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Time-dependent Hartree approximation for a one-dimensional system of bosons with attractive δ-function interactions*

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The time-dependent Hartree approximation is compared with an exact solution for the scattering between two N-particle bound states in the case of a 1-dimensional system of bosons with attractive δ-function interactions. It is shown that to leading order in N, the approximation is exact, and arguments are presented relating this asymptotic agreement to the nonsaturation of the bound states.

I. INTRODUCTION

The time-dependent Hartree-Fock (TDHF) approximation, introduced by Dirac[1] in 1930, provides a theoretical framework for the microscopic treatment of the dynamics of self-bound composite systems. Although this approximation offers the intuitive appeal of a mean-field theory, it is subject to two serious limitations. The time-dependent variational principle[2] yields a stationarity condition, rather than a minimum condition, and thus provides no bound or rigorous constraint on the true wave function. Secondly, the TDHF approximation specifies the evolution of the one-body density matrix, which is insufficient for determining S-matrix elements between asymptotic many-body eigenstates, so the formulation and interpretation of scattering theory in this approximation is conceptually unclear. Hence, in view of recent applications of TDHF to the nuclear many-body problem,[3] this work attempts to explore quantitatively the validity of analogous approximations in the context of an exceedingly simple, exactly soluble model problem.

Exact analytic solutions[4,5] have been obtained for the bound states and scattering states of one-dimensional systems of bosons or fermions interacting via attractive δ-function potentials by extension of the Bethe ansatz.[6] Since δ-function interactions require totally symmetric spatial wave functions for bound states, fermion systems possess many-body bound states only upon the addition of an extra quantum number such as spin. To avoid the added complication of spin, this initial investigation is therefore restricted to the case of bosons. The most general analog of the TDHF approximation for bosons would be to approximate the N-body wave function by a symmetrized product of N different arbitrary single-particle wave functions. In fact, guided by previous results for the bound-state problem, we shall require that all the single-particle wave functions be equal, in which case the totally symmetric trial function is a single product and is thus equivalent to the Hartree approximation.

II. STATIC HARTREE APPROXIMATION

The static Hartree approximation for this system has been studied by Calogero and Degasperis,[7] who show that the energy and one-body density are exact to the leading order in N. As a prelude to the time-dependent case, we shall review their results and show that the large N asymptotic behavior follows from the nonsaturation of the N-body bound states.

The Hamiltonian for the N-boson system is

$$H=-\frac{1}{2}\sum_{i=1}^{N}\frac{d^2}{dx_i^2}-g\sum_{i<j=1}^{N}\delta(x_i-x_j),\quad g>0. \tag{2.1}$$

This system has a single N-particle bound state,

$$\psi_N=N![(N-1)!g^{N-1}]^{1/2}\exp\left(-\frac{g}{2}\sum_{i<j=1}^{N}|x_i-x_j|\right) \tag{2.2}$$

with energy

$$E_N=-N(N^2-1)g^2/24 \tag{2.3}$$

and density

$$\rho_N(x)=\int dx_1\,dx_N\,\delta\left(\frac{1}{N}\sum_{i=1}^{N}x_i\right)|\psi_N|^2\,\delta(x_1-x)$$

$$=g(N!)^2\sum_{n=1}^{N-1}(-1)^{n+1}\frac{n\exp(-gnN|x|)}{(N+n-1)!(N-n-1)!}. \tag{2.4}$$

In the large N limit

$$\rho_N(x)\xrightarrow[N\to\infty]{}\frac{N^2g}{4\cosh^2(Ngx/2)}[1+O(1/N)+\cdots]. \tag{2.5}$$

The Hartree equation for the ground state is obtained by choosing the trial wave function,

$$\psi_N^H = \sqrt{N} \prod_{i=1}^{N} \phi_b(x_i), \tag{2.6}$$

and minimizing $\langle H \rangle$ with respect to the normalized single-particle wave function $\phi_b(x)$. The resulting single-particle equation

$$\left(-\frac{1}{2}\frac{d^2}{dx^2} - g(N-1)|\phi_b(x)|^2 - \epsilon\right)\phi_b(x) = 0 \tag{2.7}$$

has one bound-state solution

$$\phi_b(x) = [(N-1)g]^{1/2}/2\cosh[\tfrac{1}{2}(N-1)gx] \tag{2.8}$$

with energy eigenvalue

$$\epsilon_b = -(N-1)^2 g^2/8 . \tag{2.9}$$

The resulting Hartree energy

$$E_N^H = -N(N-1)^2 g^2/24 \tag{2.10}$$

and Hartree density

$$\rho_N^H(x) = N(N-1)g/4\cosh^2[\tfrac{1}{2}(N-1)gx] \tag{2.11}$$

agree with the exact results, Eqs. (2.3) and (2.5), to leading order in N.

To understand the validity of the Hartree, or mean-field approximation, for large N, it is useful to consider the leading correction to the energy in Rayleigh-Schrödinger perturbation theory. Defining the unperturbed Hamiltonian to include the mean field

$$H_0 = \sum_i \left(-\frac{1}{2}\frac{d^2}{dx_i^2} + U(x_i)\right), \tag{2.12}$$

where

$$U(x) = -(N-1)^2 g^2/4\cosh^2[\tfrac{1}{2}(N-1)gx], \tag{2.13}$$

the second-order correction to the energy is

$$\Delta E^{(2)} = \sum_i{}' \frac{\langle \psi_N^H | V | \psi_i \rangle \langle \psi_i | V | \psi_N^H \rangle}{(E_0^{(0)} - E_i^{(0)})\langle \psi_N^H | \psi_N^H \rangle \langle \psi_i | \psi_i \rangle}, \tag{2.14}$$

where

$$V = -g \sum_{i<j=1}^{N} \delta(x_i - x_j) - \sum_{i=1}^{N} U(x_i) .$$

The states ψ_i are a complete set of excited eigenstates of H_0 constructed from symmetrized products of single-particle eigenfunctions satisfying

$$\left(-\frac{1}{2}\frac{d^2}{dx^2} + U(x)\right)\phi_m(x) = \epsilon_m \phi_m(x) \tag{2.15}$$

and the energies $E_i^{(0)}$ are the sums of the eigenvalues ϵ_m of the occupied single-particle states. In addition to the bound-state eigenfunction specified by Eqs. (2.8) and (2.9), the continuum solutions to Eq. (2.15) are[8]

$$\phi_k(x) = \left(\frac{(N-1)g}{4\pi}\right)^{1/2} \exp[ik(N-1)gx/2] \times \left(\frac{\tanh[\tfrac{1}{2}(N-1)gx] - ik}{1+ik}\right) \tag{2.16}$$

with eigenvalues

$$\epsilon_k = \tfrac{1}{8}(N-1)^2 g^2 k^2 . \tag{2.17}$$

With these definitions, the form for the second-order energy correction simplifies to

$$\Delta E^{(2)} = \tfrac{1}{2} N(N-1)g^2 \int dk\, dk' \left| \int dx \phi_b^2(x) \phi_k(x) \phi_{k'}(x) \right|^2 \Big/ (2\epsilon_b - \epsilon_k - \epsilon_{k'}) . \tag{2.18}$$

As shown in Appendix A, explicit evaluation of Eq. (2.18) yields

$$\Delta E^{(2)} \cong -0.9956\, N(N-1)g^2/24 \tag{2.19}$$

accounting for almost half of the order N^2 discrepancy between Eqs. (2.3) and (2.10).

From the evaluation of $\Delta E^{(2)}$ in Eq. (2.18), it is evident that the presence of a gap in the spectrum of order N^2 between ϵ_b and the first continuum state plays a crucial role in limiting the contribution of $\Delta E^{(2)}$ to order N^2. This suggests that it should be possible to catalog systematically the N dependence of all higher-order perturbation contributions. By means of a diagrammatic technique[9] in which Goldstone diagrams are used for boson systems, the N dependence of diagrams for the expectation values of the energy and one-body density is derived in Appendix B. The main result is that, using the Hartree definition of the single-particle potential, Eq. (2.13), the contribution to the energy of any linked Goldstone diagram containing C closed boson loops and I interactions is of order N^{C+2-I}. Thus the only contribution of order N^3 is the direct Hartree term, diagram a of Fig. 1, which is included in Eq. (2.10). The contributions of order N^2 are the lowest-order exchange term (diagram b) which is included in Eq. (2.10) and the direct RPA ring diagrams, shown in diagrams c, e, and f.

Diagrams c and d of Fig. 1 correspond to the $N(N-1)$ contribution in Eq. (2.18). Combining $\Delta E^{(2)}$ from Eq. (2.19) with the spurious center-of-mass (c.m.) energy correction[7] $\Delta E_{\text{c.m.}} = -N(N-1)g^2/24$ yields a residual discrepancy of

order $2\times10^{-4}N^2g^2$, which presumably arises from the omitted ring diagrams.

Note that the choice of the Hartree potential to define the basis is essential to the present argument. Otherwise, as indicated schematically in diagram g of Fig. 1, a Hartree insertion in an arbitrary diagram introduces one closed loop and one interaction, thereby leaving the order of N unchanged. Only by systematically cancelling all Hartree insertions of the form (g) by the corresponding single-particle potential contribution (h) is the leading N^3 contribution restricted to a single diagram.

From this analysis, it is evident that the dominance of the Hartree contribution depends crucially on the N^2 gap in the single-particle spectrum which suppresses all higher-order graphs relative to the mean-field terms. This gap, in turn, arises from the fact that the Hartree potential becomes deeper and narrower with increasing N, a situation which cannot occur for saturating systems.[10] Thus the asymptotic accuracy of the static Hartree approximation is associated with the collapse of bound states for large systems in this model, and unfortunately does not generalize to saturating systems.[11]

III. SCATTERING OF BOUND STATES

Having established the asymptotic accuracy of the static Hartree approximation, we now consider the scattering between two N-body bound states.

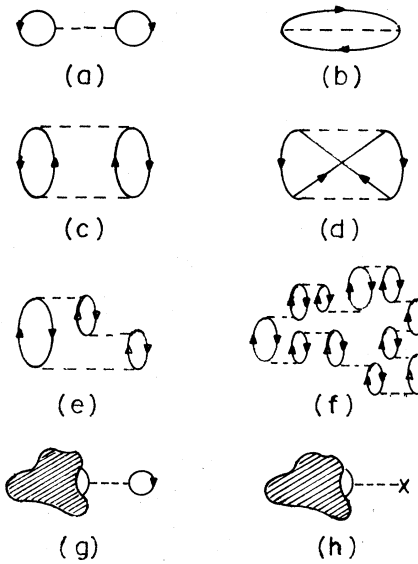

FIG. 1. Goldstone diagrams for the ground-state energy of a bose system. Boson single-particle propagators, two-body interactions, and one-body potential insertions are denoted by solid lines, dashed lines, and crosses, respectively.

The scattering matrix for distinguishable particles interacting via δ-function potentials has been derived by Yang.[5] Applying his formulation to the scattering of N bosons by N bosons, the symmetric wave function must be of the form

$$\psi(x_1,\ldots,x_{2N})=\sum_{Q=1}^{(2N)!}\sum_{P=1}^{(2N)!}\theta_Q(x_1,\ldots,x_{2N})a_P \times\exp\left(i\sum_{j=1}^{2N}k_{P(j)}x_{Q(j)}\right), \tag{3.1}$$

where Q and P are elements of the permutation group S_{2N}, and

$$\theta_Q(x_1\cdots x_{2N})=\begin{cases}1, & \text{if } x_{Q(1)}<x_{Q(2)}<\cdots<x_{Q(2N)}\\ 0, & \text{otherwise.}\end{cases}$$

Continuity of the wave function and the discontinuity of its derivative impose the following condition on the coefficients:

$$a_P=\frac{i[k_{P(i)}-k_{P(i+1)}]+g}{i[k_{P(i)}-k_{P(i+1)}]-g}a_{P'}, \tag{3.2}$$

where P and P' are related by

$$P(i)=P'(i+1),\quad P(i+1)=P'(i). \tag{3.3}$$

The $(2N)!\,(2N-1)$ equations of the form (3.2) are internally consistent and sufficient to determine all coefficients if one of them is given.

For the case of two N-particle bound states, we choose the incoming state to be in the center of mass frame

$$\psi_i=\sum_{Q=1}^{(2N)!}\theta_Q(x_1\cdots x_{2N}) \times\exp\left[iK\left(\sum_{i=1}^{N}x_{Q(i)}-\sum_{j=N+1}^{2N}x_{Q(j)}\right)\right] \times\exp\left(-\frac{g}{2}\sum_{i<j=1}^{N}|x_{Q(i)}-x_{Q(j)}|\right) \times\exp\left(-\frac{g}{2}\sum_{k<l=N+1}^{2N}|x_{Q(k)}-x_{Q(l)}|\right). \tag{3.4}$$

The final state is the outgoing wave

$$\psi_f=\sum_{Q=1}^{(2N)!}\theta_Q(x_1\cdots x_{2N})T_{2N} \times\exp\left[-iK\left(\sum_{i=1}^{N}x_{Q(j)}-\sum_{j=N+1}^{2N}x_{Q(j)}\right)\right] \times\exp\left(-\frac{g}{2}\sum_{i<j=1}^{N}|x_{Q(i)}-x_{Q(j)}|\right) \times\exp\left(-\frac{g}{2}\sum_{k<l=N+1}^{2N}|x_{Q(k)}-x_{Q(l)}|\right) \tag{3.5}$$

with transmission coefficient given by (3.2)

$$T_{2N}=\sum_{m=1}^{N-1}\left(\frac{2K-mig}{2K+mig}\right)^2\left(\frac{2K-Nig}{2K+Nig}\right). \tag{3.6}$$

Thus only elastic scattering occurs in this model and is completely specified by the phase shift $\delta(K) = (1/2i)\ln(T_{2N})$. For subsequent comparison, it will be useful to calculate the time delay,

$$\Delta t=\frac{1}{NK}\frac{\partial}{\partial K}\delta(K) \tag{3.7}$$

which, from (3.6), has the following form

$$\Delta t=-\frac{4g}{NK}\sum_{m=1}^{N-1}\frac{m}{4K^2+m^2g^2}-\frac{2g}{K(4K^2+N^2g^2)}. \tag{3.8}$$

Formulation of the analogous TDH scattering problem requires specification of an appropriate initial condition. Given the fact that an isolated system is well approximated by a product wave function in which all particles occupy the same single-particle state, two trial functions appear natural:

$$\psi_{2N}(x_1,\ldots,x_{2N};t)=\prod_{i=1}^{2N}\left(\frac{1}{\sqrt{2}}\phi^+(x_i,t)+\frac{1}{\sqrt{2}}\phi^-(x_i,t)\right) \tag{3.9}$$

and

$$\psi_{N+N}(x_1,\ldots,x_{2N};t)=\hat{S}\left(\prod_{i=1}^{N}\phi^+(x_i,t)\prod_{j=N+1}^{2N}\phi^-(x_j,t)\right), \tag{3.10}$$

where

$$\phi^+(x,t)=e^{-iKx}\phi_0(x-R), \tag{3.11}$$

$$\phi^-(x,t)=e^{iKx}\phi_0(x+R), \tag{3.12}$$

$\hat{S}$ is the symmetrization operator, K is the initial center of mass (c.m.) velocity of each system, and R is the initial displacement of each system. Both ansätze have the property that prior to the collision, the equation of motion uniformly translates the static Hartree density distributions with velocity K. The one-body density matrix,

$$\rho(x,x',t)\equiv\int dx_2\cdots dx_{2N}\psi^*(x,x_2\cdots x_{2N},t)\times\psi(x',x_2\cdots x_{2N},t), \tag{3.13}$$

however, differs for the two initial states:

$$\rho_{2N}(x,x',t)=N[\phi^+(x,t)^*\phi^+(x',t)+\phi^-(x,t)^*\phi^-(x',t)+\phi^+(x,t)^*\phi^-(x',t)+\phi^-(x,t)^*\phi^+(x',t)], \tag{3.14}$$

$$\rho_{N+N}(x,x',t)=N[\phi^+(x,t)^*\phi^+(x',t)+\phi^-(x,t)^*\phi^-(x',t)].$$

Thus as in the case of the initial state for the exact solution, $\rho_{N+N}(x,x',t)$ has no long-range off-diagonal components, whereas $\rho_{2N}(x,x',t)$ has unphysical off-diagonal correlations arising from the cross terms in Eq. (3.14). *A priori*, one would therefore prefer ψ_{N+N} as the more physical initial condition.

Unfortunately, application of the time-dependent variational principle to a wave function of the form (3.10) is intractable. In contrast to the case of determinants, the space of symmetrized products of arbitrary functions is much larger than the space of symmetrized products of orthogonal functions. In general, the equations of motion do not preserve the orthogonality of ϕ^+ and ϕ^-, and the action involves overlaps of ϕ^+ and ϕ^- raised to all powers up to N, rendering analytic solution impossible and numerical solution extremely complicated. For asymmetric collisions, it is possible to introduce constraints which enforce orthogonality, although such constraints are quite unphysical because they inhibit boson condensation. For symmetric collisions even introduction of an artificial constraint is useless, since the instantaneous change in one single-particle wave function turns out to have no component orthogonal to the other orbital. Thus we are forced to relinquish the preferred form ψ_{N+N} in favor of the much more tractable ansatz ψ_{2N}. At the present time we do not understand the implications of the unphysical off-diagonal correlations in the one-body density matrix ρ_{2N}.

The time-dependent Hartree approximation for the wave function

$$\psi_{2N}(x_1\cdots x_{2N};t)=\prod_{i=1}^{2N}\phi(x_i,t) \tag{3.15}$$

is obtained by requiring that the action

$$\langle\psi_{2N}|i\frac{\partial}{\partial t}-H|\psi_{2N}\rangle$$

be stationary with respect to variations of $\phi(x,t)$. The resulting TDH equation for the single-particle state ϕ is

$$\left(i\frac{\partial}{\partial t}+\frac{1}{2}\frac{d^2}{dx^2}+(2N-1)g\phi^*(x,t)\phi(x,t)\right)\phi(x,t)=0 \tag{3.16}$$

and has the property that the norm of ϕ is a constant of the motion. This cubic Schrödinger equation [Eq. (3.16)] admits a family of solitary wave solutions which have been studied extensively.[12] The solution which satisfies the initial condition (3.9) is the two-soliton solution[13]

$$\phi(x,t)=\tfrac{1}{2}[(2N-1)v]^{1/2}e^{-(i/2)(K^2-a^2)t}$$
$$\times\frac{(e^{iKx}\{e^{-a(x-Kt)}+[K^2/(K-ia)^2]e^{-a(3x+Kt)}\}+e^{-ix}\{e^{-a(x+Kt)}+[K^2/(K+ia)^2]e^{-a(3x-Kt)}\})}{1+2e^{-2ax}\cosh(2aKt)-8a^2e^{-2ax}\,\mathrm{Re}[e^{2iKx}/4(K+ia)^2]+[K^4/(K^2+a^2)^2]e^{-4ax}} \tag{3.17}$$

This solution is in the center of mass frame, with relative velocity $2K$, and parameter $a=\frac{1}{4}(2N-1)g$. It describes the transmission of two solitary waves through one another with time delay Δt^H defined by

$$\phi(x,t)\xrightarrow[t\to+\infty]{}\frac{1}{\sqrt{2}}[\phi^+(x,t-\Delta t^H)+\phi^-(x,t-\Delta t^H)]\,. \tag{3.18}$$

The time delay is given by[13]

$$\Delta t^H=\frac{-4}{(2N-1)gK}\ln\left(1+\frac{(2N-1)^2g^2}{16K^2}\right)\,. \tag{3.19}$$

For the collision of two large particle wave packets moving with velocity K,

$$\Delta t^H=-\frac{4}{NgK}[\ln N+O(1)]\,,\quad N\to\infty,\ K\text{ fixed}\,. \tag{3.20}$$

This result agrees with the exact time delay, to order $1/N$. Thus we see that in the large N limit, just as in the static case, the time-dependent Hartree approximation becomes asymptotically exact.

IV. DISCUSSION

We have shown that the time-dependent Hartree approximation using a wave function of the form (3.15) yields asymptotic agreement with the exact time delay for large N. Note that this result is true for all energies, and thus constitutes a much stronger statement than agreement in the weak-coupling (or high-energy) limit.[13]

Since our intent in this work has been to understand the validity of the time-dependent mean-field approximation, it is essential to evaluate the extent to which the agreement arises from artificial and unrealistic aspects of the one-dimensional δ-function model. The fact that the exact scattering solution exhibits only elastic scattering with no excitation or breakup is a special feature of the δ-function potential which seriously restricts the generality of the model. Nevertheless it is significant that the TDH approximation yielded no spurious excitation or breakup, so in this sense the problem constitutes a nontrivial test of the theory.

A far more serious limitation is the degree to which the result depended upon the nonsaturation of the bound states. In the static case, we explicitly demonstrated the fact that the ratio of two-particle–two-hole to one-particle–one-hole admixture was proportional to $1/N$, and this resulted because of the N^2 dependence of the energy denominators. In contrast, for saturating systems in which the volume is proportional to the number of particles, the magnitude of the interior single-particle potential is independent of N and the separation between single-particle bound states decreases with N. Thus the suppression of two-particle–two-hole components relative to one-particle–one-hole amplitudes with increasing N need not, in general, occur for a saturating system, but rather will depend in detail upon the nature of the two-body interaction.

For the time-dependent δ-function case, we believe, but have not explicitly proven, that the asymptotic validity of the TDH approximation arises from analogous suppression of many-particle–many-hole amplitudes relative to the one-particle–one-hole amplitudes included in the mean-field approximation. Hence there is no reason to expect that the validity of the TDH approximation in the model must necessarily generalize to saturating systems.

In spite of these limitations, it is still significant that there exists a model system for which the TDH approximation is valid. In particular, the conceptual questions[14] one may raise about the validity of the time-dependent variational principle and the interpretation of scattering in this approximation are completely well posed and resolvable in this case for which exact solutions exist, and provide insight into other applications of the TDH approximation.

The conclusion which emerges is that, just as in the time-independent case, the mean-field approximation yields an adequate approximation for the expectation value of few-body operators although it is incapable of describing the full many-body wave function. Thus the time-dependent density is as accurate as the static density, and since the time delay may be extracted from the one-body density, it is also well approximated. S-matrix elements require knowledge of the full many-body wave function, which is beyond the scope of the present theory, so scattering must be discussed only in terms of the much more restricted information available in expectation values of few-body operators. Many experiments, however, deal directly with such quantities as average excitation energy, mean numbers of particles in fragments, or the dispersion of number of particles,

so many questions of relevance may be addressed directly.[15]

Finally, it is worthwhile to comment on the semiclassical nature of the TDH approximation. The Hartree $2N$-body density describes two N-particle wave packets of fixed initial spatial extent $L \sim 1/Ng$ and c.m. uncertainty $\Delta R_0 \equiv \langle (R - \langle R \rangle)^2 \rangle^{1/2} \sim 1/Ng$ which scatter without change in shape or velocity. If the TDH approximation is to provide an accurate picture of the physical scattering process then the exact density of a relative wave packet describing the same initial condition must agree with the Hartree density during the entire measurement time t. In particular, the spreading of the wave packet must be negligible. The uncertainty in a free N-particle wave packet has the following time dependence:

$$\Delta R = \Delta R_0 [1 + t^2/(\Delta R_0)^4 N^2]^{1/2} .$$

In the case of a saturating system where the initial wave packet is specified by $L \sim \Delta R_0 \sim N$, the spread during the time $t = mL/K$ to travel several multiples m of the system's size with velocity K is negligible in the limit of large N. Thus the semiclassical approximation will never invalidate the TDH description of a large saturating system.

For the δ-function potential, however,

$$\Delta R = \Delta R_0 \left[1 + \frac{m^2 g^2}{K^2} \right]^{1/2} ,$$

where the measurement time is taken to be $t = mL/K$ since there is no positive time delay. With this criteron, which may be overly strict, the TDH approximation would only appear to be valid in the high-energy limit, i.e., for $g^2/K^2 \ll 1$.

Thus there remain two gaps in our understanding of the exact large N behavior of the TDH approximation. In addition to the fact that the correct time delay is produced by an initial condition involving incorrect off-diagonal density matrix elements, one also obtains correct results at energies for which the validity of the semiclassical approximation is not obvious.

APPENDIX A

We calculate an approximate value for the second-order energy correction

$$\Delta E^{(2)} = \tfrac{1}{2} N(N-1) g^2 \int dk\, dk' \left| \int dx\, \phi_b^2(x) \phi_k(x) \phi_{k'}(x) \right| \times (2\epsilon_b - \epsilon_k - \epsilon_{k'})^{-1} . \quad \text{(A1)}$$

Upon substituting the eigenfunctions and eigenvalues given by Eqs. (2.8), (2.9), (2.16), and (2.17), Eq. (A1) becomes

$$\Delta E^{(2)} = \frac{-N(N-1)g^2}{16} \int dk\, dk' \frac{|I(k,k')|^2}{2 + k^2 + k'^2} , \quad \text{(A2)}$$

where

$$I(k,k') \equiv \frac{1}{2\pi} \int_{-\infty}^{\infty} dx \frac{e^{i(k+k')x}}{\cosh^2(x/2)} \times \frac{(\tanh \tfrac{1}{2}x - ik)(\tanh \tfrac{1}{2}x - ik')}{(1+ik)(1+ik')} = \frac{(k+k')(k^2 + k'^2 - kk' + 1)}{3(1+ik)(1+ik') \sinh \tfrac{1}{2}(k+k')\pi} . \quad \text{(A3)}$$

With a change in variables, $k = k$, $l = k + k'$,

$$\Delta E^{(2)} = \frac{-N(N-1)g^2}{16} \int_{-\infty}^{\infty} dl \frac{I(l)}{\sinh^2(l\pi/2)} , \quad \text{(A4)}$$

where

$$I(l) \equiv \frac{1}{9} \int_{-\infty}^{\infty} dk \frac{(3k^2 + l^2 - 3kl + 1)^2}{(1+k^2)[1 + (l-k)^2][2 + k^2 + (l-k)^2]} = \frac{\pi}{9} \left(2 - \frac{[2 - (l^2+4)]^{1/2}}{l^2} \right) . \quad \text{(A5)}$$

Substituting (A5) into (A4),

$$\Delta E^{(2)} = -\tfrac{1}{27} N(N-1) g^2 (1 + \tfrac{3}{2} I) , \quad \text{(A6)}$$

where

$$I = \int_0^{\infty} dx \frac{(x^2+1)^{1/2} - 1}{\sinh^2 \pi x} < \frac{1}{2} \int_0^{\infty} dx \frac{x^2}{\sinh^2 \pi x} = \frac{1}{12\pi} . \quad \text{(A7)}$$

Thus

$$\Delta E^{(2)} > -N(N-1)g^2/24 = (E_N - E_N^H)/2 . \quad \text{(A8)}$$

Numeral evaluation of the integral I in Eq. (A7) yields the approximate value

$$\Delta E^{(2)} = -0.9956\, N(N-1) g^2/24 . \quad \text{(A9)}$$

APPENDIX B

A systematic diagrammatic expansion which manifestly displays the leading N dependence of each term is conveniently derived using the Goldstone expansion. The spatial wave function of an interacting fermion system with spin degeneracy $2S + 1 \geq N$ will be totally symmetric if a spin-independent interaction is adiabatically switched on starting from a noninteracting eigenstate in which each single-particle wave function has the same spatial dependence and a different spin projection. Thus the Goldstone expansion may be applied to an N-particle Bose system by simply introducing a fictitious spin degeneracy $2S + 1 = N$ and disregarding the totally antisymmetric spin wave function.[9]

Consider first the Goldstone expansion for the ground-state energy. Any linked graph in this expansion has I interactions and C closed loops.

Every interaction contributes a factor of $(N-1)^2$ which arises from the normalization of the particle and hole wave functions and a factor of $(N-1)^{-1}$ which arises from the integration over spatial variables. Note that this latter factor may be extracted only because all the wave functions are functions of $(N-1)x$ and the interaction is zero range, allowing one to change variables and remove all N dependence from the integrand. Every closed loop contributes a factor of N arising from the sum over spin projections. For every graph, there are $I-1$ energy denominators, each of which contributes a factor of $(N-1)^{-2}$. Hence the overall N dependence of any Goldstone diagram in the energy expansion is $(N-1)^{2-I}N^C \sim N^{C-I+2}$.

The expansion for the one-body density is derived from the energy expansion by inserting the one-body density operator into a particle or hole line. This modification contributes a factor of $N-1$ arising from the normalization of the particle and hole wave functions and a factor of $(N-1)^{-2}$ arising from the addition of an energy denominator. The overall N dependence is thus $(N-1)^{1-I}N^C \sim N^{C-I+1}$.

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