\psi(\vec{r}, t)$ to be the azimuthal angle of the magnetization vector. We are interested in planar configurations such as Bloch, or domain, walls and we shall look for structures which are independent of y and z, in planes normal to the x axis. In the absence of magnetic fields, the energy of magnetization per unit area of plane normal to the x axis, E, is

$$E=\int_{-\infty}^{+\infty}\left\{2\pi M^2 \sin^2\theta\cos^2\psi - K\cos^2\theta + A\left[\sin^2\theta\left(\frac{\partial\psi}{\partial x}\right)^2+\left(\frac{\partial\theta}{\partial x}\right)^2\right]\right\}dx\,, \qquad (2.1)$$

which is the sum of demagnetization, anisotropy, and exchange contributions, respectively. This equation serves to define the constants K and A. If we ignore Landau spin damping effects, then we can describe[8] the possible planar ($\psi=\frac{1}{2}\pi$) domain-wall structures and their dynamics by the solutions of the sine-Gordon equation

$$\phi_{tt}-c_0^2\phi_{xx}+\omega_0^2\sin\phi=0, \qquad (2.2)$$

where $\phi=2\theta$, $c_0=8\pi\gamma^2 A$, $\omega_0^2=8\pi\gamma^2 K$, and γ is the magneto-mechanical ratio. Equation (2.2) not only describes the dynamics of domain excitations but also plane linear excitations, as well as their mutual interaction. We stress that the latter excitations are not the classical magnons of the ferromagnet.

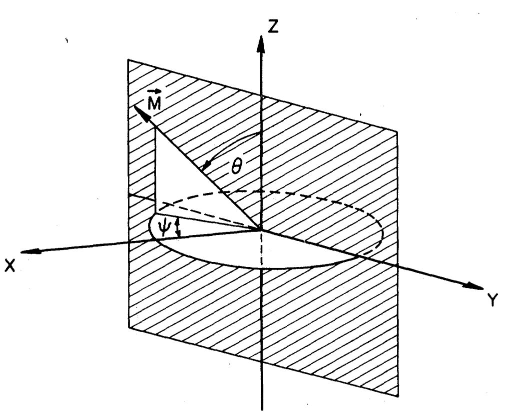

FIG. 1. Plane containing the magnetization density vector; Eq. (2.2) corresponds to $\psi=\frac{1}{2}\pi$.

In order to appreciate the richness of the possibilities of the sine-Gordon model, it is necessary to be able to obtain in a useful form the general solutions. This we have been able to do by slightly modifying, by an analytic continuation, the remarkable N-soliton solution discovered by Hirota.[7] In the remainder of this section, the details of Hirota's N-soliton solution, our modification of it, and how the derived solutions are important in describing the dynamics of the above model ferromagnet will be explained.

For convenience, one chooses units in which $\omega_0 = 1 = c_0$ and the sine-Gordon equation reads:

$$\phi_{tt}-\phi_{xx}+\sin\phi=0\,. \qquad (2.3)$$

There are at present three methods to construct the exact N-soliton solution to Eq. (2.3). The Bäcklund transformation,[9] the earliest method, is complicated and tedious in constructing solutions for $N \geq 2$. The "inverse scattering method"[10] is also unwieldy for $N \geq 2$. A third method, which is due to Hirota,[7] is notable for its simplicity. The latter method will be employed in this paper, using wherever possible the notation of Ref. 1 which has now become widely adopted in soliton calculations. Hirota's solution is given by

$$\tan[\tfrac{1}{4}\phi(x,t)]=g(x,t)/f(x,t), \qquad (2.4)$$

where

$$f(x,t)=\sum_{\mu=0,1}^{(e)}\exp\left(\sum_{i<j}^{N}B_{ij}\mu_i\mu_j+\sum_{j=1}^{N}\mu_j X_j\right), \qquad (2.5)$$

$$g(x,t)=\sum_{\mu=0,1}^{(o)}\exp\left(\sum_{i<j}^{N}B_{ij}\mu_i\mu_j+\sum_{j=1}^{N}\mu_j X_j\right), \qquad (2.6)$$

$$X_j=k_j x-\beta_j t+\gamma_j,\quad j=1,\ldots,N. \qquad (2.7)$$

k_j, γ_j, β_j are all constants satisfying:

$$k_j^2=\beta_j^2+1, \qquad (2.8)$$

and[10]

$$k_j\beta_i-k_i\beta_j\neq 0,\quad \text{if } i\neq j. \qquad (2.9)$$

By $\sum_{\mu=0,1}^{(e,o)}$, we mean a sum over all sets $(\mu_1, \ldots, \mu_N)$ where each μ_j is 0 or 1, and

$$\sum_{j=1}^{N}{}^{(e,o)}\mu_j$$

is an (even, odd) integer, respectively. The B_{ij} are defined by

$$\exp(B_{ij})=\frac{(k_i-k_j)^2-(\beta_i-\beta_j)^2}{(k_i+k_j)^2-(\beta_i+\beta_j)^2}\,. \qquad (2.10)$$

As an example of the technique, I shall construct the well-known ($N=2$) two-soliton solutions corresponding first to a soliton and antisoliton, and second to two solitons. For brevity, the frame of reference used in the construction is one in which the center of mass of the two solitons is at rest. Since Eq. (2.3) is invariant with respect to Lorentz transformations, it is possible to deduce the solutions corresponding to the two solitons traveling at arbitrary velocities by means of transforming to a different frame of reference. The solution for two solitons with velocities $+\mu$ and $-\mu$ is known[1] to be

$$\tan\tfrac{1}{4}\phi = \frac{\sinh[ut(1-u^2)^{-1/2}]}{u\cosh[x(1-u^2)^{-1/2}]}\,, \quad \text{soliton-antisoliton,} \tag{2.11}$$

$$\tan\tfrac{1}{4}\phi = \frac{u\sinh[x(1-u^2)^{-1/2}]}{\cosh[ut(1-u^2)^{-1/2}]}\,, \quad \text{soliton-soliton.} \tag{2.12}$$

Hirota's two-soliton solution is

$$\tan\tfrac{1}{4}\phi = \frac{\exp(X_1)+\exp(X_2)}{1+\exp(X_1+X_2+B_{12})}\,. \tag{2.13}$$

Let us define

$$\beta_1 = \beta_2 = uk_1 = -uk_2, \tag{2.14a}$$

$$k_1 = -k_2 = (1-u^2)^{-1/2}, \tag{2.14b}$$

$$\gamma_1 = \gamma_2 = \ln u, \tag{2.14c}$$

then $\exp(B_{12}) = -u^{-2}$ and

$$\tan\tfrac{1}{4}\phi = \frac{u(e^{k_1x}+e^{-k_1x})}{e^{-uk_1t}-e^{+uk_1t}} = \frac{-u\cosh k_1x}{\sinh uk_1t}\,. \tag{2.15}$$

Now recall the identity

$$\tan^{-1}(x) = \tfrac{1}{2}\pi - \tan^{-1}(x^{-1}). \tag{2.16}$$

Therefore,

$$\phi = 2\pi + \tan^{-1}\left(\frac{\sinh uk_1t}{u\cosh k_1x}\right).$$

Since ϕ is periodic in 2π, we see that Hirota's two-soliton solution with the choice of parameters (2.14) is the same as (2.11). Note that in the limit $u\to 0^+$, the points corresponding to $\phi(x_c,t)=\pi$ (or the centers of the soliton and antisoliton) when extremely far apart approach zero asymptotic velocity as follows: $v_c = \pm t^{-1}$ as $t\to\pm\infty$ or $v_c = \exp(-x_c)$ as $x_c\to\pm\infty$. Similarly, with the following choice of parameters

$$\beta_1 = -\beta_2 = uk_1, \tag{2.17a}$$

$$k_1 = k_2 = (1-u^2)^{-1/2}, \tag{2.17b}$$

$$\gamma_1 = \gamma_2 = -\ln u, \tag{2.17c}$$

$$\exp(B_{12}) = -u^{+2}. \tag{2.17d}$$

Eq. (2.13) reduces to Eq. (2.12).

The solutions of Eqs. (2.11) and (2.12) have a simple interpretation. The ϕ solitons are 180°-ferromagnetic domain walls, and their structure as they change from $2n\pi$ to $2(n\pm1)\pi$ describes how the magnetization changes between up and down (magnetization parallel and antiparallel, respectively) domains. The single soliton corresponds to the moving wall between up and down domains. The soliton-antisoliton ($s\bar{s}$) solution (2.11) and soliton-soliton (ss) solution (2.12) correspond to two walls separating three domains (two up domains separated by a down domain). The two solutions differ by how, in the physical model, the classical magnetization fields in the domain wall are strained. In the $s\bar{s}$ case, the angle θ increases from 0 to π and then eventually decreases to 0, whereas in the ss case the angle θ increases from 0 to π and then increases to 2π. Topologically, these two solutions are distinct. In the former solution, two walls may approach each other and annihilate leaving only one domain. However, in the latter, the walls may never come close one to another and the three domains remain distinct. In the classical picture, this corresponds to the fact that it is energetically unfavorable to have large gradients of the magnetization field and thereby increase the strain energy. The connection between the parameters u (only one in the center-of-mass frame) and the domain-wall picture is clear from the example $u=0$ described above. At $t=\pm\infty$ the domain walls were asymptotically far apart and stationary; thus u gives the asymptotic domain-wall velocity. The walls experience an exponentially small attractive force in this region for any large but finite $|t|$ and $|x_c|$; that is, they interact with one another via a short-range force.

Ablowitz, Kaup, Newell, and Segur[11] discovered the rest of the time-dependent solutions of Eq. (2.3) using the inverse scattering transformation. These solutions were classified as breathers or continuum solutions according to their pole structure. Explicit construction of general solutions in this formalism is not easy. We addressed the problem of whether the Hirota N-soliton construction might be extended to incorporate breathers and continuum excitations. We have found that it is possible to obtain exact solutions containing breathers or continuum solutions from the N-soliton solution by analytically continuing the asymptotic velocity parameters in Hirota's formalism for some of the solitons into the complex u plane. This modification is similar in spirit to a remark some time ago[12] concerning how to obtain an oscillating breather solution from the well-known $s\bar{s}$ solution Eq. (2.11). We shall repeat this well-known construction, for it represents the simplest possible application of the continuation to the Hirota formalism. As a result of continuing in the complex u plane from $u=a+i0$ to $u=0+ib$ in Eq. (2.14), we have

$$\tan\tfrac{1}{4}\phi = \frac{\sinh[ibt(1+b^2)^{-1/2}]}{ib\cosh[x(1+b^2)^{-1/2}]} = \frac{\sin[bt(1+b^2)^{-1/2}]}{b\cosh[x(1+b^2)^{-1/2}]}\,. \qquad (2.18)$$

This "breather" solution is periodic in time with angular frequency $\omega = b(1+b^2)^{-1/2}$. If the modulus of the velocity $|u| = |ib|$ takes on any real value, the frequency ω obeys $|\omega| \leq 1$.

This localized oscillating solution corresponds to a bound pair of domain walls. The energy of creation of this excitation may take any value less than that for two domain walls separating three long-lived domains. In terms of domains, this excitation represents a localized time-dependent deviation of the magnetization density from a direction parallel or antiparallel from the easy axis. This "breather" excitation has not been generally considered in sine-Gordon models of ferromagnets and should have interesting consequences. In a real ferromagnet, such localized excitations would of course have a lifetime, due to damping, and not exist indefinitely as Eq. (2.18) suggests.

The small-amplitude plane linear excitations may also be derived within the Hirota formalism. Physically, these correspond to traveling spin waves in which the classical spins oscillate in planes normal to the x axis. In the presence of domain walls, the excitations are plane waves far from the wall, but in the wall, the plane waves are modulated and undergo a phase shift [see Eq. (4.11)]. The combined wall and continuum solution presents an interesting new planar excitation of the ferromagnetic system which differs from the solutions of linear spin waves about a static Bloch wall that have been found by Winter, Janak, and others.[13] A harmonic traveling-wave solution of given frequency ω and wave vector k is related to that of any other permissible ω' and k' by a Lorentz transformation since Eq. (2.2) is Lorentz covariant. Thus, in order to know all harmonic traveling-wave (continuum) solutions, it is sufficient to know just one such solution. Note that if $b^{-1} \equiv \epsilon$ where $0 < \epsilon \ll 1$ in Eq. (2.18), then

$$\begin{aligned}\tan\tfrac{1}{4}\phi &= \epsilon\sin[t(1-\tfrac{1}{2}\epsilon^2)][1-\tfrac{1}{2}(\epsilon x)^2] + o(\epsilon^4)\\ &\approx \epsilon\sin(t\omega)\cos(kx)\\ &\approx \epsilon\sin(\omega t \pm kx),\end{aligned}$$

where

$$\left.\begin{aligned}\omega &\simeq 1-\tfrac{1}{2}\epsilon^2\\ k &\simeq \epsilon\end{aligned}\right\}\ \text{to } O(\epsilon^2)\,.$$

To first order in ϵ, we have

$$\phi \simeq 4\epsilon\sin(\omega t \pm kx)\,, \qquad (2.19)$$

which is a low-amplitude harmonic traveling-wave solution and also a solution of the linearized sine-Gordon equation $\phi_{tt} - \phi_{xx} + \phi = 0$. The dispersion relation for the continuum solutions is $\omega^2 = 1 + k^2$. Hence, in the limit $\epsilon \to 0$, the solution (2.18) is the (zero amplitude) $k=0$, $\omega = 1$ continuum state. While this example is simple, it should be emphasized that a general solution corresponding to any arbitrary number of solitons, breathers, and continuum excitations may now be constructed. Thus, we can determine the effect of the presence of a domain wall upon a given continuum mode in this classical theory by examining the appropriate classical solution. We shall see in the next section that the presence of solitons and breathers can, in fact, alter the continuum density of states.

One cautionary remark should be made. The solutions of Hirota are valid on an infinite interval $x \in [-\infty, +\infty]$, and should describe a ferromagnet whose extent is infinite, as is commonly assumed when considering properties in a thermodynamic limit. One must be careful if these results are to be used on a finite interval $x \in [-\tfrac{1}{2}L, \tfrac{1}{2}L]$ where the above solutions represent at best an approximate solution of Eq. (2.2).

III. DYNAMICS OF THE SINE-GORDON FIELD: PHASE-SHIFT ANALYSIS

In the previous section, we noticed that domain walls in the sine-Gordon model interacted via a short-range force which was exponentially small at long distance. It remains to understand the interaction of the walls at short distances. While the details of such interactions are complicated, the asymptotic effect, on one wall, of having interacted with many others is, amazingly, characterized by a single simple quantity—a phase shift. This is reminiscent of the Friedel discussion[14] of impurity states in metals. In order to be able to see this, it is necessary to reexamine the special mathematical features of the classical solutions of Sec. II.

Hirota investigated the dynamics of his exact N-soliton solutions corresponding to $k_j\beta_i \neq k_i\beta_j$ if $i \neq j$ and established the following theorem: On the spatial interval $[-\infty, +\infty]$ the N-soliton solution of the sine-Gordon equation (2.3) splits apart into N single solitons in the limit $|t| \to \infty$. The effect of the collision of the jth soliton with the other $N-1$ solitons is only a relative phase shift $\delta(j)$; otherwise the jth soliton conserves its identity (k_j, β_j) after the collisions. Suppose l solitons satisfy

$$\lim_{t\to\infty}(X_i - X_j) = +\infty\,,$$

while N-1-l solitons satisfy

$$\lim_{t\to\infty}(X_i - X_j) = -\infty .$$

Defining the phase shifts $\delta_\pm(j)$ by

$$\lim_{t\to\pm\infty}\phi(x,t) = 4\tan^{-1}\{\exp[X_j - \delta_\pm(j)]\},$$

the relative phase shift $\delta(j)$ is given by

$$\delta(j) \equiv \delta_+(j) - \delta_-(j) = \sum_{m=1;m\neq j}^{l}\delta_{m,j} - \sum_{m=l+1;m\neq j}^{N-1}\delta_{m,j}, \qquad (3.1)$$

where

$$\delta_{m,j} = -\ln[-\exp(B_{m,j})] = -(B_{m,j}+i\pi).$$

One observation of considerable computational importance in the calculation of the density of continuum states and of the classical statistical mechanics arises from Eq. (3.1) in the following corollary: The phase shift $\delta(j)$ of Eq. (3.1) is independent of $\gamma_1,\ldots,\gamma_N$ and depends on the parameters k_i and β_i, $i=1,\ldots,N$ only.

The physical importance of this phase-shift result can be seen if we pause to consider the effect of N domain walls with all different velocities $\{u_j\}$, $j=1,\ldots,N$, on the continuum density of states. Suppose the ferromagnet extends a distance L along the x axis ($L\to\infty$ to allow us to use Hirota's theorem). In the absence of domain walls, the continuum states parametrized by wave vector k are allowed if they satisfy the Born–von Karman boundary conditions in the form

$$Lk = 2\pi n \quad \text{for } n = 0, \pm 1, \ldots .$$

However, in the presence of domain walls, this condition is changed to become

$$Lk + \delta(k) = 2\pi n \quad \text{for } n = 0, \pm 1, \ldots ,$$

where $\delta(k)$ is, in the limit $L\to\infty$, the sum of the phase shifts of the continuum state (k,ω) by the N domain walls. The density of states D is easily calculated

$$D(k) = \frac{dn}{dk} = \frac{L}{2\pi} + \frac{1}{2\pi}\delta'(k), = D^\circ(k) + \frac{1}{2\pi}\delta'(k).$$

where $D^\circ(k)$ is the density of states in the absence of domain walls. Once the analytic form of the phase shift $\delta(k)$ is known, then one has a result for $D(k)$ which is exact in the limit $L\to\infty$. The analytic expressions of all the pair-collision phase shifts will be derived and listed in Sec. IV. We have made particular use of the phase-shift formulas and resulting density of states in order to calculate the classical partition function corresponding to the model ferromagnet. Details of the calculation will be presented elsewhere.[5] We have shown how it is necessary to include the domain walls as elementary excitations[15] along with the continuum modes for an accurate low-temperature theory of the thermodynamics. This is quite reasonable for Sec. II demonstrated how closely the domain-wall classical solutions are related to the continuum classical solutions.

A heuristic proof of Hirota's theorem follows. By assumption, $k_i\beta_j \neq k_j\beta_i$ if $i\neq j$. This means that in the limit $t\to=\infty$, the phase difference $X_i - X_j$, $i\neq j$, can be made arbitrarily large or small. In fact, requiring that $X_j = \gamma_j$, a finite constant, then

$$k_j x - \beta_j t = 0 \qquad (3.2)$$

and

$$X_i - X_j = k_i(\beta_j/k_j - \beta_i/k_i)t + \gamma_i - \gamma_j . \qquad (3.3)$$

Combining (2.9) and (3.3), we can assume, without using lengthy and formal limit analysis but retaining generality, that

$$\left.\begin{array}{l} X_1,\ldots,X_l \to +\infty \\ X_j = \text{finite} \\ X_{l+1},\ldots,X_{N-1}\to -\infty \end{array}\right\} \text{ as } t\to+\infty . \qquad (3.4)$$

Now multiplying the right-hand side of Eq. (2.4) by

$$1 = \exp[-X_1 -,\ldots,-X_l]/\exp[-X_1^-,\ldots,-X_l],$$

we have in the limit $t\to+\infty$

$$\tan\tfrac{1}{4}\phi = \frac{\sum_\mu^{(o)}\exp[\sum_{i<m}^{l}(B_{im}\mu_i\mu_m + \mu_j X_j) + \sum_{i=1}^{l}(\mu_i - 1)X_i]}{\sum_\mu^{(e)}\exp[\sum_{i<m}^{l}(B_{im}\mu_i\mu_m + \mu_j X_j) + \sum_{i=1}^{l}(\mu_i - 1)X_i]} . \qquad (3.5)$$

Equation (3.5) may be evaluated for two cases:

(i) l is even: $$\tan\tfrac{1}{4}\phi = \exp\left(\sum_{i=1}^{l}B_{ij} + X_j\right), \qquad (3.6)$$

(ii) l is odd: $$\tan\tfrac{1}{4}\phi = \exp\left(-\sum_{i=1}^{l}B_{ij} - X_j\right). \qquad (3.7)$$

Using the result (2.16) and the periodicity of $\phi(x,t)$, we may rewrite (3.7) as

(ii) l is odd: $$\tan\tfrac{1}{4}\phi = -\exp\left(\sum_{i=1}^{l}B_{ij} + X_j\right) = \exp\left(\sum_{m=1}^{l}B_{mj} + i\pi l + X_j\right). \qquad (3.8)$$

From (3.6) and (3.8) and the definition of $\delta_+(j)$, we see

$$\delta_{m,j} = -(B_{m,j} + i\pi)$$

and

$$\delta_+(j) = \sum_{m=1}^{l} \delta_{m,j}\,.$$

The result for $\delta_-(j)$ follows *mutatis mutandis*, proving both the theorem and corollary.

It will be convenient in the following section to differentiate between the real and imaginary parts of δ. Therefore, define

$$\delta' \equiv \mathrm{Re}(\delta)\,, \tag{3.9a}$$

$$\delta'' \equiv \mathrm{Im}(\delta)\,, \tag{3.9b}$$

$$\delta \equiv \delta' + i\delta''\,. \tag{3.9c}$$

IV. DERIVATION OF PHASE SHIFTS FOR GENERAL COLLISIONS OF SINE-GORDON SOLUTIONS

Hirota's phase-shift formula as defined in Sec. III together with observations in Sec. II (concerning analytic continuation of the velocity and soliton parameters) and the properties of solutions to Eq. (2.3) under Lorentz transformation, allow one to derive the phase shift of an arbitrary collision between any number of solitons, breathers, or continuum states. The technique is readily introduced and illustrated by the following example.

Ultimately one wishes to find the phase shift δ of a continuum state of wave number k by a soliton with velocity v. Let us begin by considering the three-soliton solution viewed in a frame of reference in which one soliton travels with velocity $u = v' + i0$, and the remaining two solitons are bound so as to form a breather whose center of mass has zero velocity. Let

$$\beta_1 = i\omega = \beta_2\,, \tag{4.1}$$

$$\beta_3 = v'(1 - v'^2)^{-1/2}\,, \tag{4.2}$$

then,

$$k_1 = (1 - \omega^2)^{+1/2} = -k_2\,, \quad k_3 = (1 - v'^2)^{-1/2}\,. \tag{4.3}$$

Treating the bound solitons constituting the breather as one excitation, the phase shift of the breather by the soliton is

$$e^{-\delta} = -\frac{(k_1 - k_3)^2 - (\beta_1 - \beta_3)^2}{(k_1 + k_3)^2 - (\beta_1 + \beta_3)^2}\,. \tag{4.4}$$

Let $d \equiv [(1 - \omega^2)(1 - v'^2)]^{1/2}$. Then,

$$e^{-\delta} = -\frac{(d-1)^2 - [i\omega(1 - v'^2)^{1/2} - v']^2}{(d+1)^2 - [i\omega(1 - v'^2)^{1/2} + v']^2}\,, \quad v' \neq 1\,. \tag{4.5}$$

In order to find the phase shift of a continuum state of frequency 1, and wave number 0, let $\omega \to 1$. Then $d \to 0$ and

$$e^{-\delta} = -\frac{(1 - v'^2)^{1/2} + iv'}{(1 - v'^2)^{1/2} - iv'} = e^{-i\delta''} \tag{4.6}$$

or

$$\delta''(v') = \pi - 2\tan^{-1}[v'(1 - v'^2)]^{-1/2}\,. \tag{4.7}$$

This is the phase shift of a $k = 0$ continuum state by a soliton of velocity v'.

Now perform a Lorentz transformation to the frame of reference in which the soliton is at rest. Under this transformation, the frequency and wave vector of the continuum solution transform as

$$k' = (1 - V^2)^{-1/2}(k - V\omega)\,, \tag{4.8}$$

$$\omega' = (1 - V^2)^{-1/2}(\omega - Vk)\,, \tag{4.9}$$

$$\omega^2 - k^2 = 1 = \omega'^2 - k'^2\,.$$

If one chooses $V = v'$, then $\omega' = (1 - v'^2)^{-1/2}$ and $k' = -v'(1 - v'^2)^{-1/2}$. In this new frame, a phase shift $\delta_1 = i\delta''$ will remain the same as Eq. (4.7). Thus, the phase shift of a continuum solution of frequency ω' and wave vector k' in the presence of a soliton at rest is

$$\delta'' = \pi - 2\tan^{-1}(k') = 2\tan^{-1}(k'^{-1})\,. \tag{4.10}$$

The foregoing result may be easily checked independently. J. Rubinstein[16] first showed that the exact form of the continuum solution in the presence of a single soliton at rest is[17]

$$\psi_{k'}(x) = (2\pi)^{-1/2}(1 + k'^2)^{-1/2} e^{ik'x}(k' + i\tanh x)\,. \tag{4.11}$$

Defining the phase shift δ'' to agree with Hirota's use of the term

$$\lim_{x\to\infty} \psi_{k'}(x) \equiv (2\pi)^{-1/2} e^{i(k'x - \delta''/2)} \tag{4.12}$$

yields the same result for δ'' as Eq. (4.10).

Finally the general phase shift of a continuum state with ω and k by a soliton with velocity v can be deduced by performing the Lorentz transformation of either Eq. (4.10) or Eq. (4.2) and (4.3), and recalculating δ. In the former case,

$$\delta''(v; k) = 2\tan^{-1}\{(1 - v^2)^{1/2}[k + v(1 + k^2)^{1/2}]^{-1}\}, \tag{4.13}$$

and in the latter case

$$\beta_1 = u_1 k_1, \quad \beta_2 = u_2 k_2, \quad \beta_3 = v(1 - v^2)^{-1/2},$$

$$u_1 = \frac{+ib + V}{1 + ibV}, \quad u_2 = \frac{-ib + V}{1 - ibV} = u_1^*,$$

$$b = \omega(1 - \omega^2)^{-1/2}, \quad V = k(1 + k^2)^{-1/2},$$

$$k_1 = -k_2^* = (1 - u_1^2)^{-1/2}, \quad k_3 = (1 - v^2)^{-1/2}. \tag{4.14}$$

and algebraically one can deduce Eq. (4.13).

Before leaving this example, let us note two interesting limits of Eq. (4.13). First, let $v \to 1$, corresponding to a soliton which is traveling at the limiting speed and whose width $d = (1 - v^2)^{1/2}$ vanishes: $\delta(1, k) = 0$ for all k. Second, let $k \to \infty$; once again $\delta(v, \infty) = 0$ for all k. In both cases, $v \to 1$ and $k \to \infty$, the continuum solution is insensitive to presence of the soliton.

In the following paragraph, I catalog phase shifts according to the N-soliton formula that is required when constructing the solution for ϕ.

A. Two-soliton formula

The phase shift for an arbitrary collision of two like solitons for velocities u_1 and u_2 is

$$\delta(s, s \,|\, u_1, u_2) = \delta(\bar{s}, \bar{s} \,|\, u_1, u_2) = -\log\left(\frac{(u_1 u_2 - 1) + [(1 - u_1^2)(1 - u_2^2)]^{1/2}}{(u_1 u_2 - 1) - [(1 - u_1^2)(1 - u_2^2)]^{1/2}}\right). \tag{4.15}$$

Choosing the frame of reference in which $u_1 = u$ and $u_2 = -u$, Eq. (4.15) reduces to

$$\delta(s, s \,|\, u, -u) = \delta(\bar{s}, \bar{s} \,|\, u, -u) = -2\log|u|. \tag{4.16}$$

Similarly, for the general soliton-antisoliton collision

$$\delta(s, \bar{s} \,|\, u_1, u_2) = \delta(\bar{s}, s \,|\, u_1, u_2) = -\delta(s, s \,|\, u_1, u_2), \tag{4.17}$$

and hence

$$\delta(s\bar{s} \,|\, u, -u) = +2\log|u|. \tag{4.18}$$

B. Three-soliton formula

The general phase shift for a soliton with velocity u, and breather of frequency ω, and velocity u_2 is

$$\delta(s, br \,|\, u_1; \omega, u_2) = -\log\left(\frac{[(1 - \omega^2)^{1/2}(u_1 u_2 - 1) - i\omega(u_1 - u_2)] + [(1 - u_1^2)(1 - u_2^2)]^{1/2}}{-[(1 - \omega^2)^{1/2}(u_1 u_2 - 1) + i\omega(u_1 - u_2)] + [(1 - u_1^2)(1 - u_2^2)]^{1/2}}\right). \tag{4.19}$$

It is readily shown that (4.19) reduces to the formula for the soliton (v) – continuum (k) phase-shift result of Eq. (4.14) when the substitutions $u_1 = v$, $u_2 = -k(1 + k^2)^{-1/2}$, and $\omega = 1$ are made. For completeness, we repeat Eq. (4.13)

$$\delta(s, c \,|\, v, k) = \delta(s, br \,|\, v; 1, k(1 + k^2)^{-1/2}) = 2\tan^{-1}\{(1 - v^2)^{1/2}[k + v(1 + k^2)^{1/2}]^{-1}\}.$$

Results for the phase shifts in antisoliton-breather and antisoliton-continuum collisions are merely negatives one of another:

$$\delta(\bar{s}, br \,|\, u_1; \omega, u_2) = -\delta(s, br \,|\, u_1; \omega, u_2), \quad \delta(\bar{s}, c \,|\, v, k) = -\delta(s, c \,|\, v, k).$$

C. Four-soliton formula

The phase shift in a general breather-breather collision specified by (ω_1, u_1) and (ω_2, u_2) is given by the complicated expression

$$\begin{aligned}
&\delta(br; br \,|\, \omega_1, u_1; \omega_2, u_2) \\
&= -\log\left(\frac{[(1 - u_1^2)(1 - u_2^2)]^{1/2} - (1 - u_1 u_2)\{\omega_1\omega_2 + [(1 - \omega_1^2)(1 - \omega_2^2)]^{1/2}\} + i[\omega_2(1 - \omega_1^2)^{1/2}(u_1 - u_2) + \omega_1(1 - \omega_2^2)^{1/2}(u_2 - u_1)]}{[(1 - u_1^2)(1 - u_2^2)]^{1/2} + (1 - u_1 u_2)\{\omega_1\omega_2 + [(1 - \omega_1^2)(1 - \omega_2^2)]^{1/2}\} - i[\omega_2(1 - \omega_1^2)^{1/2}(u_1 - u_2) + \omega_1(1 - \omega_2^2)^{1/2}(u_2 - u_1)]}\right. \\
&\quad \left.\times \frac{[(1 - u_1^2)(1 - u_2^2)]^{1/2} - (1 - u_1 u_2)\{-\omega_1\omega_2 + [(1 - \omega_1^2)(1 - \omega_2^2)]^{1/2}\} + i[\omega_2(1 - \omega_1^2)^{1/2}(u_1 - u_2) - \omega_1(1 - \omega_2^2)^{1/2}(u_2 - u_1)]}{[(1 - u_1^2)(1 - u_2^2)]^{1/2} + (1 - u_1 u_2)\{-\omega_1\omega_2 + [(1 - \omega_1^2)(1 - \omega_2^2)]^{1/2}\} - i[\omega_2(1 - \omega_1^2)^{1/2}(u_1 - u_2) - \omega_1(1 - \omega_2^2)^{1/2}(u_2 - u_1)]}\right).
\end{aligned} \tag{4.20}$$

From this expression we find the result for the breather-continuum collision

$$\begin{aligned}
\delta(br, c \,|\, \omega, u; k) &= \delta(br, br \,|\, \omega, u; 1, k(1 + k^2)^{-1/2}) \\
&= -\log\left[\frac{(1 - u^2) - [(1 + k^2)^{1/2} - uk]\omega + i(1 - \omega^2)^{1/2}[u(1 + k^2)^{1/2} - k]}{(1 - u^2) + [(1 + k^2)^{1/2} - uk]\omega - i(1 - \omega^2)^{1/2}[u(1 + k^2)^{1/2} - k]}\right. \\
&\quad \left.\times \frac{(1 - u^2) + [(1 + k^2)^{1/2} - uk]\omega + i(1 - \omega^2)^{1/2}[u(1 + k^2)^{1/2} - k]}{(1 - u^2) - [(1 + k^2)^{1/2} - uk]\omega - i(1 - \omega^2)^{1/2}[u(1 + k^2)^{1/2} - k]}\right].
\end{aligned} \tag{4.21}$$

Equation (4.21) reduces to

$$\delta''(br, c\,|\,\omega_1, 0; k) = 4\tan^{-1}\left(\frac{(1-\omega^2)^{1/2}(1-u^2)}{k-u(1+k^2)^{1/2}}\right), \quad u \neq 1. \quad (4.22)$$

The last type of collision is that of the trivial two-continuum-solution collision; from (4.19),

$$\delta(c, c\,|\,k_1, k_2) = \delta(br, br\,|\,1, k_1(1+k_1^2)^{-1/2};\ 1, k_2(1+k_2^2)^{-1/2}) = 0 \text{ identically.} \quad (4.23)$$

In order to treat collisions involving more than two nonlinear solutions, one adds the pairwise phase shifts presented in this section, according to Eq. (3.1). Following from the corollary to Hirota's theorem, the initial and final spatial configuration of "scattering" solitons are not important in detail as long as these configurations are all sufficiently far from the "scattered" soliton.

V. CONCLUSION

This paper demonstrates how Hirota's results can be significantly extended, by application of analytic continuation, from a simple N-soliton construction to one which usefully incorporates breather and continuum solutions as well. The present work details this extension by considering generalizations of Hirota's two-soliton solution. Section III presents Hirota's phase-shift theorem, provides a heuristic proof, and in an important corollary to it points out that the phase shift is independent of the detailed spatial configuration of the scattering solutions. By applying analytic continuation to the original phase-shift formula of Hirota, one can greatly extend the range of solutions, include breathers and continuum solutions, and obtain expressions for the phase shifts.

We have discussed these results in the context of a classical uniaxial ferromagnet of infinite extent. The phase-shift analysis of Sec. IV and V was shown to be of importance in establishing the continuum density of states. This density of states will be, in turn, important for the calculation of the classical statistical mechanics of the model.

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