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Parametric rotation of the principal polarization axes and other effects due to four transverse waves in a plasma

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Four elliptically polarized waves acting in a cold and unmagnetized plasma are shown to be responsible for the parametric evolution in time of the rotation of the principal polarization axes (the precession frequency) and some other nonlinear effects in two evolving waves by two other powerful waves. The investigation requires the transition from the macroscopic to a new type of more general and vivid microscopic study of parametric processes due to transverse waves in continuous media. The parametric precession frequency (PPF), when complex, is a new source of instability in plasmas. Complex PPF and complex parametric frequency shift occur in some near-resonant interactions in which the sum or difference between two frequencies is close to the characteristic frequency of electrons in a plasma. It is found that when the plasma is subjected to two strong fields, sharp bands of monochromatic noise transform into growing continuous spectra.

I. INTRODUCTION

Using a plane-polarized transverse wave Sluijter and Montgomery[1] (1965) considered the problem of the nonlinear interaction of a wave with a plasma having relativistically moving electrons, static ions, and no static magnetic field. They obtained an expression for the shift in the frequency in the temporal problems and wave-number shift in spatial problems. Arons and Max[2] (1974) reconsidered the same problem using an elliptically polarized wave. Assuming that the nonlinearities in the medium, due to interaction with some fields of unequal amplitudes, are responsible for the evolution of the precession frequency (which is the rate of rotation of the principal polarization axes about the direction of propagation) simultaneously with the evolution of the shift in the frequency or wave number, they evaluated these quantities. Thus a physically interesting and important quantity — the precession of the polarization ellipse —was predicted as a nonlinear effect. Consequently the results obtained with elliptically polarized waves[3,4] can now be rectified, and those with two plane polarized waves[5] can be generalized. Moreover, the work of Arons and Max has also opened a few other important avenues of research. This paper deals with some of them.

When the precession and shift in frequency are complex, the plasma becomes unstable inspite of the fact that it is transparent to high-frequency waves in the linear approximation. This makes the study of propagation of high-frequency waves important and useful.

In Ref. 4 the interaction of four elliptically polarized waves, neglecting precession, was studied to find the evolution of parametrically excited frequency shift ω (PFS) of two weak noises by two powerful pump waves; and it was shown that the interaction of four transverse waves propagating in the same direction is necessary to study the type of parametric effects considered here.

The motivation for the present paper is the generalization of that work through the investigation of the parametrically excited precession frequency $\dot{\rho}$ (PPF) in addition to the PFS. Since the concept of the evolution of $\dot{\rho}$ is new a number of interesting questions and problems have been observed and are clarified in this paper. The earlier obtained equations and their solutions now appear to be degenerate forms of the qualitatively somewhat different and more comprehensive set of equations and their solutions.

In Sec. II the main novelties of the parametric problems involving precession in contrast to the problems where precession is either ignored or is not manifested are demonstrated with the help of a simple model set of equations and their solutions. In Sec. III the basic equations and relations of the actual physical problem are discussed. These relations are true in a frame which is rotating with the precession frequency about the common direction of propagation. The general expressions for the parametrically excited $\dot{\rho}$ and ω are derived and discussed in general terms in Sec. IV. In Sec. V some near-resonant interactions are investigated. A few of them yield complex values of $\dot{\rho}$ and ω. The basic equations for a case of exact resonance are given, and the nature of their solutions discussed briefly.

The expression for $\dot{\rho}$ becomes indeterminate when the elliptically polarized waves tend to be circularly polarized. We searched for a method of removing this indeterminacy mathematically, and ultimately found the conditions for it. It became evident that certain types of orientations of the wave polarization are permissible and others not, in order that the limiting values of the indeterminate quantities become finite and determinate. These conditions have therefore been regarded as essential for the evolution of the waves in a plasma; and the polarizations violating these conditions cannot be parametrically evolved. The reason for this behavior is not yet clear. One very plausible surmise is that this is merely a demonstration of the conservation of angular momentum. To see that, it seems that instead of using our system of equations, which is for the interaction between waves and a continuum, it would be better to start with the equations for the interaction between waves and a charged particle. Though the solution by this approach is difficult, it can as well be important and useful. Katz *et al.*[6] (1975) have worked on one such simple problem, the interaction of an elliptically polarized wave of long wavelength on an electron. They have shown that the polarization ellipse of the radiation field rotates with a frequency which, in the long-wavelength approximation, equals the precession frequency of Arons and Max.

Two types of rotation of the fields about the common direction of propagation are referred to here. One is the rotation of the field vectors of the electromagnetic fields, the rate of which is the wave frequency, and the other is the rotation of the direction of their extreme values, or, in other words, of the principal polarization axes, the rate of which is the precession frequency. See Fig. 1.

If, in the laboratory system, the frequency shift and precession are complex, the maximum intensity of the field grows exponentially, changes color, and rotates with exponentially increasing angular frequency in time about the direction of propagation. Some near-resonant interactions are important because in these PPF and PFS are complex. They have been discussed in Sec. V.

The planetary and other astrophysical bodies have axial symmetry, and they usually experience specific rotations about their own axis. We find that as an electromagnetic field enters a body, the direction of its maximum force begins to rotate about the direction of propagation. Moreover, we find that the direction of the maximum forces of parametrically excited electromagnetic fields inside a body also begins to rotate about the direction of propagation. This phenomenon seems to be very much relevant to the search for the cause of natural rotation of bodies in space about their own axial direction. The rotating maximum of the force and $\dot{\rho}$, when growing in time, can give rise to some new quantum-mechanical effects. These can occur as resonance effects when this force is of the order of forces holding the electrons to the respective inner cores in atoms and molecules and when $\dot{\rho}$ equals a characteristic frequency of oscillation inside them. This rotation can also be manifested in transverse waves through nonconducting media. So it may be of some use in seismic research.

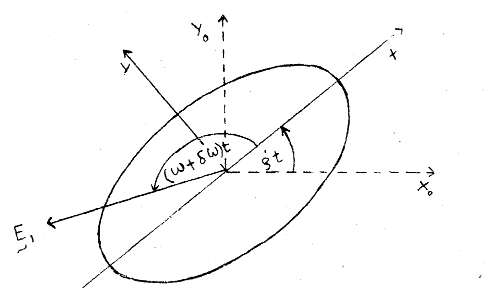

FIG. 1. Two types of rotation of an elliptically polarized field in time.

Chakraborty and Chandra[7] (1977) have considered the two-wave interaction problem. They have shown that if one of the waves is plane polarized, its plane of polarization is rotated by the second wave if that is elliptically polarized, and that, this rotation can be enhanced in near-resonant interactions. This phenomenon seems to be similar to Faraday rotation of a plane-polarized wave by an axial static magnetic field, and like it, if detected, may be useful in plasma diagnostic research.

II. A MODEL SET

Some parametric problems have been considered in the recent past with the help of mathematical models. Rosenbluth[8] (1972), Wilhelmsson[9] (1973), and Nishikawa[10] (1968) are some examples of these studies. We have also considered first of all a model set of equations and then the actual problem. The reason is that all the equations of our problem are big and contain many terms, some of which are qualitatively similar. So though all of them contribute to the determination of the nature of the final results, they separately do not contribute to the demonstration of the nature of the main features and difficulties of the work undertaken. That is why the model will serve as a good introduction.

We first write these for the parametrically excited fields of two plane-polarized waves when they are interacting with two powerful plane-polarized fields. Let a_1, a_2, a_3, a_4 be the complex amplitudes of the four waves, and $\omega_1, \omega_2, \omega_3, \omega_4$ their frequencies. Then

$$\begin{aligned} \frac{\partial a_1}{\partial t} - ip_1 a_1(a_3\bar{a}_3 + a_4\bar{a}_4) &= iq_1\bar{a}_2\bar{a}_3 a_4 , \\ \frac{\partial a_2}{\partial t} - ip_2 a_2(a_3\bar{a}_3 + a_4\bar{a}_4) &= iq_2\bar{a}_1\bar{a}_3 a_4 \end{aligned} \tag{2.1}$$

where p_1, p_2, q_1, q_2 are parameters and $\bar{a}$ is the complex conjugate of a. Let the phase-matching condition between the frequencies be

$$\omega_4 = \omega_1 + \omega_2 + \omega_3 . \tag{2.2}$$

It ensures that the interaction is parametric and corresponds to a coupling of the lowest order between the waves. Using (2.2) the solution of (2.1) was considered in Ref. 4.

When the waves are elliptically polarized the field variables can be expressed in the form (3.5). To avoid some complications let initially all the major axes be taken along the same direction parallel to the x axis and all the minor axes parallel to the y axis. Now denoting the PPE by $\dot{\rho}$, the model equations are

$$\frac{\partial a_1}{\partial t} + ib_1\dot{\rho} - ip_1\lambda a_1 = iq_1\bar{a}_3(\bar{a}_2 a_4 + \bar{b}_2 b_4) , \tag{2.3}$$

$$\frac{\partial b_1}{\partial t} + ia_1\dot{\rho} - ip_1\lambda b_1 = -iq_1\bar{b}_3(\bar{a}_2 a_4 + \bar{b}_2 b_4) , \tag{2.4}$$

$$\frac{\partial a_2}{\partial t} + ib_2\dot{\rho} - ip_2\lambda a_2 = iq_2 a_4(\bar{a}_1\bar{a}_3 - \bar{b}_1\bar{b}_3) , \tag{2.5}$$

$$\frac{\partial b_2}{\partial t} + ia_2\dot{\rho} - ip_2\lambda b_2 = iq_2 b_4(\bar{a}_1\bar{a}_3 - \bar{b}_1\bar{b}_3) , \tag{2.6}$$

with

$$\lambda = a_3\bar{a}_3 + b_3\bar{b}_3 + a_4\bar{a}_4 + b_4\bar{b}_4 . \tag{2.6a}$$

Equations (2.3) and (2.4) clearly follow from (4.3a) and (4.3b), and (2.5) and (2.6) from (4.6) if (4.7a) and (4.7b) are used with it. When the field orientation is specified by (2.25), in the limit of circular polarization, only one of the three terms in the right-hand side of (4.3a), (4.3b), (4.7a), (4.7b) does not vanish. In the model (2.3) to (2.6) the right-hand sides contain only terms of the type which do not vanish in the said limit.

The most significant new additions to the equations (2.3)–(2.6), compared to the corresponding two equations of (2.1), are those proportional to $\dot{\rho}$. Besides these p_1, p_2, q_1, q_2 contain more terms due to the b_j's. If $\dot{\rho}$ was absent, or ignored, these additional terms could not effectively alter the pattern of solution indicated by (2.1) and obtained earlier.[4] Equations (2.3)–(2.6) give

$$\frac{\partial}{\partial t}(a_1^2 - b_1^2) = 2ip_1\lambda(a_1^2 - b_1^2) + 2iq_1(a_1\bar{a}_3 + b_1\bar{b}_3)(\bar{a}_2 a_4 + \bar{b}_2 b_4) , \tag{2.7}$$

$$\frac{\partial}{\partial t}(a_2^2 - b_2^2) = 2ip_2\lambda(a_2^2 - b_2^2) + 2iq_2(\bar{a}_1\bar{a}_3 - \bar{b}_1\bar{b}_3)(a_2 a_4 - b_2 b_4) . \tag{2.8}$$

Equation (2.7) has been obtained from (2.3) and (2.4), and (2.8) from (2.5) and (2.6). These will give the frequency shift if the $\partial(a_j, b_j)/\partial t$ are replaced by $-i\omega(a_j, b_j)$. To obtain $\dot{\rho}$ from (2.3) and (2.4) the first term in the left-hand side of these equations is eliminated. The result is

$$\dot{\rho}\frac{\partial}{\partial t}(a_1^2 - b_1^2) = 2p_1\lambda(\dot{a}_1 b_1 - a_1\dot{b}_1) - 2q_1(\dot{a}_1\bar{b}_3 + \bar{a}_3\dot{b}_1)(\bar{a}_2 a_4 + \bar{b}_2 b_4) . \tag{2.9}$$

Similarly Eqs. (2.5) and (2.6) give

$$\dot{\rho}\frac{\partial}{\partial t}(a_2^2 - b_2^2) = 2p_2\lambda(\dot{a}_2 b_2 - a_2\dot{b}_2) + 2q_2(\dot{a}_2 b_4 - a_4\dot{b}_2)(\bar{a}_1\bar{a}_3 - \bar{b}_1\bar{b}_3) . \tag{2.10}$$

The equations for PFS and PPF, (2.7)–(2.10), are very different from the hitherto known linear equations for ω only. Their investigations have led us to the transition from the usual type, which can now be termed the macroscopic study, to a new type of more general and vivid microscopic study of propagation of transverse waves in a continuum.

Replacing $\partial(a_1, b_1, a_2, b_2)/\partial t$ by $-i\omega(a_1, b_1, a_2, b_2)$, etc., all the terms of (2.7) and (2.9) are divided by $a_1^2 - b_1^2$, and of (2.8) and (2.10) by $a_2^2 - b_2^2$. Then dividing all the terms of the numerators and denominators of the resulting relations containing p_1 and q_1 by a_1^2 and those containing p_2 and q_2 by a_2^2, we get

$$\omega + p_1\lambda = -q_1\frac{\bar{a}_2}{a_1}\left(1 + \frac{\bar{b}_2 b_4}{\bar{a}_4 a_4}\right)\left(1 + \frac{b_1\bar{b}_3}{a_1\bar{a}_3}\right)\left(1 - \frac{b_1^2}{a_1^2}\right)^{-1}\bar{a}_3 a_4 , \tag{2.11}$$

$$\omega + p_2\lambda = -q_2\frac{\bar{a}_1}{a_2}\left(1 - \frac{\bar{b}_1\bar{b}_3}{\bar{a}_1\bar{a}_3}\right)\left(1 - \frac{b_2 b_4}{a_2 a_4}\right)\left(1 - \frac{b_2^2}{a_2^2}\right)^{-1}\bar{a}_3 a_4 , \tag{2.12}$$

$$\dot{\rho} = -q_1\frac{\bar{a}_2}{a_1}\left(\frac{\bar{b}_3}{\bar{a}_3} + \frac{b_1}{a_1}\right)\left(1 + \frac{\bar{b}_2 b_4}{\bar{a}_2 a_4}\right)\left(1 - \frac{b_1^2}{a_1^2}\right)^{-1}\bar{a}_3 a_4 , \tag{2.13}$$

$$\dot{\rho} = q_2\frac{\bar{a}_1}{a_2}\left(\frac{b_4}{a_4} - \frac{b_2}{a_2}\right)\left(1 - \frac{\bar{b}_1\bar{b}_3}{\bar{a}_1\bar{a}_3}\right)\left(1 - \frac{b_2^2}{a_2^2}\right)^{-1}\bar{a}_3 a_4 . \tag{2.14}$$

The equation for ω is obtained by eliminating $\bar{a}_2/a_1$ between (2.11) and (2.12) and for $\dot{\rho}$ between (2.13) and (2.14).

$$\frac{(\omega + p_1\lambda)(\omega + p_2\lambda)}{q_1 q_2 a_3^2 a_4^2} = \left(1 + \frac{b_1 b_3}{a_1 a_3}\right)\left(1 + \frac{b_2 b_4}{a_2 a_4}\right)\left(1 - \frac{b_1 b_3}{a_1 a_3}\right) \times \left(1 - \frac{b_2 b_4}{a_2 a_4}\right)\left(1 - \frac{b_1^2}{a_1^2}\right)^{-1}\left(1 - \frac{b_2^2}{a_2^2}\right)^{-1} , \tag{2.15}$$

$$\frac{\dot{\rho}^2}{q_1 q_2 |a_3|^2 |a_4^2|} = -\left(\frac{\bar{b}_3}{\bar{a}_3} + \frac{b_1}{a_1}\right)\left(\frac{\bar{b}_4}{\bar{a}_4} - \frac{\bar{b}_2}{\bar{a}_2}\right)\left(1 + \frac{\bar{b}_2 b_4}{\bar{a}_2 a_4}\right) \times \left(1 - \frac{b_1 b_3}{a_1 a_3}\right)\left(1 - \frac{b_1^2}{a_1^2}\right)^{-1}\left(1 - \frac{\bar{b}_2^2}{\bar{a}_2^2}\right)^{-1} \tag{2.16}$$

Equation (2.15) can be simplified to

$$\frac{(\omega+p_1\lambda)(\omega+p_2\lambda)}{q_1q_2|a_3|^2|a_4|^2}=\left(1-\frac{|b_1|^2|b_3|^2}{|a_1|^2|a_3|^2}\right)\left(1-\frac{|b_2|^2|b_4|^2}{|a_2|^2|a_4|^2}\right)\times\left(1-\frac{|b_1|^2}{|a_1|^2}\right)^{-1}\left(1-\frac{|b_2|^2}{|a_2|^2}\right)^{-1}, \tag{2.17}$$

because $a_1=|a_1|e^{-i\omega t}$, etc.

For further calculations we put

$$\left|\frac{b_j}{a_j}\right|^2=1-e_j^2 \quad (j=1,2,3,4), \tag{2.18}$$

in (2.17) and obtain

$$\frac{(\omega+p_1\lambda)(\omega+p_2\lambda)}{q_1q_2|a_3|^2|a_4|^2}=\frac{(e_1^2+e_3^2-e_1^2e_3^2)(e_2^2+e_4^2-e_2^2e_4^2)}{e_1^2e_2^2}. \tag{2.19}$$

Here e_1, e_2, e_3, e_4 are the eccentricities of the polarization ellipses of the waves. So $0\le e_j\le 1$. Now let

$$e_j=l_je_4, \tag{2.20}$$

where l_j are constants and $j=1,2,3,4$. Then for the limits leading to circular polarization of the waves we find that $e_1\to 0$, $e_2\to 0$, $e_3\to 0$, as $e_4\to 0$. Moreover, since for the plane-polarized waves

$$e_j=1, \tag{2.21}$$

we find that we must take $l_j=1$ because for them $e_j/e_4=1$. Hence

$$\lim_{e_i\to 0,\, e_j\to 0}\frac{e_i}{e_j}=1, \tag{2.21a}$$

and (2.19) reduces to

$$(\omega+p_1\lambda)(\omega+p_2\lambda)=4q_1q_2|a_3|^2|a_4|^2. \tag{2.22}$$

To consider (2.16) for these limits, we find that the relations (2.18) are not sufficient and so we write

$$|b_j|/|a_j|=\pm(1-e_j^2)^{1/2}. \tag{2.23}$$

The field at the frequency ω_j is

$$\vec{E}_{1j}=(\vec{x}a_j-i\vec{y}b_j)\tfrac{1}{2}e^{i\theta_j}+\text{c.c.},\quad \theta_j=k_jz-\omega_jt, \tag{2.24}$$

where $\vec{x}$ and $\vec{y}$ are unit vectors along the rectangular axes $0x$ and $0y$, respectively, and c.c. refers to the complex conjugate. Looking antiparallel to $0z$ the projection of $\vec{E}_{1j}$ in the xy plane rotates in time in the clockwise sense if both $\text{Re}\, b_j>0$ and $\text{Re}\, a_j>0$, and in the anticlockwise sense if $\text{Re}\, b_j<0$ when $\text{Re}\, a_j>0$ provided $\omega_j/k_j>0$, where $\text{Re}\, a_j$ is the real part of a_j.

For the evaluation of $\dot\rho$ the imposition of the nonsymmetric further prescription that the electric vectors of the waves at the frequencies $\omega_1, \omega_3, \omega_4$ rotate clockwise and that at ω_2 counterclockwise becomes necessary, because otherwise $\dot\rho$ is indeterminate. Hence we must choose from (2.23) the following four relations:

$$\frac{|b_1|}{|a_1|}=(1-e_1^2)^{1/2},\quad \frac{|b_2|}{|a_2|}=-(1-e_2^2)^{1/2},$$
$$\frac{|b_3|}{|a_3|}=(1-e_3^2)^{1/2},\quad \frac{|b_4|}{|a_4|}=(1-e_4^2)^{1/2}. \tag{2.25}$$

These are the allowed polarizations. Those which exclude (2.25) but which are given by (2.23) for $j=1,2,3,4$ are not allowed. Equation (2.16) can be written as

$$\dot\rho^2=q_1q_2|a_3|^2|a_4|^2\left(\frac{\bar b_3}{\bar a_3}-\frac{b_1}{a_1}\right)\left(\frac{\bar b_4}{\bar a_4}-\frac{\bar b_2}{\bar a_2}\right)\left[\frac{(1-|b_2/a_2|^2|b_4/a_4|^2)(1-|b_1/a_1|^2|b_3/a_3|^2)}{(1-|b_1/a_1|^2)(1-|b_2/a_2|^2)(1-|b_2/a_2|\,|b_4/a_4|)(1+|b_1/a_1|\,|b_3/a_3|)}\right].$$

Using (2.23) the quantity within square brackets reduces to

$$\frac{(e_2^2+e_4^2-e_2^2e_4^2)(e_1^2+e_3^2-e_1^2e_3^2)}{e_1^2e_2^2(1-|b_2/a_2|\,|b_4/a_4|)(1+|b_1/a_1|\,|b_3/a_3|)}. \tag{2.26}$$

Because of (2.25) the denominator is not zero and the indeterminacy in (2.16) is removed for circular polarization. Hence using in (2.26) the relations (2.20) to (2.21a), we find that for circular polarization Eq. (2.16) becomes

$$\dot\rho^2=-4q_1q_2|a_3|^2|a_4|^2. \tag{2.27}$$

Hence by (2.25) the waves 1 and 2 should have electric vectors rotating in opposite senses. This is a nonsymmetric condition, although (2.2) is symmetric with respect to the waves 1 and 2. Hence the effective parts of the equations for the evolution of their amplitudes, namely (2.3)–(2.6), do not have any symmetry. The obtainment of ω, (2.22), does not depend on these finer details.

The allowed polarizations can interact with the gyration of electrons and ions in the presence of a static magnetic field in the direction of wave propagation, and so may provide a source of transforming energy of strong fields (e.g., laser fields) to ions or electrons for the purpose of heating a plasma. This seems to be an interesting unsolved problem.

The expression (2.26) and the formulas (4.14) and (4.15) show that the ellipticities of the pump waves 3 and 4 are given. But since those of the evolving waves 1 and 2 are not given, their frequency shifts depend on e_1 and e_2.

For example, the wave at the frequency ω_1 evolves into that at $\omega_1 + \omega$ where ω has two sets of continuous and generally complex values. These are the solutions of the quadratic (4.14) and are regular functions of e_1 and e_2. Therefore, the sharp bands of the two evolving monochromatic noises transform into continuous spectra having all frequencies lying between certain maxima and minima.

The electric field of the wave 1 is given by

$$\vec{E}_{\parallel} = \tfrac{1}{2}|a_1|[\vec{x} - i\,\vec{y}(1-e_1^2)^{1/2}] \times \exp\{i[\theta_1 - \omega(e_1, e_2)t]\} + \text{c.c.} \qquad (2.28)$$

where $\theta_1 = k_1 z - \omega_1 t$. Now summing over all e_1 and e_2 in the system rotating with the angular velocity $\dot{\rho}$ we find that

$$\sum_{e_1, e_2} \vec{E}_{\parallel}(e_1, e_2) = \int_0^1\int_0^1 \frac{|a_1|}{2}([\vec{x} - i(1-e_1^2)^{1/2}\vec{y}]\exp\{i[\theta_1 - \omega(e_1,e_2)t]\} + \text{c.c.})\,de_1\,de_2\,. \qquad (2.29)$$

In the fixed laboratory frame if $\vec{x}^0$ and $\vec{y}^0$ are the unit vectors along the rectangular Cartesian coordinate axes, we get

$$\sum_{e_1, e_2} \vec{E}_{\parallel}(e_1, e_2) = \int_0^1\int_0^1 \frac{|a_1|}{2}([(\vec{x}^0\cos\rho(e_1,e_2) + \vec{y}^0\sin\rho) - i(1-e_1^2)^{1/2}(-\vec{x}^0\sin\rho + \vec{y}^0\cos\rho)] \times \exp\{i[\theta_1 - \omega(e_1,e_2)t]\} + \text{c.c.})\,de_1\,de_2\,. \qquad (2.30)$$

Relations of transformation between $\vec{x}, \vec{y}$ and $\vec{x}^0$, $\vec{y}^0$ are given in Eq. (4.18).

III. STARTING BASIC EQUATIONS AND RELATIONS

The basic equations for electron motion in a cold and homogeneous plasma in the absence of any static magnetic field are

$$\left(\frac{\partial}{\partial t} + (\vec{v}\cdot\vec{\nabla})\right)\frac{\vec{v}}{(1-v^2/c^2)^{1/2}} = -\frac{e\vec{E}}{m} - \frac{e}{mc}[\vec{v}\times\vec{H}]\,, \qquad (3.1)$$

$$\text{curl}\,\vec{E} = -\frac{1}{c}\frac{\partial\vec{H}}{\partial t}\,, \qquad (3.2)$$

$$\text{curl}\,\vec{H} = \frac{1}{c}\frac{\partial\vec{E}}{\partial t} - \frac{4\pi e}{c}(n_0 + n)\vec{v}\,, \qquad (3.3)$$

$$\text{div}\,\vec{E} = -4\pi e n\,, \quad \text{div}\,\vec{H} = 0\,. \qquad (3.4)$$

As in Refs. 1, 2, 4, and 5, the relativistic electron momentum is also used here, although the plasma is cold and the perturbation field is of a nonrelativistic nature. Thus according to the terminology of Ferrari *et al.*,[11] ours is a problem of the weakly relativistic type.

As in (2.24) let $\vec{E}_{1j}$ denote the first harmonic field at the frequency ω_j and $\vec{E}_1$ denote the sum field

$$\sum_{j=1}^{4}\vec{E}_{1j}\,.$$

Moreover, let the waves propagate parallel to the z axis and oscillate in the xy plane, where initially the x axis is parallel to the major axis and y axis parallel to the minor axis of all the waves. Thus we can write

$$\vec{E}_1 = \sum_{j=1}^{4}\vec{E}_{1j} = \tfrac{1}{2}\sum_{j=1}^{4}\{\vec{x}(a_j e^{i\theta_j} + \bar{a}_j e^{-i\theta_j}) - i\,\vec{y}(b_j e^{i\theta_j} - \bar{b}_j e^{-i\theta_j})\}\,, \qquad (3.5)$$

where

$$\theta_j = k_j z - \omega_j t \qquad (3.6)$$

and $\bar{a}_j$ is the complex conjugate of a_j. Actually the total electric field $\vec{E}$ is the sum

$$\vec{E} = \vec{E}_1 + \vec{E}_2 + \vec{E}_3 + \cdots\,, \qquad (3.7)$$

where $\vec{E}_2$ contains the vector sum of all the double harmonics $\theta_j \pm \theta_k$, and $\vec{E}_3$ contains the vector sum of the third harmonics provided by the combinations in $\theta_i \pm \theta_j \pm \theta_k$ for all integral values from 1 to 4 for i, j, k.

The linear approximation will be called the first approximation, and represents only the first harmonic fields whose amplitudes are independent of time. The dispersion relation is

$$k_j^2 c^2 = \omega_j^2 - \omega_p^2\,. \qquad (3.8)$$

The second harmonic solutions will be regarded as those of the second-order approximation; the third harmonics as those of the third-order solutions, and so on for the higher-order solutions.

The second-order velocity $\vec{v}_2$ and the electric field $\vec{E}_2$ are determined from

$$\frac{\partial\vec{v}_2}{\partial t} = -\frac{e\vec{E}_2}{m} - \frac{e}{mc}[\vec{v}_1 + \vec{H}_1]\,, \quad \frac{\partial\vec{E}_2}{\partial t} = 4\pi e n_0\vec{v}_2\,. \qquad (3.9)$$

They are longitudinal waves and so

$$\vec{H}_2 = 0\,, \quad \text{div}\,\vec{E}_2 = -4\pi e n_2\,. \qquad (3.10)$$

In this paper we mainly consider the problems for which

$$(\omega_i \pm \omega_j)^2 \neq \omega_p^2 \tag{3.11}$$

and briefly mention at the end of Sec. V one case of the type

$$(\omega_i \pm \omega_j)^2 = \omega_p^2 . \tag{3.11a}$$

Using, therefore, (3.11) to solve (3.9) we get

$$\vec{v}_2 = -\frac{e^2\vec{z}}{4m^2}\left(\frac{2k_1[(a_1^2-b_1^2)e^{2i\theta_1}+\text{c.c.}]}{\omega_1(4\omega_1^2-\omega_p^2)} + \frac{(k_1+k_2)(\omega_1+\omega_2)}{\omega_1\omega_2[(\omega_1+\omega_2)^2-\omega_p^2]}[(a_1a_2-b_1b_2)e^{i(\theta_1+\theta_2)}+\text{c.c.}] - \frac{(k_1-k_2)(\omega_1-\omega_2)}{\omega_1\omega_2\{(\omega_1-\omega_2)^2-\omega_p^2\}}[(a_1a_2+b_1b_2)e^{i(\theta_1-\theta_2)}+\text{c.c.}]+\ldots\right) . \tag{3.12}$$

$\vec{E}_2$ is evaluated from the second relation of (3.9) and n_2 from $\vec{E}_2$ with the help of the second equation of (3.10).

A. Third-order source

The third-order source for the electric field is

$$\frac{m\omega_p^2}{e}\left\{-\frac{\partial}{\partial t}\left(\frac{v_1^2\vec{v}_1}{2c^2}\right)+\frac{1}{n_0}\frac{\partial}{\partial t}(n_2\vec{v}_1) - \frac{c}{mc}[\vec{v}_2\times\vec{H}_1]-(\vec{v}_2\cdot\vec{\nabla})\vec{v}_1\right\}. \tag{3.13}$$

All the terms of it being transverse waves, the third-order fields must be transverse and divergence free. Hence

$$\operatorname{div}\vec{E}_3 = 0 . \tag{3.14}$$

Moreover, the calculations show that

$$(e/mc)[\vec{v}_2\times\vec{H}_1]+(\vec{v}_2\cdot\vec{\nabla})\vec{v}_1 = 0 . \tag{3.15}$$

The remaining two terms of (3.13) simplify to a linear combination of $\exp(2i\theta_j \pm i\theta_j)$ and $\exp(i\theta_i \pm i\theta_j \pm i\theta_k)$.

Some of these are first harmonic sources because they are sines and cosines of $\theta_j+\theta_j-\theta_j$ for $j=1,2,3,4$. Moreover, because of (2.2), the linear combination $\theta_4-\theta_3-\theta_2$, for example, will be reduced to $(k_4-k_3-k_2)z-\omega_1 t$ and so will add to the sources containing linear combinations of sines and cosines of $\omega_1 t$. Similarly all the first harmonic sources of the third order at ω_1 and ω_2, both parametric and nonparametric, can be obtained.

Since $\omega_j^2 \gg \omega_p^2$ we have

$$k_j^2c^2 \approx \omega_j^2 . \tag{3.15a}$$

Let us now impose the condition

$$k_4 \approx k_1+k_2+k_3 . \tag{3.15b}$$

Relations (2.2) and (3.15b) can not hold simultaneously when some of the waves propagate in the positive direction of the z axis and some in the negative direction. So all the waves propagate only in one of these two directions. Consequently ω_j and and k_j should have the same sign:

$$\omega_j/k_j>0;\ k_j>0 \text{ when } \omega_j>0, \quad k_j<0 \text{ when } \omega_j<0 . \tag{3.15c}$$

B. Basic third-order equation

Let us here express the third-order fields as the sum $\vec{E}_1+\vec{E}_3+\cdots$ where $\vec{E}_1$ represents the evolution of the first harmonic fields correct up to third order, $\vec{E}_3$ represents the third harmonic fields and the terms in $+\cdots$ contain the electric field of the other combination harmonics of the type $\theta_1\pm\theta_2\pm\theta_3$. Since we are interested only in the first harmonic fields in the third order, we use (3.15) and (2.2) to find the first harmonic terms of (3.13). Writing only the relevant first harmonic sources at ω_1, we get

$$\ddot{\vec{E}}_1 - c^2\nabla^2\vec{E}_1+\omega_p^2\vec{E}_1 = \frac{\omega_p^2}{16}\Bigg(\sum_j 2(\alpha_j\bar{\alpha}_j+\beta_j\bar{\beta}_j)(\vec{x}a_1-i\vec{y}b_1)+A_{1+1}(\alpha_1^2-\beta_1^2)(\vec{x}\bar{a}_1+i\vec{y}\bar{b}_1)+2A_{1-2}(a_1\bar{\alpha}_2+b_1\bar{\beta}_2)(\vec{x}\alpha_2-i\vec{y}\beta_2)$$
$$+2A_{1-3}(a_1\bar{\alpha}_3+b_1\bar{\beta}_3)(\vec{x}\alpha_3-i\vec{y}\beta_3)+2A_{1-4}(a_1\bar{\alpha}_4+b_1\bar{\beta}_4)(\vec{x}\alpha_4-i\vec{y}\beta_4)$$
$$+2A_{1+2}(a_1\alpha_2-b_1\beta_2)(\vec{x}\bar{\alpha}_2+i\vec{y}\bar{\beta}_2)+2A_{1+3}(a_1\alpha_3-b_1\beta_3)(\vec{x}\alpha_3+i\vec{y}\beta_3)$$
$$+2A_{1+4}(a_1\alpha_4-b_1\beta_4)(\vec{x}\alpha_4+i\vec{y}\beta_4)-2A_{2-4}\frac{(\bar{a}_2\alpha_4+\bar{b}_2\beta_4)(\vec{x}\bar{\alpha}_3+i\vec{y}\beta_3)}{\omega_2/\omega_1}$$
$$-2A_{3-4}\frac{(\bar{\alpha}_3\alpha_4+\bar{\beta}_3\beta_4)(\vec{x}\bar{a}_2+i\vec{y}\bar{b}_2)}{\omega_2/\omega_1}-2A_{2+3}\frac{(\bar{a}_2\bar{\alpha}_3-\bar{b}_2\bar{\beta}_3)(\vec{x}\alpha_4-i\vec{y}\beta_4)}{\omega_2/\omega_1}\Bigg)e^{i\theta_1}+\text{c.c.}+\cdots, \tag{3.16}$$

where

$$A_{i\pm j} = 1-\frac{(k_i\pm k_j)^2c^2}{(\omega_i\pm\omega_j)^2-\omega_p^2},\quad \alpha_j=\frac{ea_j}{mc\omega_j},\quad \beta_j=\frac{eb_j}{mc\omega_j},\quad \bar{\alpha}_j=\frac{e\bar{a}_j}{mc\omega_j},\quad \bar{\beta}_j=\frac{e\bar{b}_j}{mc\omega_j}. \tag{3.17}$$

α_j and β_j are dimensionless amplitudes. Since we are interested in the evolution of the waves 1 and 2, we use from the right-hand side of (3.16) only the sources containing $\exp[\pm i(\theta_1, \theta_2)]$ and do not use those containing $\exp[\pm i(\theta_3, \theta_4)]$.

C. Temporal and spatial problems

In the next section we only consider the temporal problem in which ω and $\dot{\rho}$ are determined. There is also the corresponding spatial problem, in which the parametrically excited wave number shift and space rate of variation of the precession angle along the common direction of propagation can be found by properly switching from the use of evolution in time to space. However, the field equations, which are partial differential equations, are not identical in space and time. Particularly the difference between the involvement in space and time is most evident in the equations of second order. The solutions for (3.11) are nonsecular; those for (3.11a) are secular. Since the dispersion relations are not linear in ω_j and k_j, corresponding to (3.11) and (3.11a) no relations involving $k_i \pm k_j$ exist. Therefore, only the temporal problem can be defined and investigated in the near-resonant and exactly resonant interaction which we have considered in Sec. V.

IV. PARAMETRIC EQUATIONS AND THEIR GENERAL SOLUTIONS

Equation (3.17) is valid in the rotating frame of reference in which the unit vectors $\vec{x}$ and $\vec{y}$ coincide with the principal polarization directions of all the waves of $\vec{E}$. The first- and second-order time derivatives of $\vec{x}$ and $\vec{y}$ are

$$\frac{\partial\vec{x}}{\partial t}=\dot{\rho}\vec{y}\,,\quad \frac{\partial^2\vec{x}}{\partial t^2}=-\dot{\rho}^2\vec{x}+\ddot{\rho}\vec{y}\,,$$
$$\frac{\partial\vec{y}}{\partial t}=-\dot{\rho}\vec{x}\,,\quad \frac{\partial^2\vec{x}}{\partial t^2}=-\dot{\rho}^2\vec{y}-\ddot{\rho}\vec{x}\,. \tag{4.1}$$

Using these and (3.8) and (3.5) in the right-hand side of (3.17), we get

$$\left(\frac{\partial^2}{\partial t^2}-c^2\nabla^2+\omega_p^2\right)\{(\vec{x}a_1-i\vec{y}b_1)e^{i\theta_1}+\text{c.c.}\}$$
$$\equiv\{\vec{x}(-2i\omega_1\dot{a}_1+\ddot{a}_1-a_1\dot{\rho}^2+2b_1\omega_1\dot{\rho}+2i\dot{b}_1\dot{\rho}+ib_1\ddot{\rho})+\vec{y}(-2ia_1\omega_1\dot{\rho}+2\dot{a}_1\dot{\rho}+a_1\ddot{\rho}-2\dot{b}_1\omega_1-i\ddot{b}_1+ib_1\dot{\rho}^2\}e^{i\theta_1}+\text{c.c.}+\cdots, \tag{4.2}$$

where $\dot{a}_1=\partial a_1/\partial t$, etc. We retain only $\dot{\rho}, \dot{a}_1, \dot{b}_1$ and ignore $\ddot{\rho}, \ddot{a}_1, \ddot{b}_1, \dot{a}_1\dot{\rho}, \dot{b}_1\dot{\rho}, \dot{\rho}^2$ because these are negligible in the nonsingular cases in which $a_1, b_1, \dot{\rho}$ vary slowly in time. Coefficients of $\vec{x}e^{i\theta_1}$ and $\vec{y}e^{i\theta_1}$ are equated from both sides of (3.17). Then $ea_j/m\omega_j c$ and $eb_j/m\omega_j c$ are replaced by α_j and β_j in those factors in which these replacements have not yet been made. These operations result in the following pair of equations:

$$\dot{\alpha}_1+i\beta_1\dot{\rho}-ip_1\{\mu\alpha_1+\alpha_3(\alpha_1\overline{\alpha}_3+\beta_1\overline{\beta}_3)A_{1-3}+\overline{\alpha}_3(\alpha_1\alpha_3-\beta_1\beta_3)A_{1+3}+\overline{\alpha}_4(\alpha_1\alpha_4+\beta_1\beta_4)A_{1-4}+\overline{\alpha}_4(\alpha_1\alpha_4-\beta_1\beta_4)A_{1+4}\}$$
$$=iq_1\overline{\alpha}_2(\overline{\alpha}_3\alpha_4+\overline{\beta}_3\beta_4)+ir_1\overline{\alpha}_3(\overline{\alpha}_2\alpha_4+\overline{\beta}_2\beta_4)+is_1\alpha_4(\overline{\alpha}_2\overline{\alpha}_3-\overline{\beta}_2\overline{\beta}_3)+(\text{nonparametric negligible terms}), \tag{4.3a}$$
$$\dot{\beta}_1+i\alpha_1\dot{\rho}-ip_1\{\mu\beta_1+\beta_3(\alpha_1\overline{\alpha}_3+\beta_1\overline{\beta}_3)A_{1-3}-\overline{\beta}_3(\alpha_1\alpha_3+\beta_1\beta_3)A_{1+3}+\overline{\beta}_4(\alpha_1\overline{\alpha}_4+\beta_1\overline{\beta}_4)A_{1-4}-\overline{\beta}_4(\alpha_1\alpha_4-\beta_1\beta_4)A_{1+4}\}$$
$$=-iq_1\beta_2(\overline{\alpha}_3\alpha_4+\overline{\beta}_3\beta_4)-ir_1\overline{\beta}_3(\overline{\alpha}_2\alpha_4+\overline{\beta}_2\beta_4)+is_1\beta_4(\overline{\alpha}_2\overline{\alpha}_3-\overline{\beta}_2\overline{\beta}_3)+(\text{nonparametric negligible terms}), \tag{4.3b}$$

where

$$\mu=\sum_{j=3,4}(\alpha_j\overline{\alpha}_j+\beta_j\overline{\beta}_j),\quad p_1=\frac{\omega_p^2}{8\omega_1},\quad q_1=-\frac{\omega_p^2}{8\omega_1}A_{3-4},\qquad r_1=-\frac{\omega_p^2}{8\omega_1}A_{2-4},\quad s_1=-\frac{\omega_p^2}{8\omega_1}A_{2+3}. \tag{4.3c}$$

We can write these equations concisely as

$$2\omega_1(-i\dot{\alpha}_1+\beta_1\dot{\rho})=A_1,\quad 2\omega_1(-i\dot{\beta}_1+\alpha_1\dot{\rho})=B_1, \tag{4.3d}$$

where

$$A_1/2\omega_1=p_1\{\mu\alpha_1+\alpha_3(\alpha_1\overline{\alpha}_3+\beta_1\overline{\beta}_3)A_{1-3}+\overline{\alpha}_3(\alpha_1\alpha_3-\beta_1\beta_3)A_{1+3}+\alpha_4(\alpha_1\overline{\alpha}_4+\beta_1\overline{\beta}_4)A_{1-4}+\overline{\alpha}_4(\alpha_1\alpha_4-\beta_1\beta_4)A_{1+4}\}$$
$$+q_1\overline{\alpha}_2(\overline{\alpha}_3\alpha_4+\overline{\beta}_3\beta_4)+r_1\overline{\alpha}_3(\overline{\alpha}_2\alpha_4+\overline{\beta}_2\beta_4)+s_1\alpha_4(\overline{\alpha}_2\overline{\alpha}_3-\overline{\beta}_2\overline{\beta}_3)+\cdots, \tag{4.3e}$$

$$B_1/2\omega_1=p_1\{\mu\beta_1+\beta_3(\alpha_1\overline{\alpha}_3+\beta_1\overline{\beta}_3)A_{1-3}-\overline{\beta}_3(\alpha_1\alpha_3-\beta_1\beta_3)A_{1+3}+\beta_4(\alpha_1\overline{\alpha}_4+\beta_1\overline{\beta}_4)A_{1-4}-\overline{\beta}_4(\alpha_1\alpha_4-\beta_1\beta_4)A_{1+4}\}$$
$$-q_1\beta_2(\overline{\alpha}_3\alpha_4+\overline{\beta}_3\beta_4)-r_1\overline{\beta}_3(\overline{\alpha}_2\alpha_4+\overline{\beta}_2\beta_4)+s_1\beta_4(\overline{\alpha}_2\overline{\alpha}_3-\overline{\beta}_2\overline{\beta}_3)+\cdots. \tag{4.3f}$$

Eliminating $\dot{\rho}$ between the two equations of (4.3d) we get

$$\alpha_1\dot{\alpha}_1 - \beta_1\dot{\beta}_1 = (i/2\omega_1)(A_1\alpha_1 - B_1\beta_1)\,. \tag{4.4}$$

This is the equation for the frequency shift ω if $\partial/\partial t$ is replaced by $-i\omega$. Eliminating next the first term in the left-hand sides of the two equations of (4.3d) we get

$$\dot{\rho}(\beta_1\dot{\beta}_1 - \alpha_1\dot{\alpha}_1) = (1/2\omega_1)(A_1\dot{\beta}_1 - B_1\dot{\alpha}_1)\,. \tag{4.5}$$

ω cancels out because it occurs as a factor in both sides and leaves (4.5) as the equation for PPF.

Equating similarly the coefficients of $\vec{x}e^{i\theta_2}$ and $\vec{y}e^{i\theta_2}$ from both sides of (3.17) with the help of (4.2), we obtain

$$2\omega_2(-i\dot{\alpha}_2 + \beta_2\dot{\rho}) = A_2, \quad 2\omega_2(-i\dot{\beta}_2 + \alpha_2\dot{\rho}) = B_2 \tag{4.6}$$

with

$$A_2/2\omega_2 = p_2\{\mu\alpha_2 + \alpha_3(\alpha_2\bar{\alpha}_3 + \beta_2\bar{\beta}_3)A_{2-3} + \bar{\alpha}_3(\alpha_2\alpha_3 - \beta_2\beta_3)A_{2+3} + \alpha_4(\alpha_2\bar{\alpha}_4 + \beta_2\bar{\beta}_4)A_{2-4} + \bar{\alpha}_4(\alpha_2\alpha_4 - \beta_2\beta_4)A_{2+4}\}$$
$$+ q_2\bar{\alpha}_1(\bar{\alpha}_3\alpha_4 + \bar{\beta}_3\beta_4) + r_2\bar{\alpha}_3(\bar{\alpha}_1\alpha_4 + \bar{\beta}_1\beta_4) + s_2\alpha_4(\bar{\alpha}_1\bar{\alpha}_3 - \bar{\beta}_1\bar{\beta}_3) + \cdots\,. \tag{4.7a}$$

$$B_2/2\omega_2 = p_2\{\mu\beta_2 + \beta_3(\alpha_2\alpha_3 + \beta_2\beta_3)A_{2-3} - \beta_3(\alpha_2\alpha_3 - \beta_2\beta_3)A_{2+3} + \beta_4(\alpha_2\alpha_4 + \beta_2\beta_4)A_{3-4} - \beta_4(\alpha_2\alpha_4 - \beta_2\beta_4)A_{2+4}\}$$
$$- q_2\beta_2(\alpha_3\alpha_4 + \beta_3\beta_4) - r_2\beta_3(\alpha_1\alpha_4 + \beta_1\beta_4) + s_2\beta_4(\alpha_1\alpha_3 - \beta_1\beta_3) + \cdots, \tag{4.7b}$$

where

$$p_2 = \frac{\omega_p^2}{8\omega_2}, \quad q_2 = -\frac{\omega_p^2}{8\omega_2}A_{3-4}, \quad r_2 = -\frac{\omega_p^2}{8\omega_2}A_{1-4}, \quad s_2 = -\frac{\omega_p^2}{8\omega_2}A_{1+3}\,. \tag{4.8}$$

Hence for the second wave

$$\alpha_2\dot{\alpha}_2 - \beta_2\dot{\beta}_2 = -\frac{i(A_2\alpha_2 - B_2\beta_2)}{2\omega_2}, \quad \dot{\rho}(\beta_2\dot{\beta}_2 - \alpha_2\dot{\alpha}_2) = \frac{A_2\dot{\beta}_2 - B_2\dot{\alpha}_2}{2\omega_2}. \tag{4.9}$$

Replacing $\partial(\alpha_j, \beta_j)/\partial t$ by $-i\omega(\alpha_j, \beta_j)$ in (4.4), (4.5), and (4.9) and using (3.18), we get

$$\left(\frac{8\omega\omega_1}{\omega_p^2} + \mu\right)e_1^2 + \left(|\alpha_3|^2 - \left|\frac{\beta_1}{\alpha_1}\right|^2|\beta_3|^2\right)(A_{1-3} + A_{1+3}) + \left(|\alpha_4|^2 - \left|\frac{\beta_1}{\alpha_1}\right|^2|\beta_4|^2\right)(A_{1-4} + A_{1+4})$$
$$= \frac{\bar{\alpha}_2}{\alpha_1}\left[\left(\alpha_4 + \frac{|\beta_2|}{|\alpha_2|}\beta_4\right)\left(\bar{\alpha}_3 + \frac{|\beta_1|}{|\alpha_1|}\bar{\beta}_3\right)A_{2-4} + (\bar{\alpha}_3\alpha_4 + \bar{\beta}_3\beta_4)\left(1 + \frac{|\beta_1||\beta_2|}{|\alpha_1||\alpha_2|}\right)A_{3-4} + \left(\bar{\alpha}_3 - \frac{|\beta_2|}{|\alpha_2|}\bar{\beta}_3\right)\left(\alpha_4 - \frac{|\beta_1|}{|\alpha_1|}\beta_4\right)A_{2+3}\right], \tag{4.10}$$

$$\left(\frac{8\omega\omega_2}{\omega_p^2} + \mu\right)e_2^2 + \left(|\alpha_3|^2 - \left|\frac{\beta_2}{\alpha_2}\right|^2|\beta_3|^2\right)(A_{2-3} + A_{2+3}) + \left(|\alpha_4|^2 - \left|\frac{\beta_2}{\alpha_2}\right|^2|\beta_4|^2\right)(A_{2-4} + A_{2+4})$$
$$= \frac{\bar{\alpha}_1}{\alpha_2}\left[\left(\alpha_4 + \left|\frac{\beta_1}{\alpha_1}\right|\beta_4\right)\left(\bar{\alpha}_3 + \left|\frac{\beta_2}{\alpha_2}\right|\bar{\beta}_3\right)A_{1-4} + (\bar{\alpha}_3\alpha_4 + \bar{\beta}_3\beta_4)\left(1 + \left|\frac{\beta_1}{\alpha_1}\right|\left|\frac{\beta_2}{\alpha_2}\right|\right)A_{3-4} + \left(\bar{\alpha}_3 - \left|\frac{\beta_1}{\alpha_1}\right|\bar{\beta}_3\right)\left(\alpha_4 - \left|\frac{\beta_2}{\alpha_2}\right|\beta_4\right)A_{1+3}\right], \tag{4.11}$$

$$\frac{8\omega_1\dot{\rho}e_1^2}{\omega_p^2} + \left(\bar{\alpha}_3 + \frac{\beta_1}{\alpha_1}\bar{\beta}_3\right)\left(\alpha_3\frac{\beta_1}{\alpha_1} - \beta_3\right)A_{1-3} + \left(\bar{\alpha}_4 + \frac{\beta_1}{\alpha_1}\bar{\beta}_4\right)\left(\alpha_4\frac{\beta_1}{\alpha_1} - \beta_4\right)A_{1-4}$$
$$+ \left(\alpha_3 - \frac{\beta_1}{\alpha_1}\beta_3\right)\left(\bar{\alpha}_3\frac{\beta_1}{\alpha_1} + \bar{\beta}_3\right)A_{1+3} + \left(\alpha_4 - \frac{\beta_1}{\alpha_1}\beta_4\right)\left(\bar{\alpha}_4\frac{\beta_1}{\alpha_1} + \bar{\beta}_4\right)A_{1+4}$$
$$= \frac{\bar{\alpha}_2}{\alpha_1}\left[\left(\alpha_4 + \frac{\bar{\beta}_2}{\bar{\alpha}_2}\beta_4\right)\left(\alpha_3\frac{\beta_1}{\alpha_1} + \beta_3\right)A_{2-4} + (\bar{\alpha}_3\alpha_4 + \bar{\beta}_3\beta_4)\left(\frac{\beta_1}{\alpha_1} + \frac{\bar{\beta}_2}{\bar{\alpha}_2}\right)A_{3-4} + \left(\bar{\alpha}_3 - \frac{\bar{\beta}_2}{\bar{\alpha}_2}\beta_3\right)\left(\alpha_4\frac{\beta_1}{\alpha_1} - \beta_4\right)A_{2+3}\right., \tag{4.12}$$

$$\frac{8\omega_2\dot{\rho}e_2^2}{\omega_p^2}+\left(\bar{\alpha}_3+\frac{\beta_2}{\alpha_2}\bar{\beta}_3\right)\left(\alpha_3\frac{\beta_2}{\alpha_2}-\beta_3\right)A_{2-3}+\left(\bar{\alpha}_4+\frac{\beta_2}{\alpha_2}\bar{\beta}_4\right)\left(\alpha_4\frac{\beta_2}{\alpha_2}-\beta_4\right)A_{2+4}$$

$$+\left(\alpha_3-\frac{\beta_2}{\alpha_2}\beta_3\right)\left(\bar{\alpha}_3\frac{\beta_2}{\alpha_2}+\bar{\beta}_3\right)A_{2+3}+\left(\alpha_4-\frac{\beta_2}{\alpha_2}\beta_4\right)\left(\alpha_4\frac{\beta_2}{\alpha_2}+\beta_4\right)A_{2+4}$$

$$=\frac{\bar{\alpha}_1}{\alpha_2}\left(\bar{\alpha}_3-\frac{\bar{\beta}_1}{\bar{\alpha}_1}\bar{\beta}_3\right)\left(\alpha_4\frac{\beta_2}{\alpha_2}-\beta_4\right)A_{1+3}+\left(\frac{\beta_2}{\alpha_2}+\frac{\bar{\beta}_1}{\bar{\alpha}_1}\right)(\bar{\alpha}_3\alpha_4+\bar{\beta}_3\beta_4)A_{3-4}+\left(\alpha_4+\frac{\bar{\beta}_1}{\bar{\alpha}_1}\beta_4\right)\left(\bar{\alpha}_3\frac{\beta_2}{\alpha_2}+\bar{\beta}_3\right)A_{1-4}\Big]. \quad (4.13)$$

As in (2.18), here also $|\beta_j/\alpha_j|^2=1-e_j^2$.

A. Equations for PFS and PPF

By eliminating $\bar{\alpha}_1/\alpha_2$ between (4.10) and (4.11) the equation for the PFS is obtained:

$$\left[\left(\frac{8\omega\omega_1}{\omega_p^2}+\mu\right)e_1^2+\left(|\alpha_3|^2-\left|\frac{\beta_1}{\alpha_1}\right|^2|\beta_3|^2\right)(A_{1-3}+A_{1+3})+\left(|\alpha_4|^2-\left|\frac{\beta_1}{\alpha_1}\right|^2|\beta_4|^2\right)(A_{1-4}+A_{1+4})\right]$$

$$\times\left[\left(\frac{8\omega\omega_2}{\omega_p^2}+\mu\right)e_2^2+\left(|\alpha_3|^2-\left|\frac{\beta_2}{\alpha_2}\right|^2|\beta_3|^2\right)(A_{2-3}+A_{2+3})+\left(|\alpha_4|^2-\left|\frac{\beta_2}{\alpha_2}\right|^2|\beta_4|^2\right)(A_{2-4}+A_{2+4})\right]$$

$$=\left[\left(\alpha_4+\frac{\bar{\beta}_2}{\bar{\alpha}_2}\beta_4\right)\left(\bar{\alpha}_3+\frac{\beta_1}{\alpha_1}\bar{\beta}_3\right)A_{2-4}+(\bar{\alpha}_3\alpha_4+\bar{\beta}_3\beta_4)\left(1+\left|\frac{\beta_1}{\alpha_1}\right|\left|\frac{\beta_2}{\alpha_2}\right|\right)A_{3-4}+\left(\bar{\alpha}_3-\frac{\bar{\beta}_2}{\bar{\alpha}_2}\bar{\beta}_3\right)\left(\alpha_4-\frac{\beta_1}{\alpha_1}\beta_4\right)A_{2+3}\right]$$

$$\times\left[\left(\bar{\alpha}_4+\frac{\beta_1}{\alpha_1}\bar{\beta}_4\right)\left(\alpha_3+\frac{\bar{\beta}_2}{\bar{\alpha}_2}\beta_3\right)A_{1-4}+(\alpha_3\bar{\alpha}_4+\beta_3\bar{\beta}_4)\left(1+\left|\frac{\beta_1}{\alpha_1}\right|\left|\frac{\beta_2}{\alpha_2}\right|\right)A_{3-4}+\left(\alpha_3-\frac{\beta_1}{\alpha_1}\beta_3\right)\left(\bar{\alpha}_4-\frac{\bar{\beta}_2}{\bar{\alpha}_2}\bar{\beta}_4\right)A_{1+3}\right]. \quad (4.14)$$

Similarly by eliminating α_1/α_2 between (4.12) and (4.13), PPF is obtained as the solution of the quadratic

$$\left[\frac{8\omega_1e_1^2\dot{\rho}}{\omega_p^2}+\left(\bar{\alpha}_3+\frac{\beta_1}{\alpha_1}\bar{\beta}_3\right)\left(\alpha_3\frac{\beta_1}{\alpha_1}-\beta_3\right)A_{1-3}+\left(\bar{\alpha}_4+\frac{\beta_1}{\alpha_1}\bar{\beta}_4\right)\left(\alpha_4\frac{\beta_1}{\alpha_1}-\beta_4\right)A_{1-4}\right.$$

$$\left.+\left(\alpha_3-\frac{\beta_1}{\alpha_1}\beta_3\right)\left(\bar{\alpha}_3\frac{\beta_1}{\alpha_1}+\bar{\beta}_3\right)A_{1+3}+\left(\alpha_4-\frac{\beta_1}{\alpha_1}\beta_4\right)\left(\bar{\alpha}_4\frac{\beta_1}{\alpha_1}+\bar{\beta}_4\right)A_{1+4}\right]$$

$$\times\left[\frac{8\omega_2e_2^2\dot{\rho}}{\omega_p^2}+\left(\bar{\alpha}_3+\frac{\beta_2}{\alpha_2}\bar{\beta}_3\right)\left(\alpha_3\frac{\beta_2}{\alpha_2}-\beta_3\right)A_{2-3}+\left(\alpha_4-\frac{\beta_2}{\alpha_2}\beta_4\right)\left(\bar{\alpha}_4\frac{\beta_2}{\alpha_2}+\bar{\beta}_4\right)A_{2+4}\right.$$

$$\left.+\left(\bar{\alpha}_4+\frac{\beta_2}{\alpha_2}\bar{\beta}_4\right)\left(\alpha_4\frac{\beta_2}{\alpha_2}-\beta_4\right)A_{2-4}+\left(\bar{\alpha}_3+\frac{\beta_2}{\alpha_2}\bar{\beta}_3\right)\left(\alpha_3-\frac{\beta_2}{\alpha_2}\beta_3\right)A_{2+3}\right]$$

$$=\left[\left(\alpha_4+\frac{\beta_2}{\alpha_2}\beta_4\right)\left(\bar{\alpha}_3\frac{\beta_1}{\alpha_1}+\bar{\beta}_3\right)A_{2-4}+(\bar{\alpha}_3\alpha_4+\bar{\beta}_3\beta_4)\left(\frac{\beta_1}{\alpha_1}+\frac{\bar{\beta}_2}{\bar{\alpha}_2}\right)A_{3-4}+\left(\bar{\alpha}_3-\frac{\bar{\beta}_2}{\bar{\alpha}_2}\bar{\beta}_3\right)\left(\alpha_4\frac{\beta_1}{\alpha_1}-\beta_4\right)A_{2+3}\right]$$

$$\times\left[\left(\alpha_4+\frac{\bar{\beta}_1}{\bar{\alpha}_1}\beta_4\right)\left(\bar{\alpha}_3\frac{\beta_2}{\alpha_2}+\bar{\beta}_3\right)A_{1-4}+(\bar{\alpha}_3\alpha_4+\bar{\beta}_3\beta_4)\left(\frac{\beta_2}{\alpha_2}+\frac{\bar{\beta}_1}{\bar{\alpha}_1}\right)A_{3-4}+\left(\bar{\alpha}_3-\frac{\bar{\beta}_1}{\bar{\alpha}_1}\bar{\beta}_3\right)\left(\alpha_4\frac{\beta_2}{\alpha_2}-\beta_4\right)A_{1+3}\right]. \quad (4.15)$$

These two equations depend upon the parameters $A_{i\pm j}, e_j, \omega_j, \omega_p, \alpha_3, \beta_3, \alpha_4, \beta_4$. Using permissible values of these, equations (4.14) and (4.15) can have both real and complex roots. Their implications have been discussed at the end of this section.

Using the eccentricities defined by (2.25) and then taking limits for circular polarization, as in the deduction of (2.27), we find that (4.14) and (4.15) reduce to

$$\left[\frac{4\omega\omega_1}{\omega_p^2}+\frac{\mu}{2}+|\alpha_3|^2(A_{1-3}+A_{1+3})+|\alpha_4|^2(A_{1-4}+A_{1+4})\right]\left[\frac{4\omega\omega_2}{\omega_p^2}+\frac{\mu}{2}+|\alpha_3|^2(A_{2-3}+A_{2+3})+|\alpha_4|^2(A_{2-4}+A_{2+4})\right]$$

$$=|\alpha_3|^2|\alpha_4|^2(A_{2-3}+A_{3-4}+A_{2+3})(A_{1-4}+A_{3-4}+A_{1+3}), \quad (4.16)$$

$$\left(\frac{4\omega_1\dot{\rho}}{\omega_p^2}+|\alpha_3|^2A_{1+3}+|\alpha_4|^2A_{1+4}\right)\left(\frac{4\omega_2\dot{\rho}}{\omega_p^2}-|\alpha_3|^2A_{2-3}-|\alpha_4|^2A_{2-4}\right)=-|\alpha_3|^2|\alpha_4|^2A_{2-4}A_{1+3}. \quad (4.17)$$

Equations (4.16) and (4.17) depict a significant distinction between ω and $\dot{\rho}$. In the two sides of (4.16) in both factors the coefficients of $|\alpha_3|^2$ and $|\alpha_4|^2$ contain some linear combinations of $A_{i\pm j}$ such that if A_{1+j} occurs in one factor A_{2+j} will occur in the corresponding term of the other factor. But this rule is not followed in (4.17), because PPF depends on the nonsymmetric allowed polarizations and so on some microscopic details of the wave process which do not follow the rules of symmetry. But ω does not depend on these details.

The formulas for circular polarization, being simple, will be discussed mainly in the first part of Sec. IV B. They should be valid for the nearly-circular elliptical polarizations and are usually of the same type as the latter.

For the highest of the frequencies $\omega_j^2 \gg \omega_p^2$, $(\omega_i \pm \omega_j)^2 \gg \omega_p^2$, $k_j^2 c^2 \approx \omega_j^2$, $A_{i\pm j} \approx 0$; so (4.14) and (4.15) reduce to

$$\omega \approx -\frac{\mu\omega_p^2}{8\omega_1}, -\frac{\mu\omega_p^2}{8\omega_2}; \quad \dot{\rho} \approx 0 . \tag{4.17a}$$

This ω is real.

For plane polarized fields putting $\beta_j = 0$, we get $\dot{\rho} = 0$. If, for simplicity, we take $|\alpha_3|^2 \approx |\alpha_4|^2 = \alpha^2$, $\omega_j > 0$, and $\omega_i \approx \omega_j$ then ω is determined from the quadratic

$$4[(4\omega_j\omega/\alpha^2\omega_p^2) + 1 + A_{i-j} + A_{i+j}]^2 = (2A_{i-j} + A_{i+j})^2. \tag{4.17b}$$

ω has here two real and nonidentical values.

B. The two frames of reference

In this paper we deal with two frames of reference. One is the laboratory frame of the observer, or, the fixed inertial frame, in which the principal axes of the polarization ellipses are rotating at the rate of the precession frequency $\dot{\rho}$. The other, which can be called the principal polarization reference frame (or, in short, the PPRF) is fixed with the principal polarization axes. The field equations used, (3.1) to (3.4), and (3.5) are true in this frame.

Let $\vec{x}^0$ and $\vec{y}^0$ be the unit vectors along the rectangular coordinate axes $0x_0$ and $0y_0$ of the laboratory frame, and $\vec{x}, \vec{y}$ be those along $0x$ and $0y$ of the PPRF. Since the two frames coincide initially instantaneously, we have

$$\vec{x} = \vec{x}^0 \cos\rho + \vec{y}^0 \sin\rho ,$$
$$\vec{y} = -\vec{x}^0 \sin\rho + \vec{y}^0 \cos\rho . \tag{4.18}$$

In the laboratory frame the first harmonic electric field is given by

$$\vec{E}_1 = \frac{1}{2}\sum_{j=1}^{4}[\{\vec{x}^0(a_j \cos\rho_j + ib_j \sin\rho_j) + \vec{y}^0(a_j \sin\rho_j - ib_j \cos\rho_j)\} e^{i\theta_j} + \text{c.c.}] . \tag{4.19}$$

In the laboratory frame the determination of the density of energies and their fluxes in different forms should be useful, but according to the general theory of relativity (cf. Alfvén and Falthammar[12]) the electrodynamical equations do not have the usual form for rotating systems. Thus it seems that investigation of the energies in the laboratory frame should not proceed from our basic equations (3.1)–(3.4). As such and because this and other relevant points can be the theme of a separate paper, they have not been considered here.

C. Significance of complex ρ

In the moving frame the basic relations for $\dot{\rho}$ are (4.1), but in the fixed frame in which the linear relations of transformation, (4.18) and (4.19), are used, θ and ρ have similar involvements in the expression for $\vec{E}_1$.

To explain this point clearly, we write the x component of (4.19) in the complex form

$$E_{1x} = \frac{1}{4}\sum_j [-i(a_j e^{i\theta_j} - \bar{a}_j e^{-i\bar{\theta}_j})(e^{i\rho_j} + e^{-i\bar{\rho}_j}) + i(b_j e^{i\theta_j} + \bar{b}_j e^{-i\bar{\theta}_j})(e^{i\rho_j} - e^{-i\bar{\rho}_j})] , \tag{4.20}$$

where $\bar{\theta}_j$ and $\bar{\rho}_j$ are the complex conjugates of θ_j and ρ_j. Now let $\theta_j = \theta_j^0 + i\theta_j^1$, $\rho_j = \rho_j^0 + i\rho_j^1$ where θ_j^0, θ_j^1, ρ_j^0, ρ_j^1 are real. Then

$$E_{1x} = \sum_j \{e^{-(\theta_j^1 + \rho_j^1)}(|a_j| \sin(\theta_j^0 + \Omega_j t)\cos\rho_j^0 - |b_j| \cos(\theta_j^0 + \Omega_j t) \sin\rho_j^0\} , \tag{4.21}$$

where Ω_j contains the total frequency shift in the jth wave.

Hence a complex $\dot{\rho}$, just like a complex θ_j due to a complex ω will lead to exponentially growing fields of instability and PPF is complementary to ω. Evaluation of $\dot{\rho}$ is therefore at least as important as that of ω. For the location of the conditions for instability in a plasma, complex values of both ω and $\dot{\rho}$, or for at least one of them, should be searched for.

V. NEAR-RESONANT AND EXACTLY RESONANT INTERACTIONS

We find that complex values of $\dot{\rho}$ and ω occur in some cases of interaction in which the sum or difference of two of the wave frequencies is close to the characteristic plasma frequency ω_p. These are called the near-resonant interactions and can be studied by neglecting entirely the relativistic nonlinear corrections in the momentum-transfer equation (3.1).

For the evaluation of the roots of (4.14) and (4.15) it becomes necessary to approximately evaluate the dimensionless quantity $A_{i\pm j}$. Since $\omega_j^2 \gg \omega_p^2$ using (3.15a) and (3.15c) we can write

$$A_{i\pm j} \approx -\frac{\omega_p^2}{(\omega_i \pm \omega_j)^2 - \omega_p^2}\left[1 + \frac{(\omega_i \pm \omega_j)^2}{\omega_i \omega_j} + \frac{\omega_p^2(\omega_i \pm \omega_j)^2}{4\omega_j^2 \omega_j} + \cdots\right]. \quad (5.1)$$

If further $\omega_i \pm \omega_j = \omega_p(1-\delta)$, $0<\delta \ll 1$,

$$A_{i\pm j} \approx 1/2\delta . \quad (5.2)$$

A. Some near-resonant interactions

We consider the near-resonance defined by

$$\omega_4 - \omega_1 = \omega_2 + \omega_3 = \omega_p(1-\delta), \quad 0<\delta \ll 1, \quad (5.3)$$

and further specified by the following two sets of conditions:

$$\text{(a)}\quad \omega_1>0, \quad \omega_2>0, \quad \omega_3<0, \quad \omega_4>0, \quad \omega_1 \approx \omega_2 . \quad (5.4)$$

$$\omega_1 = \omega_0, \quad \omega_2 = \omega_0, \quad \omega_3 = -\omega_0 + \omega_p(1-\delta),$$
$$\omega_4 = \omega_0 + \omega_p(1-\delta), \quad \omega_0>0, \quad \omega_0^2 \gg \omega_p^2 ; \quad (5.5)$$

$$\text{(b)}\quad \omega_1>0, \quad \omega_2<0, \quad \omega_3>0, \quad \omega_4>0, \quad (5.6)$$

$$\omega_1 = \omega_0, \quad \omega_2 = -\omega_0 + \omega_p(1-\delta), \quad \omega_3 = \omega_0,$$
$$\omega_4 = \omega_0 + \omega_p(1-\delta), \quad \omega_0>0, \quad \omega_0^2 \gg \omega_p^2 . \quad (5.7)$$

Case (a): In this case by (5.1) we have

$$A_{1+4} = A_{2+4} \approx -5\omega_p^2/4\omega_0^2, \quad A_{1-3} = A_{2-3} = A_{3-4} \approx 3\omega_p^2/4\omega_0^2, \quad (5.8)$$

and by (5.2)

$$A_{1-4} = A_{2+3} = A_{1+3} = A_{2-4} \approx 1/2\delta . \quad (5.9)$$

The circular polarization formulas (4.16) and (4.17) give

$$\omega \approx \frac{\alpha^2\omega_p^2}{8\omega_0\delta}(-2 \pm \sqrt{2}),$$
$$\dot{\rho} \approx \frac{\alpha^2\omega_p^3}{4\omega_0^2}\left(\frac{\omega_p}{\omega_0} \pm \frac{i}{2\sqrt{\delta}}\right), \quad (5.10)$$

where

$$\alpha_3^2 \approx \alpha_4^2 = \alpha^2 . \quad (5.11)$$

This $\dot{\rho}$ is complex and ω is real.

For elliptical polarizations using (5.9), (5.10), and (5.11), dropping those terms which have in the numerator $e_j^2 - e_i^2$ (for $j \neq i$), and assuming $e_j^2 \ll 1$, $e_1^2 = e_2^2 = e_0^2$, $e_3^2 = e_4^2 = e'^2$, we find that (4.15) reduces to

$$\dot{\rho} \approx \frac{\alpha^2\omega_p^3}{4\omega_0^2}\frac{1}{2}\left(1 + \frac{e'^2}{e_0^2}\right)\left(\frac{\omega_p}{\omega_0} \pm \frac{i}{2\sqrt{\delta}}\right). \quad (5.12)$$

As in (5.10), ω of (4.14) will be real.

Case (b): In this case by (5.1) we have

$$A_{1-3} \approx 1, \quad A_{1-2} = A_{2-3} = A_{2-4} \approx 3\omega_p^2/4\omega_0^2,$$
$$A_{1+3} = A_{1+4} \approx -5\omega_p^2/4\omega_0^2, \quad A_{2+4} \approx -\tfrac{1}{3}, \quad (5.13)$$

and by (5.2)

$$A_{1+2} = A_{1-4} = A_{2+3} = A_{3-4} \approx 1/2\delta . \quad (5.14)$$

The circular polarization formulas (4.16) and (4.17) give

$$\omega \approx \frac{\alpha^2\omega_p^2}{4\omega_0}\left(-\frac{2}{3} \pm \frac{i\sqrt{3}}{2\delta}\right), \quad \dot{\rho} \approx \frac{\alpha^2\omega_p^4}{16\omega_0^3}(2 \pm 7). \quad (5.15)$$

This ω is complex and $\dot{\rho}$ real.

1. Connection with the Poynting flux

Let P_j be the static part of the Poynting flux of the wave at the frequency ω_j. Then

$$P_j = \left|\langle (c/4\pi)\,[\vec{E}_j \times \vec{H}_j]\rangle\right| = (k_j c/\omega_j)(c/8\pi)(\alpha_j^2 + b_j^2) . \quad (5.16)$$

When $k_j c \approx \omega_j$ and $\alpha_j^2 \approx \beta_j^2$ we get

$$\alpha_j^2 \approx \frac{P_j}{\omega_j^2} \times 1.5 \times 10^5 . \quad (5.17)$$

To overcome losses due to friction, the pumps must be greater than their threshold values. The threshold power estimation in Ref. 4 [Secs. II and IV, and particularly Eq. (4.6)] can be used here because the expressions for ω obtained there and in Ref. 4 are quantitatively of the same order. Hence we find that

$$\nu \approx \frac{8\pi e^2}{m^2c^3}\frac{(P_3P_4)^{1/2}}{4\sqrt{2}\,\omega_0\delta}, \quad (5.18)$$

where ν is the collision frequency of momentum transfer per unit mass from electrons to ions.

2. Numerical estimation

For numerical calculations we take

$$\omega_0 = 10^{15}\ \text{rad/sec}, \quad \omega_p = 10^{11}\ \text{rad/sec}, \quad \delta = 0.1 . \quad (5.19)$$

Then the threshold power, when $P_3 \approx P_4$, is about $\nu \times 1.9 \times 10^9$ erg/cm^2 sec.

For the evaluation of ω and $\dot{\rho}$ we further put

$$P_3 \approx P_4 = 10^{23}\ \text{erg/cm}^2\,\text{sec} \quad (5.20)$$

and obtain the following values in units of rad/sec: for Eq. (4.17a)

$$\omega \approx -3.7 \times 10^4 ;$$

for (5.10)

$$\omega \approx (-1.1, -6.5) \times 10^5, \quad \dot{\rho} \approx 3.7 \times (10^{-4} \pm i1.6);$$

for (5.15)

$$\omega \approx (-0.26 \pm i3.24) \times 10^5, \quad \dot{\rho} \approx (0.85, -0.47) \times 10^{-3}. \tag{5.21}$$

B. A case of exact resonance

In the resonant interactions defined by

$$(\omega_4 - \omega_2)^2 = (\omega_1 + \omega_3)^2 = \omega_p^2 \tag{5.22}$$

some terms of the solution for $\vec{v}_2$, obtained from (3.9), become secular and so proportional to $t \sin\omega_p t$ and $t \cos\omega_p t$. Retaining therefore only the secular terms, we get

$$\vec{v}_2 = + \frac{\vec{z} e^2 t}{8m^2} \left(\frac{i(a_1 a_3 - b_1 b_3)(k_1 + k_3)}{\omega_1 \omega_3} e^{i(\theta_1 + \theta_2)} + \text{c.c.} - \frac{i(a_2 a_4 + b_2 b_4)(k_4 - k_2)}{\omega_2 \omega_4} e^{i(\theta_4 - \theta_2)} + \text{c.c.} \right) + \cdots . \tag{5.23}$$

Using (3.9) and (3.10) with (5.23), n_2 and $\vec{E}_2$ are obtained. The third-order secular sources proportional to $te^{\pm i\theta_1}$ and $te^{\pm i\theta_2}$ can then be evaluated easily. Some of these are parametrically excited because (2.2) is used. The relativistic third-order source, $(\partial/\partial t)v_1^2 \vec{v}_1/2c^2$, is entirely nonsecular. This and all other nonsecular sources are totally neglected because the secular terms grow indefinitely in time.

The equation for the amplitudes of the first harmonic fields, correct up to third order, is

$$\begin{aligned} &\{\vec{x}[\omega_1(-2i\omega_1\dot{\alpha}_1 + \ddot{\alpha}_1 - \alpha_1\dot{\rho}^2 + 2\beta_1\omega_1\dot{\rho} + 2i\dot{\beta}_1\dot{\rho} + i\beta_1\ddot{\rho})e^{i\theta_1} + \text{c.c.}] \\ &\quad + \vec{y}[\omega_1(-2i\omega_1\alpha_1\dot{\rho} + 2\dot{\alpha}_1\dot{\rho} + \alpha_1\ddot{\rho} - 2\dot{\beta}_1\omega_1 - i\ddot{\beta}_1 + i\beta_1\dot{\rho}^2)e^{i\theta_1} + \text{c.c.}] + \cdots\} \\ &\quad = \frac{te^2\omega p}{8}\{[\omega_1(-i\vec{x}\bar{\alpha}_3 + \vec{y}\bar{\beta}_3)e^{i\theta_1} + \omega_2(-i\vec{x}\bar{\alpha}_4 - \vec{y}\bar{\beta}_4)e^{i\theta_2}][(\alpha_1\alpha_3 - \beta_1\beta_3)(k_1 + k_3)^2 - (\alpha_2\alpha_4 + \beta_2\beta_4)(k_4 - k_2)^2] + \text{c.c.}\}. \end{aligned} \tag{5.24}$$

When the coefficients of $\vec{x}e^{i\theta_1}, \vec{y}e^{i\theta_1}, \vec{x}e^{i\theta_2}, \vec{y}e^{i\theta_2}$ are equated, complicated nonlinear equations for $\alpha_1, \beta_1, \alpha_2, \beta_2, \dot{\rho}$ will be obtained. Since the sources are secular the quantities neglected earlier, $\ddot{\alpha}_j$, $\ddot{\beta}_j$, $\ddot{\rho}$, $\dot{\rho}^2$, $\dot{\alpha}_j\dot{\rho}$, $\dot{\beta}_j\dot{\rho}_g$ are no longer negligible. Hence the resulting equations and their solutions are not simple like (4.2), (4.3d), (4.6) and their solutions. Actually the approximate solutions of (5.26) are valid in a time which is very small and of the order of the time estimated in Sec. V of Ref. 4, but within this time instability sets in. As a result the system breaks up and so any more rigorous estimate of it is felt unnecessary.

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