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Ray theoretic analysis of spatial and temporal self-focusing in general weakly nonlinear media

Dan Censor
Department of Electrical Engineering, Ben Gurion University of the Negev, Beer-Sheva, Israel
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A general method is presented for describing self-focusing phenomena in various media by means of a geometrical-optics approach. The general formalism applies to a wide class of media: inhomogeneous, anisotropic, dispersive, and time varying. In contradistinction to previous methods, which derive pertinent wave equations and analyze them by assuming intensity-dependent wave speeds, here the appropriate Hamilton equations are derived, from which self-focusing phenomena evolve directly. Simple examples illustrate the feasibility of the method to deal with spatial and temporal effects in various media.

I. INTRODUCTION AND GENERAL THEORY

Since Askar'yan[1] predicted in the early sixties the existence of the self-focusing phenomenon, a vast body of literature has evolved describing experiments and analyzing the problem theoretically. Space limitations rule out the possibility of giving credit to numerous authors who contributed to our present knowledge of the subject. To link the present discussions with the existing literature, the reader is referred to a recent paper by Miyagi and Nishida[2] and the book by Akhmanov and Khokhlov.[3] Henceforth only work directly related to the present subject will be cited.

The problem of describing self-focusing phenomena is approached here from the point of view of geometrical optics. Cumberbatch[4] (who also gives a review of the subject with many citations) considers self-focusing by means of geometrical optics theory. His approach will be compared to the present method. Ray theory is discussed by Brandstatter[5] and Kline and Kay[6]; hence the basic theory will be presented here very succintly.

The basis for the present argument is a ray-tracing formalism for weakly nonlinear media recently derived by the author.[7] This is summarized here using the extended Fermat principle of Synge[8] and a compact four-vector notation.

The behavior of the physical system is governed by Maxwell's equations in sourceless domains. In the conventional notation (e.g., see Ref. 9), we have

$$\frac{\partial}{\partial \vec{x}} \times \vec{E} = -\frac{\partial}{\partial t}\vec{B}, \quad \frac{\partial}{\partial \vec{x}} \times \vec{H} = \frac{\partial}{\partial t}\vec{D},$$
$$\frac{\partial}{\partial \vec{x}} \cdot \vec{B} = \frac{\partial}{\partial \vec{x}} \cdot \vec{D} = 0 , \tag{1}$$

which must be supplemented with constitutive relations to make the system of equations determinate. Essentially, geometrical optics is based on the applicability of the eikonal approximation, which describes the solutions for (1) as quasi-monochromatic waves having slowly varying amplitude and phase functions. Here we assume periodic, rather than harmonic waves, of the form

$$E_i = \sum_{m=-\infty}^{\infty} E_{m,i} \exp(im\theta) , \tag{2}$$

where $E_{m,i}(\vec{X})$ and $\theta(\vec{X})$ are the amplitude of the mth harmonic and the phase, respectively; $i=1, 2, 3$ denotes the space coordinates $\vec{x}=(x_1, x_2, x_3)$, and $\vec{X}=(\vec{x}, ict)$ is the space-time radius vector. A structure similar to Eq. (2) exists for $\vec{D}$, $\vec{B}$, and $\vec{H}$. The phase is represented as a line integral in four-space,

$$\theta(\vec{X}) = \int_{\vec{X}_0}^{\vec{X}} \vec{K} \cdot d\vec{X} , \tag{3}$$

where $\vec{K}=(\vec{k}, i\omega/c)$ and $\vec{k}$ and ω are the propagation vector and frequency, respectively. The self-consistency condition

$$\frac{\partial K_\alpha}{\partial X_\beta} - \frac{\partial K_\beta}{\partial X_\alpha} = 0, \quad \alpha \neq \beta, \ \alpha, \beta = 1, 2, 3, 4, \tag{4}$$

prescribes

$$\vec{K} = \frac{\partial \theta}{\partial \vec{X}} ,$$
$$\frac{\partial}{\partial \vec{X}} = \left(\frac{\partial}{\partial \vec{x}}, \frac{\partial}{\partial t} \Big/ ic \right) . \tag{5}$$

The constitutive relations for weakly nonlinear media[4,7] are written as

$$D_i = \epsilon_{ij}^{(1)} E_j + \epsilon_{ijk}^{(2)} E_j E_k + \cdots ,$$
$$B_i = \mu_{ij}^{(1)} H_j + \mu_{ijk}^{(2)} H_j H_k + \cdots , \tag{6}$$

where i, j, etc. indicate spatial Cartesian components, and all quantities depend on $\vec{K}$ and $\vec{X}$. Substituting (2), (5), and (6) into (1), we obtain for each harmonic wave

$$(\vec{k}\times\vec{E})_i - \omega\mu^{(1)}_{ij}H_j - \omega\mu^{(2)}_{ijk}H_jH_k - \cdots = 0\,,$$
$$(\vec{k}\times\vec{H})_i + \omega\epsilon^{(1)}_{ij}E_j + \omega\epsilon^{(2)}_{ijk}E_jE_k + \cdots = 0\,, \qquad (7)$$

where m, indicating the harmonic, is suppressed. The notation in (7) is compacted in the form

$$G_r(\vec{K}, \vec{a};\ \vec{X}) = 0\,, \quad r = 1, \ldots, 6\,, \qquad (8)$$

where $\vec{X}$ denotes that the constitutive parameters and amplitudes are slowly varying functions of $\vec{X}$, and $\vec{a} = (E_1, E_2, E_3, H_1, H_2, H_3)$ is a six-component vector. Similarly to the case of linear systems, (8) can be rewritten as

$$G_r = F_{rs}a_s = 0, \quad r, s = 1, \ldots, 6, \qquad (9)$$

where F_{rs} is a 6×6 matrix, depending on $\vec{K}$, $\vec{a}$, $\vec{X}$. For a nontrivial solution of (9) the determinant of F_{rs} must vanish

$$\det F_{rs} = F(\vec{K}, \vec{a}; \vec{X}) = 0\,. \qquad (10)$$

All nonsingular representations of F_{rs} lead to the same $F = 0$. Thus we have derived a dispersion relation between $\vec{K}, \vec{a}, \vec{X}$. It is recalled that (8) describes all harmonics. To conform with the linear case, for which only $m=1$ in (2) exists, we consider (10) as the dispersion relation only for $m=1$; for all other harmonics, (8) and (10) constitute equations for finding $\vec{a}$ of the relevant harmonic. It is observed that θ in (2) prescribes identical phase velocities for all harmonics. This satisfies the physical requirement for coherent harmonic generation which is essential for weakly nonlinear media.

The existence of a dispersion relation (10) and the use of the eikonal approximation, (2) and (3), together with the set (8) of field equations suffices to derive the Hamilton equations for the class of problems discussed here.

II. THE FIELD OF A BUNDLE OF RAYS

We consider the problem of deriving the appropriate Hamilton equations. As a variant of the derivation in Ref. 7, this is done here by using the extended Fermat principle of Synge,[8] which stipulates that (3) is an extremum. Since $\vec{K}$, $\vec{a}$, and $\vec{X}$ are related by means of (9) and (10), the integrand is augmented by using Lagrange-multiplier functions. The variational principle is, therefore,

$$0 = \delta\int\Big(\vec{K}\cdot\frac{d\vec{X}}{d\tau} + \lambda(\tau)F(\vec{K}, \vec{a}; \vec{X}) + \lambda(\tau)\lambda_r(\tau)[a_r - A_r(\vec{K}; \vec{X})]\Big)\,d\tau\,, \qquad (11)$$

where the form $\lambda\lambda_r$ has been chosen for convenience, and $a_r = A_r(\vec{K}; \vec{X})$ is a solution of (9), which we assume to exist, although we need not find it explicitly. The corresponding Euler equations are

$$\frac{d\vec{X}}{d\tau} = -\lambda\left[\frac{\partial F}{\partial\vec{K}} + \lambda_r\frac{\partial A_r}{\partial\vec{K}}\right]\,,$$
$$\frac{d\vec{K}}{d\tau} = \lambda\left[\frac{\partial F}{\partial\vec{X}} + \lambda_r\frac{\partial A_r}{\partial\vec{X}}\right]\,, \qquad (12)$$
$$\frac{\partial F}{\partial a_r} = \lambda_r\,.$$

Splitting (12) into space and time components and using t as the parameter along the ray path, we obtain

$$\frac{d\vec{x}}{dt} = -\frac{\partial F/\partial\vec{k} + (\partial F/\partial a_r)\partial A_r/\partial\vec{k}}{\partial F/\partial\omega + (\partial F/\partial a_r)\partial A_r/\partial\omega}\,,$$
$$\frac{d\vec{k}}{dt} = \frac{\partial F/\partial\vec{x} + (\partial F/\partial a_r)\partial A_r/\partial\vec{x}}{\partial F/\partial\omega + (\partial F/\partial a_r)\partial A_r/\partial\omega}\,, \qquad (13)$$
$$\frac{d\omega}{dt} = -\frac{\partial F/\partial t + (\partial F/\partial a_r)\partial A_r/\partial t}{\partial F/\partial\omega + (\partial F/\partial a_r)\partial A_r/\partial\omega}\,.$$

The derivatives $\partial A_r/\partial l_i$, l_i standing for k_i, x_i, t, or ω, are related to derivatives of G_r [Eq. (9)] according to

$$\frac{\partial A_s}{\partial l_i} = -\left(\frac{\partial G_r}{\partial a_s}\right)^{-1}\frac{\partial G_r}{\partial l_i}\,. \qquad (14)$$

Equations (13) and (14) suffice for ray tracing in weakly nonlinear media; however, for discussion of self-focusing we have to consider the structure of rays which form a beam, or bundle. This provides the necessary interaction mechanism which leads to self-focusing and pulse compression effects. We demand that (8) be satisfied consistently throughout the bundle of rays, which are described by (13). We therefore add the constraints

$$\frac{\partial a_s}{\partial x_i} = \frac{\partial A_s}{\partial x_i} + \frac{\partial A_s}{\partial k_j}\frac{\partial k_j}{\partial x_i} + \frac{\partial A_s}{\partial\omega}\frac{\partial\omega}{\partial x_i}\,,$$
$$\frac{\partial a_s}{\partial t} = \frac{\partial A_s}{\partial t} + \frac{\partial A_s}{\partial k_j}\frac{\partial k_j}{\partial t} + \frac{\partial A_s}{\partial\omega}\frac{\partial\omega}{\partial t}\,, \qquad (15)$$

prescribing the relations between spatial and temporal distributions of amplitude frequency, and propagation vector. These constraints are equivalent to $dG_r/dt = dG_r/dx_i = 0$.

The framework (13)–(15) suffices to deal with self-focusing phenomena in a variety of media. Unlike more specialized models, it is feasible to investigate anisotropic, inhomogeneous, time varying, and dispersive media. Other methods, e.g., that of Cumberbatch,[4] require that a wave equation be given, which is usually much more complicated than dealing with the field equations directly. The "effective refractive index," or the intensity-dependent wave velocity assumed by many authors is more heuristic than (15), which takes into account amplitude and polarization of all field components.

III. SIMPLE EXAMPLES

It is the aim of the present study to formulate a general theory rather than to investigate special cases. It is, however, felt that the feasibility of discussing various media should be demonstrated, at least for the simplest possible cases. We therefore consider spatial and temporal effects in simple media, and discuss the influence of dispersion and time and space inhomogeneities.

A. Simple media

The medium is considered to be homogeneous, isotropic, and time constant. This means that the constitutive parameters in (7) are constants with respect to $\vec{x}$, t. Also the medium is isotropic, so that all indices in (7) become i. In addition, it will be assumed here that the constitutive parameters are nondispersive, i.e., independent of $\vec{k}$, ω. For simplicity the nonlinearity will be restricted to the dielectric properties, i.e., only $\mu^1 = \mu = \text{const.}$ is considered. Only $\epsilon^{(2)}$, the first nonlinear effect, will be retained. Further simplification results due to a proper choice of the fields. In the present simple medium, transverse fields are possible. To reduce the problem to a scalar two-dimensional case the $\vec{E}$ field is assumed to be polarized in the x_3 direction. The beam is incident and stays in the x_1, x_2 plane.

Because the nonlinearity exists only for the dielectric parameters, the $\vec{H}$ field can be eliminated from (7), yielding only one equation,

$$G = [-k^2 + \omega^2\mu(\epsilon^{(1)} + E\epsilon^{(2)})]\,E = FE = 0 \,,$$
$$k^2 = k_1^2 + k_2^2 \,. \tag{16}$$

Clearly this is a vast simplification of the general case that can be discussed in the frame of the general theory. We can write

$$F = -k^2 + \omega\mu\epsilon_e = 0 \,, \tag{17}$$

where $\epsilon_e = \epsilon^{(1)} + E\epsilon^{(2)}$ is the "effective" value of the dielectric parameter. This is used here only as a convenient notation.

We wish now to compute (13) and (15) for the present case. It is noted that because G and F are proportional, the first equation, (13), becomes indeterminate, i.e., 0/0. By applying a limiting process and l'Hospital's rule we obtain

$$\frac{dx_i}{dt} = \frac{k_i}{\omega\mu\epsilon_e} \,, \quad i = 1, 2 \,. \tag{18}$$

This means that the group velocity is directed parallel to $\vec{k}$, as expected for a homogeneous, isotropic, and time-invariant medium. The medium properties also render $d\omega/dt \equiv 0$. Next consider dk_i/dt. Since $\partial F/\partial x_i = \partial A/\partial x_i \equiv 0$ in a homogeneous medium, we have dk_i/dt indeterminate. This is not caused by the fact that $G = FE$; hence we get

$$\frac{\partial E}{\partial x_i} = \frac{\partial A}{\partial k_j}\,\frac{\partial k_j}{\partial x_i} = \frac{2k_j}{\omega^2\mu\epsilon^{(2)}}\,\frac{\partial k_j}{\partial x_i} \,. \tag{19}$$

The time invariance and $(\partial/\partial\vec{x})\times\vec{k} = 0$, Eq. (4), prescribe

$$\frac{dk_i}{dt} = \frac{\partial k_i}{\partial t} + \frac{\partial k_i}{\partial x_j}\,\frac{dx_j}{dt} = \frac{\partial k_j}{\partial x_i}\,\frac{k_j}{\omega\mu\epsilon_e} \,. \tag{20}$$

Finally we get from (19) and (20)

$$\frac{dk_i}{dt} = \frac{\omega\epsilon^{(2)}}{2\epsilon_e}\,\frac{\partial E}{\partial x_i} \,. \tag{21}$$

The profile of the field within the beam tapers off away from the center; therefore we have $\partial E/\partial x_2 < 0$ for $x_2 > 0$. Consequently we get a negative dk_2/dt for $\epsilon^{(2)} > 0$; i.e., the $\vec{k}$ vector changes direction towards the center of the beam. Since the group velocity (18) remains parallel to $\vec{k}$, *the rays converge towards the center of the beam*. This is a demonstration of the spatial self-focusing phenomenon. Of course negative $\epsilon^{(2)}$ will give rise to a divergent beam. The field along the ray path behaves according to

$$\frac{dE}{dt} = \frac{\partial A}{\partial k_i}\,\frac{dk_i}{dt} = \frac{2k_i}{\omega^2\mu\epsilon^{(2)}}\,\frac{dk_i}{dt}$$
$$= (\omega^2\mu\epsilon^{(2)})^{-1}\frac{d}{dt}\,k^2 \,. \tag{22}$$

This yields E along the ray path, but since the present problem is time invariant, results from various rays can be used to compute $\partial E/\partial x_i$ for the next computational step. Hence the whole process of spatial self-focusing can be computed for the present case. As the rays get closer, $\partial E/\partial x_2$ will become smaller, slowing down the process of self-focusing. If, for some reason, E is not a monotonic function of x_2, the beam will break up into filaments, as observed by Brewer and Lifsitz.[10]

The structure of (15) suggests that there should also be a temporal self-focusing phenomenon. To show this for a simple example, let us assume an infinitely wide pulse, for which we assume $\partial a_s/\partial x_2 = 0$. On the other hand, $\partial a_s/\partial t$ is taken as a constraint. The analog of $\omega = \text{const.}$ for the spatial case is replaced now with $\vec{k} = \text{const.}$ This prescribes through (4) that $\partial\omega/\partial x_i = 0$. Consequently Eq. (15) becomes

$$\frac{\partial E}{\partial t} = \frac{\partial A}{\partial\omega}\,\frac{\partial\omega}{\partial t} = -\frac{2\epsilon_e}{\omega\epsilon^{(2)}}\,\frac{d\omega}{dt} \,, \tag{23}$$

which is the analog of (19). Equation (18) is the same for this problem. Thus Eq. (18) and

$$\frac{d\omega}{dt} = -\frac{\omega\epsilon^{(2)}}{2\epsilon_e}\frac{\partial E}{\partial t} \tag{24}$$

are the equations needed for tracing the ray. Usually a pulse will start with $\partial E/\partial t > 0$ and have a trailing edge with $\partial E/\partial t < 0$. For $\epsilon^{(2)} > 0$ and positive $\partial E/\partial t$, the frequency will diminish, according to (24); hence the group velocity (18) will increase. For the trailing part of the pulse the group velocity will decrease. Consequently $\epsilon^{(2)} > 0$ will cause the pulse to expand. On the other hand, $\epsilon^{(2)} < 0$ will cause a time-focusing, i.e., compression of the pulse. Note that for spatial self-focusing, $\epsilon^{(2)} > 0$ was needed. It appears that space and time focusing are working in opposite directions. The field intensity which builds up in a spatial focusing process will therefore be counteracted by the pulses becoming more extended in the direction of propagation. The last statement is of course speculative, as we address ourselves to two different and highly specialized examples.

B. Effect of spatial and temporal dispersion

The influence of dispersion on self-focusing is of great interest because the phenomena may be competitive. Relevant studies have been published by Hasegawa and Tappert.[11,12] It must be kept in mind that the present model, with first-order derivatives of the various parameters, can only cope with small dispersion effects. Cases where the wave packets change form rapidly, as considered by Anderson and Askne,[13] are out of the scope of the present discussion.

To include the effect of dispersion in the simplest form, replace ϵ_e by $\epsilon_e(k^2, \omega^2)$. The quadratic dependence is what we expect for simple lossless media. For discussion of spatial dispersion (dependence on $\vec{k}$) see Ginzburg.[14]

Repeating the first problem, i.e., spatial self-focusing, we obtain

$$\frac{dx_i}{dt} = \frac{k_i \Lambda_k}{\omega\mu\epsilon_e\Lambda_\omega}, \quad \frac{dk_i}{dt} = \frac{\omega\epsilon^{(2)}}{2\epsilon_e\Lambda_\omega}\frac{\partial E}{\partial x_i},$$
$$\Lambda_k = 1 - \omega^2\mu\frac{\partial\epsilon_e}{\partial k^2}, \quad \Lambda_\omega = 1 + \frac{\omega^2}{\epsilon_e}\frac{\partial\epsilon_e}{\partial\omega^2}. \tag{25}$$

For the temporal self-focusing problem, the first equation of (25) is obtained, and

$$\frac{d\omega}{dt} = -\frac{\omega\epsilon^{(2)}}{2\epsilon_e\Lambda_\omega}\frac{\partial E}{\partial t}. \tag{26}$$

Considering temporal dispersion only, $\Lambda_k = 1$, then Eq. (25) and (26) indicate that the ray tracing is speeded or slowed, compared to the nondispersive case, depending on Λ_ω. Thus, for $\Lambda_\omega > 1$ we replace dt by $dt' = dt/\Lambda_\omega$, i.e., the parameter along the ray is smaller and the effects take place sooner. Identifying the phase velocity $V_p = \omega/k = (\mu\epsilon_e)^{-1/2}$ according to (17), the first equation of (25) becomes

$$V = \left|\frac{d\vec{x}}{dt}\right| = \frac{V_p}{\Lambda_\omega}. \tag{27}$$

For $\Lambda_\omega \gtrless 1$, $V \gtrless V_p$ corresponding to anomalous and normal dispersion. Thus our conclusion that anomalous dispersion is beneficial for self-focusing agrees with the results of Hasegawa and Tappert.[11,12] The inclusion of spatial dispersion is not expected to change this conclusion, since temporal dispersion is usually predominant.

C. Effect of inhomogeneity and time variance

The ray theoretic approach is especially suitable for media involving slowly varying parameters. It might be necessary in certain problems to neutralize self-focusing, or it might be desirable to add the self-focusing effect in such a way that filaments (produced by undesirable field gradients within the beam) will tend to merge.

We return to the first problem of spatial self-focusing without dispersion. It is assumed that $\epsilon_e(\vec{x}, t)$ now depends on position and time. For $\epsilon_e(\vec{x})$ depending on position only, the group velocity (18) remains unchanged, but

$$\frac{dk_i}{dt} = \frac{\omega}{2\epsilon_e}\left(\epsilon^{(2)}\frac{\partial E}{\partial x_i} - \frac{\partial\epsilon_e}{\partial x_i}\right) \tag{28}$$

is obtained. Hence the change of direction of the rays can be compensated by a spatial inhomogeneity.

Analogously, for temporal focusing $\epsilon_e(t)$ will be assumed. Again (18) is preserved, and instead of (24) we find

$$\frac{d\omega}{dt} = -\frac{\omega}{2\epsilon_e}\left(\epsilon^2\frac{\partial E}{\partial t} + \frac{\partial\epsilon_e}{\partial t}\right), \tag{29}$$

demonstrating how in a time-varying medium the nonlinear effects may be neutralized.

IV. EFFECT OF HIGHER-ORDER NONLINEAR TERMS

Inasmuch as many materials display nonlinear effects due to the cubic term, let us examine the simple wave equation

$$G = [-k^2 + \omega^2\mu(\epsilon^{(1)} + E^2\epsilon^{(3)})]E = FE = 0, \tag{30}$$

which should be compared to (16), with $\epsilon_e = \epsilon^{(1)} + E^2\epsilon^{(3)}$, for this case. Retracing our steps which led to (21), we find again (18), with the present definition of ϵ_e, and

$$\frac{dk_i}{dt}=\frac{\omega\epsilon^{(3)}E}{\epsilon_e}\frac{\partial E}{\partial x_i}=\frac{\omega\epsilon^{(3)}}{2\epsilon_e}\frac{\partial E^2}{\partial x_i}, \tag{31}$$

as the analog of (21). Similarly, the analog of (24) involves $\partial E^2/\partial t$ for temporal self-focusing. The effects depend now on $\epsilon^{(3)}$ and its sign and the profile of the square of the field.

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