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Comments on the polarization of the quenched radiation from the 2s state of hydrogen*

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This paper examines two theoretical aspects of the measurement of polarization of the quenched radiation which comes from the application of a weak electric field to the $2s$ state of hydrogen. Specifically, it outlines the justification of the Bethe-Lamb phenomenological analysis of this problem from the external field approximation of quantum electrodynamics. Further, it includes the lowest-order effects of the J-breaking part of the hyperfine structure in Rayleigh-Schrödinger perturbation theory. While it was expected that these terms would contribute to the polarization fraction a correction to the lowest order of the relative size of the ratio of the hyperfine splittings to the fine-structure splitting of the $n = 2$ state or about one part in 10^3, cancellations occur and these J-breaking terms give a correction in the fourth place of the polarization fraction.

I. INTRODUCTION

The measurement of the polarization of the quenched radiation that comes from the application of a weak electric field to the $2s$ state of hydrogen has been suggested as an indirect method of measuring the Lamb shift in hydrogen and hydrogenlike ions.[1] It appears that four-place accuracy in the polarization fraction and thus in the derived Lamb-shift values will be possible from this technique. The theoretical framework which has been prescribed to determine the Lamb shift from the polarization fraction is the Bethe-Lamb phenomenological quenching theory.[2]

This paper examines two problems. First, we look at whether the basic concept of the Bethe-Lamb phenomenological analysis is right, and we justify it from the external field approximation of quantum electrodynamics. Second, we calculate the polarization fraction, including in our treatment the lowest-order J-breaking parts of the hyperfine interaction, which were ignored by Casalese and Gerjuoy.[3] We expected that these parts would give a contribution of the relative size to the lowest-order polarization fraction of about the ratio of the hyperfine splittings to the fine-structure splittings of the $n=2$ state or one part in 10^3. Instead, we find that cancellations occur for which we do not have a simple reason, and the J-breaking terms give a correction in the fourth place of the polarization fraction.[4]

II. THEORY

We give here an outline for the justification from first principles of the Bethe-Lamb phenomenological quenching theory, including $2p_{3/2}$ mixing but neglecting the hyperfine effects. The use of energy denominators which include the Lamb shift while employing nonrelativistic wave functions in second-order perturbation theory does not follow directly from the ordinary Rayleigh-Schrödinger analysis. The requisite addition is the demonstration that the radiative corrections shift the poles in the complete electron propagator. This point was discussed originally by Low[5] and recently in a more complete way by Fox and Yennie.[6] We use the notation of Fox and Yennie.

The significant part of the perturbation series for the decay of a $2s$ state in the presence of a weak electric field is given by

$$\mathfrak{M} = e(\hbar/2kc)^{1/2}\langle u_{1s_{1/2}}|\not{\epsilon}e^{-i\vec{k}\cdot\vec{x}} \times [\not{\pi}' - m - \Sigma_c(\pi', m, e)]^{-1} e\not{A}\,|u'_{2s_{1/2}}\rangle\,, \tag{1}$$

where $\Sigma_c(\pi', m, e)$ is the mass- and charge-renormalized self-energy operator with e and m the physical charge and mass. The symbol $\not{\epsilon}$ means $\epsilon_\alpha\gamma^\alpha = \epsilon_0\gamma^0 - \vec{\epsilon}\cdot\vec{\gamma}$, where $\epsilon^\mu = (0, \vec{\epsilon})$; $\not{A} = z\mathcal{E}_0\gamma^0$; $\not{\pi} = \pi_\alpha\gamma^\alpha$, where $\pi_0 = E'_{2s} - e^2/r$ and $\pi_i = p_i$; and $k = (E'_{2s} - E_{1s})/\hbar c$. The wave function $u_{1s_{1/2}}$ is a solution of the Dirac equation for the hydrogen atom, and E_{1s} is its eigenvalue. The wave function $u'_{2s_{1/2}}$ is the wave function for the $2s_{1/2}$ state, including all the radiative corrections. This wave function may be written in terms of a perturbation expansion in Σ_c acting on $u_{2s_{1/2}}$, the Dirac wave function. For the purposes of this discussion, we replace $u'_{2s_{1/2}}$ with $u_{2s_{1/2}}$, since the wave-function correction gives a negligible correction to the transition amplitude. Note that E'_{2s} consists of the value of the Dirac $2s_{1/2}$-state eigenvalue plus the complex level shift due to all the radiative corrections. The real part is the energy-level shift, and the imaginary part is minus one-half of the linewidth of the $2s_{1/2}$ state. The state $\langle u_{1s_{1/2}}| = |u_{1s_{1/2}}\rangle^\dagger\gamma^0$ is the Dirac adjoint vector rather than the Hermitian conjugate vector.

We insert two complete sets of hydrogen eigenstates of the Dirac equation and write the transition amplitude $\mathfrak{M}$ as a double sum:

$$\mathfrak{M} \approx e^2\left(\frac{\hbar}{2kc}\right)^{1/2} \sum_{m,n} \langle u_{1s_{1/2}} | \not{\epsilon} e^{-i\vec{k}\cdot\vec{x}} | m\rangle \langle m | \gamma_0[\not{\pi}' - m - \Sigma_c(\pi', m, e)]^{-1}\gamma^0 | n\rangle \langle n | z \mathcal{E}_0 \gamma^0 | u_{2s_{1/2}}\rangle . \tag{2}$$

By neglecting retardation and intermediate states other than $m,n = 2p_{1/2}$ and $m,n = 2p_{3/2}$, we introduce an order α^2 error to the leading term:

$$\begin{aligned}\mathfrak{M} \approx -e^2(\hbar/2kc)^{1/2}\mathcal{E}_0\{&\langle u_{1s_{1/2}} | \vec{\gamma}\cdot\vec{\epsilon} | u_{2p_{1/2}}\rangle \langle u_{2p_{1/2}} | \gamma_0[\not{\pi}' - m - \Sigma_c(\pi', m, e)]^{-1}\gamma^0 | u_{2p_{1/2}}\rangle \langle u_{2p_{1/2}} | \gamma^0 z | u_{2s_{1/2}}\rangle \\ &+ \langle u_{1s_{1/2}} | \vec{\gamma}\cdot\vec{\epsilon} | u_{2p_{3/2}}\rangle \langle u_{2p_{3/2}} | \gamma_0[\not{\pi}' - m - \Sigma_c(\pi', m, e)]^{-1}\gamma^0 | u_{2p_{3/2}}\rangle \langle u_{2p_{3/2}} | \gamma^0 z | u_{2s_{1/2}}\rangle\} .\end{aligned} \tag{3}$$

Using the result of Fox and Yennie,[6] we can establish that $\langle u_{2p_{1/2}} | \gamma_0[\not{\pi}' - m - \Sigma_c(\pi', m, e)]^{-1} \times \gamma^0 | u_{2p_{1/2}}\rangle$ equals $\Delta E(2s_{1/2}, 2p_{1/2})^{-1}$ up to a relative order α^3 correction, where $\Delta E(2s_{1/2}, 2p_{1/2})$ is the complex Lamb shift. The real part is the actual energy difference of the $2s_{1/2}$ and $2p_{1/2}$ states, and the imaginary part is minus one-half of the difference in linewidths. Similarly, $\langle u_{2p_{3/2}} | \gamma_0[\not{\pi}' - m - \Sigma_c(\pi', m, e)]^{-1}\gamma^0 | u_{2p_{3/2}}\rangle$ is, to the same approximation, $\Delta E(2s_{1/2}, 2p_{3/2})^{-1}$. Using the techniques given in Bethe and Salpeter,[7] we reduce the Dirac hydrogen wave functions to their Schrödinger-Pauli counterparts (ψ) leaving corrections of relative order α^2. Finally, we employ the commutation relation $\vec{p} = (im/\hbar)\,[H_0, \vec{r}]$, where H_0 is the nonrelativistic Hamiltonian for hydrogen to convert $\mathfrak{M}$ to its conventional "length" form:

$$\mathfrak{M} \approx [ie^2(\hbar k/2c)^{1/2}]\mathcal{E}_0 \times \left(\frac{\langle \psi_{1s_{1/2}} | \vec{r}\cdot\vec{\epsilon} | \psi_{2p_{1/2}}\rangle \langle \psi_{2p_{1/2}} | z | \psi_{2s_{1/2}}\rangle}{\Delta E(2s_{1/2}, 2p_{1/2})} + \frac{\langle \psi_{1s_{1/2}} | \vec{r}\cdot\vec{\epsilon} | \psi_{2p_{3/2}}\rangle \langle \psi_{2p_{3/2}} | z | \psi_{2s_{1/2}}\rangle}{\Delta E(2s_{1/2}, 2p_{3/2})} \right) . \tag{4}$$

III. CALCULATION

In this section we calculate the polarization fraction of the light emitted when an arbitrarily weak electric field is applied to the $2s$ state of hydrogen. In this calculation we take care of the changes in the energy levels due to hyperfine splitting as does Casalese and Gerjuoy, but we also take into account the lowest-order effect of the quantum-number J breaking of the hyperfine Hamiltonian. We find that this term is non-negligible but small, and needs to be included to obtain a value of the fractional polarization correct to four places to match the experiments which have been proposed.[8]

The polarization fraction P is given by

$$P = (I_{\parallel} - I_{\perp})/(I_{\parallel} + I_{\perp}) \tag{5}$$

where $I_{\parallel}$ and $I_{\perp}$ are the intensities of radiation observed at 90° to the electric field with polarization parallel and perpendicular to the field, respectively.

Both $I_{\parallel}$ and $I_{\perp}$, to the accuracy desired, come from averaging over the initial states and summing over the final states of a quantity proportional to $|A(F, F'', M_F, M_{F''})|^2$ for the appropriate direction of polarization. The amplitude $A(F, F'', M_F, M_{F''})$ is the sum of four parts which we find using Rayleigh-Schrödinger perturbation theory:

$$A_{\mathrm{I}} = \sum_{F'} \langle 1s_{1/2} F'' M_{F''} | \vec{r}\cdot\vec{\epsilon} | 2p_{1/2} F' M_F\rangle \langle 2p_{1/2} F' M_F | z | 2s_{1/2} F M_F\rangle G_1(F, F') , \tag{6}$$

$$A_{\mathrm{II}} = \sum_{F'} \langle 1s_{1/2} F'' M_{F''} | \vec{r}\cdot\vec{\epsilon} | 2p_{3/2} F' M_F\rangle \langle 2p_{3/2} F' M_F | z | 2s_{1/2} F M_F\rangle G_2(F, F') , \tag{7}$$

$$\begin{aligned}A_{\mathrm{III}} = &\langle 1s_{1/2} F'' M_{F''} | \vec{r}\cdot\vec{\epsilon} | 2p_{3/2} F' = 1, M_F\rangle \langle 2p_{3/2} F' = 1, M_F | H_{\mathrm{hyp}} | 2p_{1/2} F' M_F\rangle \\ &\times \langle 2p_{1/2} F' = 1, M_F | z | 2s_{1/2} F M_F\rangle G_1(F, 1) G_2(F, 1) ,\end{aligned} \tag{8}$$

$$\begin{aligned}A_{\mathrm{IV}} = &\langle 1s_{1/2} F'' M_{F''} | \vec{r}\cdot\vec{\epsilon} | 2p_{1/2} F' = 1, M_F\rangle \langle 2p_{1/2} F' = 1, M_F | H_{\mathrm{hyp}} | 2p_{3/2} F' = 1, M_F\rangle \\ &\times \langle 2p_{3/2} F' = 1, M_F | z | 2s_{1/2} F M_F\rangle G_1(F, 1) G_2(F, 1) ,\end{aligned} \tag{9}$$

where

$$H_{\mathrm{hyp}} = g_I \alpha^2 a_0^3 \left(\frac{m}{m_p}\right) \frac{1}{r^3} Ry[\vec{L}\cdot\vec{I} - \vec{S}\cdot\vec{I} + 3(\vec{S}\cdot\hat{r})(\vec{I}\cdot\hat{r})] , \tag{10}$$

$$G_1(F, F') = (E_{2s_{1/2}, F} - E_{2p_{1/2}, F'})^{-1} , \tag{11}$$

$$G_2(F, F') = (E_{2s_{1/2}, F} - E_{2p_{3/2}, F'})^{-1} . \tag{12}$$

We label our states $|nL_J FM_F\rangle$ such as in $|1s_{1/2}F''M_{F''}\rangle$, where we surpress the $I=\frac{1}{2}$ index. The energies $E_{2s_{1/2},F}$, $E_{2p_{1/2},F'}$, and $E_{2p_{3/2},F'}$ are given including the Lamb shift and hyperfine splittings. We take our values from Brodsky and Parsons,[9] employing the revised value for the fine-structure constant.[10]

We simplify our transition amplitudes to a form involving standard $3j$ and $6j$ coefficients:

$$A_{\mathrm{I}}=\frac{a_0^2 2^8\sqrt{2}}{3^4}(-1)^{M_F+M_{F''}}[(2F+1)(2F''+1)]^{1/2}$$
$$\times\sum_{F'}\begin{Bmatrix}F'' & F' & 1\\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2}\end{Bmatrix}(2F'+1)\begin{Bmatrix}F & F' & 1\\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2}\end{Bmatrix}\sum_q(-1)^q\epsilon_{-q}\begin{pmatrix}F'' & 1 & F'\\ -M_{F''} & q & M_F\end{pmatrix}\begin{pmatrix}F' & 1 & F\\ -M_F & 0 & M_F\end{pmatrix}G_1(F,F'), \qquad (13)$$

$$A_{\mathrm{II}}=\frac{a_0^2 2^9\sqrt{2}}{3^4}(-1)^{M_F+M_{F''}}[(2F+1)(2F''+1)]^{1/2}$$
$$\times\sum_{F'}\begin{Bmatrix}F'' & F' & 1\\ \frac{3}{2} & \frac{1}{2} & \frac{1}{2}\end{Bmatrix}(2F'+1)\begin{Bmatrix}F' & 1 & F\\ \frac{1}{2} & \frac{1}{2} & \frac{3}{2}\end{Bmatrix}\sum_q(-1)^q\epsilon_{-q}\begin{pmatrix}F'' & 1 & F'\\ -M_{F''} & q & M_F\end{pmatrix}\begin{pmatrix}F' & 1 & F\\ -M_F & 0 & M_F\end{pmatrix}G_2(F,F'), \qquad (14)$$

$$A_{\mathrm{III}}=\frac{a_0^2 2^5\sqrt{2}}{3^5}(-1)^{M_F+M_{F''}}\begin{Bmatrix}\frac{1}{2} & F'' & \frac{1}{2}\\ 1 & \frac{3}{2} & 1\end{Bmatrix}\begin{Bmatrix}1 & 1 & F\\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2}\end{Bmatrix}\sum_q(-1)^q\epsilon_{-q}\begin{pmatrix}F'' & 1 & 1\\ -M_{F''} & q & M_F\end{pmatrix}\begin{pmatrix}1 & 1 & F\\ -M_F & 0 & M_F\end{pmatrix}$$
$$\times[(2F''+1)(2F+1)]^{1/2}G_1(F,1)G_2(F,1)\alpha^2 g_I\left(\frac{m}{m_p}\right)Ry, \qquad (15)$$

$$A_{\mathrm{IV}}=\frac{a_0^2 2^5\sqrt{2}}{3^5}(-1)^{M_F+M_{F''}}\begin{Bmatrix}F'' & 1 & 1\\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2}\end{Bmatrix}\begin{Bmatrix}1 & 1 & F\\ \frac{1}{2} & \frac{1}{2} & \frac{3}{2}\end{Bmatrix}[(2F+1)(2F''+1)]^{1/2}$$
$$\times\sum_q(-1)^q\epsilon_{-q}\begin{pmatrix}F'' & 1 & 1\\ -M_{F''} & q & M_F\end{pmatrix}\begin{pmatrix}1 & 1 & F\\ -M_F & 0 & M_F\end{pmatrix}G_1(F,1)G_2(F,1)\alpha^2 g_I\left(\frac{m}{m_p}\right)Ry. \qquad (16)$$

The value for polarization fraction P we obtain by including just A_{I} and A_{II} in the spin sum is -32.32%, which compares with Casalese and Gerjuoy's value of -32.33%. Including the effects of the leading J-breaking part of the hyperfine structure, that is, amplitudes A_{III} and A_{IV}, we obtain -32.31%.

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