Statistical properties of a one-dimensional radiant cavity*

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The statistical properties of a classical electromagnetic field in interaction with matter are investigated. To this end a nonlinear extension of a model proposed elsewhere is studied by numerically solving the Newton-Maxwell equations of motion. The time-average energy distribution of the electromagnetic normal modes is also computed. It is shown that nonlinearity, no matter how large, does not completely destroy the dependence of the final energy distribution on initial conditions. One is therefore led to the conclusion that, as far as one is concerned with electrodynamical systems of finite total energy, no statistical behavior is to be expected. In particular, the Rayleigh-Jeans distribution law is not a general consequence of classical physics. The dependence on initial conditions can be removed, however, by the introduction of white noise delivering an infinite amount of energy to the radiation field. In this case equipartition of energy is reached, but in accordance with an old conjecture by Jeans, this process takes place at a nonuniform rate, the energy transfer being slower at higher frequencies.

I. INTRODUCTION

Since 1954 there has been renewed interest in the ergodic problem of classical physics. The main impulse to these studies was given by Fermi's numerical experiments on an anharmonic one-dimensional chain of atoms[1] and by the analytical results culminating in the celebrated theorem by Kolmogorov, Arnold, and Moser on the permanence of invariant tori in the phase space of integrable systems under small perturbations.[2] In particular, several systems of interacting particles have been extensively studied,[3-16] in order to check the traditional idea that such systems should behave ergodically under any nonlinear perturbation, no matter how small. The ultimate goal of these investigations was to establish whether the classical dynamics of an ideal crystal necessarily entails equipartition of energy together with its troublesome consequences (e.g., temperature-independent specific heats).

As a matter of fact, according to the numerical experiments hitherto performed, a fairly large set of invariant surfaces seems to persist in phase space at least at sufficiently low excitation energies, thus precluding ergodicity. Whether these facts have any relevant influence on the thermodynamics of macroscopic systems still remains an open question.

On the other hand, the point perhaps most critical in classical statistical physics, namely, the problem of black-body radiation, has not yet been reconsidered on similar grounds. Indeed, the problem of building the statistical mechanics of a system of charges interacting with a radiation field inside a cavity exhibits as an essential feature indefinitely many degrees of freedom. In the phase space of such a system there is no invariant measure at hand by means of which to define a microcanonical ensemble. Even the usual notion of ergodicity, which is an essential tool in the finite dimensional case, cannot be transferred to this problem. It is very doubtful, therefore, that the equipartition theorem is a necessary consequence of a statistical theory built on purely classical grounds. On the other hand, one could request ergodicity in the restricted sense that time averages should be to some extent independent of the initial conditions, since this would be enough to justify statistical methods. Should this be the case, at least in a suitable range of energies, then the problem could be posed, whether equipartition or another distribution law follows.

The free electromagnetic field inside a perfectly reflecting box wall is a linear system: it possesses a spectrum of proper vibrations, the so-called normal modes of the cavity, each of which represents an independent degree of freedom. Therefore, it obviously cannot exhibit ergodicity in the above sense, since the initial conditions are not wiped out in the course of the system motion. Some mechanism of energy exchange between the normal modes is needed in order to bring about statistical behavior. One might, therefore, consider a cavity inside which a number of charged particles interact with their own radiation field. Formidable difficulties, however, arise in describing this interaction, which can only be circumvented by setting up simplified models amenable to effective computations.

In the present paper, we study a model[17] which, though extremely simplified, still retains one essential feature: it provides a nonlinear interaction between the normal modes of the field. In Sec. II, the model is described and discussed; in Sec. III the linear case is analytically solved. The numerical results referring to the nonlinear case are presented in Sec. IV. Section V is devoted to a stochastic generalization of the model. Finally, discussion of the results and concluding remarks are contained in Sec. VI and VII.

II. DISCUSSION OF THE MODEL

We consider a uniformly charged infinite plane moving vertically in the z-y plane of a fixed reference frame. The plane is situated midway between two perfectly reflecting plane mirrors a distance $2l$ apart, and is subject to an external restoring force per unit area $F(z)$, where z is the displacement from the equilibrium position.

The relevant equations of motion in Gaussian units are easily found to be[17]

$$\frac{\partial^2 A}{\partial x^2}-\frac{1}{c^2}\frac{\partial^2 A}{\partial t^2}=-\frac{4\pi}{c}\sigma\delta(x)\dot{z}, \tag{1}$$

$$m\ddot{z}=-(\sigma/c)\partial A(0,t)/\partial t+F(z), \tag{2}$$

where A is the z component of the vector potential, $\dot{z}$ is the volocity, σ and m are the charge and mass densities of the oscillating plane, and $\delta(x)$ is the Dirac delta function. The boundary conditions on the surfaces of the mirrors can be satisfied, without loss of generality, by imposing that

$$A(l,t)=A(-l,t)=0. \tag{3}$$

Equations (1) and (2) are obtained on the assumption that

$$A_z(\vec{r},0)=f(x), \quad \partial A_z(\vec{r},0)/\partial t=g(x). \tag{4}$$

With initial conditions (4) the model is one dimensional, since at any later time A will be a function of x, t only. The basic objection that can be raised against the physical significance of the model is that the Lorentz force, being counterbalanced by the constraints, does not play any role. This situation is of course common to every one-dimensional model; in particular one should note that even in Planck's model the motion of a harmonic oscillator in a radiation field was studied by means of an energy balance without taking into account the Lorentz force. On the other hand, as can be seen from Eqs. (B1), our model provides a nonlinear interaction between the normal modes of the field. Let us now put Eqs. (1) and (2) into a form more suitable to numerical integration.

Expanding the functions $f(x)$, $g(x)$ defined in (4) into Fourier series in the interval $(-l, l)$ we get

$$\begin{aligned} f(x)&=\sum_n a_n \sin[\omega_n(x+l)/c], \\ g(x)&=\sum_n b_n \sin[\omega_n(x+l)/c], \end{aligned} \tag{5}$$

where the $\omega_n=\pi cn/2l$ are the angular frequencies of the unperturbed normal modes of the "cavity." Thus, the general integral $A^0(x,t)$ of the homogeneous equation

$$\frac{\partial^2 A^0}{\partial x^2}-\frac{1}{c^2}\frac{\partial^2 A^0}{\partial t^2}=0 \tag{6}$$

satisfying the initial and boundary conditions can be written as follows:

$$A^0(x,t)=\sum\left(a_n\cos(\omega_n t)+\frac{b_n}{\omega_n}\sin(\omega_n t)\right)\times\sin\left(\frac{\omega_n}{c}(x+l)\right). \tag{7}$$

Equation (1) can be written in the integral form

$$A(x,t)=A^0(x,t)+2\pi\sigma\int_0^t \dot{z}(t-\tau)H(c\tau,|x|)\,d\tau, \tag{8}$$

with

$$H(c\tau,|x|)=\theta(c\tau-|x|)+\sum(-1)^n\{\theta[c\tau-(2ln-|x|)]+\theta[c\tau-(2ln+|x|)]\},$$

where θ is the Heaviside step function. From Eq. (8) one gets

$$\frac{\partial A(0,t)}{\partial t}=\frac{\partial A^0(0,t)}{\partial t}+2\pi\sigma\times\left[\dot{z}(t)+2\sum(-1)^n\dot{z}\left(t-\frac{2ln}{c}\right)\theta\left(t-\frac{2ln}{c}\right)\right], \tag{9}$$

and, substituting Eq. (9) into Eq. (2), one obtains

$$m\ddot{z}=F(z)-\frac{\sigma}{c}\frac{\partial A^0(0,t)}{\partial t}+\frac{2\pi\sigma^2}{c}\times\left[\dot{z}(t)+2\sum(-1)^n\theta\left(t-\frac{2ln}{c}\right)\dot{z}\left(t-\frac{2ln}{c}\right)\right]. \tag{10}$$

Thus the solution to Eqs. (1) and (2) reduces to the solution of the simpler Eq. (10). Actually the sum on the right-hand side of Eq. (10) involves, at any time, a finite number of terms. This feature makes the present model particularly simple to numerically analyze.

Indeed if one defines the sequence of functions in the interval $(0,\ \tau=2l/c)$

$$z_j(t-j\tau)=z(t) \quad \text{with } j\tau \le t \le (j+1)\tau, \quad j=0,1,\ldots, \tag{11}$$

then Eq. (10) is equivalent to the following sequence of equations:

$$m\ddot{z}_j+F(z_j)=-\frac{2\pi\sigma^2}{c}\left[\dot{z}_j(t)+S_j(t)\right]-\frac{\sigma}{c}\frac{\partial A^0(t+j\tau)}{\partial t} \quad (0\le t\le\tau), \tag{12a}$$

$$S_0(t)=0, \tag{12b}$$

$$S_{j+1}(t)=-2\dot{z}_j(t)-S_j(t). \tag{12c}$$

This system of ordinary differential equations can be solved recursively.

III. LINEAR CASE

Despite the fact that the physically significant case is the nonlinear one, the linear case, in which $F(z)=-m\omega_0^2 z$, deserves some attention since it is analytically integrable and may provide a useful test of the accuracy of numerical computations.

In this case the term $-(\sigma/c)\partial A^0(0,t)/\partial t$ represents an arbitrary driving force, whose effect superimposes linearly to the solution of the equations. Therefore, this term may be dropped without loss of physical generality.

Then the system of Eqs. (1) and (2) can be solved (through lengthy calculations relegated to Appendix A) by means of Laplace transforms. One finds

$$z(t)=\sum_n z_n^*\cos(\omega_n^* t+\xi_n^*)$$

$$z_n^*=C/D,$$

where

$$C=2(v_0^2+z_0^2\omega_0^4/\omega_n^{*2})^{1/2},$$

$$D=2\omega_n^*+\frac{\omega_0^2-\omega_n^{*2}}{\omega_n^*}+\gamma\omega_n^*\left[1+\frac{l^2}{\gamma^2c^2}\left(\frac{\omega_0^2-\omega_n^{*2}}{\omega_n^*}\right)^2\right],$$

$$\xi_n^*=\arctan\left(-\frac{\omega_n^* v_0}{z_0\omega_0^2(v_0^2+z_0^2\omega_0^4/\omega_n^{*2})^{1/2}}\right), \tag{13}$$

where $\gamma=2\pi l\sigma^2/mc^2$ and ω_n^* are the angular frequencies of the normal modes of the system charged plane plus field and are given by the solutions of

$$\tan(\omega^* l/c)=(l/\gamma c)(\omega_0^2-\omega^{*2})/\omega^*. \tag{14}$$

v_0 and z_0 are the initial velocity and the initial displacement, respectively, of the plane.

Solution (13) describes a quasiperiodic motion of the system, resulting from the superposition of oscillations of frequencies ω_n^*. One can easily verify (see Appendix B) that these frequencies are associated with stationary solutions of the problem of the form

$$u_n=A_n\sin[\omega_n^*(|x|-l)/c]. \tag{15}$$

The u_n can be thought of as the "normal modes" of the total system charges plus field. To each u_n is associated an independent constant of the motion. One sees therefore that under a linear interaction the constants of the motion are not destroyed but merely transformed into new constants. The exchange of energy introduced by linear interaction cannot yield the desired statistical properties, and one is thus led to consider a nonlinear interaction.

IV. THE NONLINEAR CASE

Let us consider now the physically more interesting case when $F(z)$ is nonlinear, e.g.,

$$F(z)=-m\omega_0^2 z-\alpha z^3. \tag{16}$$

The equations of motion are now no longer analytically solvable. A numerical approach is however possible relying upon Eq. (12).

Note that once initial conditions are specified, $A^0(x,t)$ and therefore $\partial A^0(0,t)\partial t$ can be analytically computed from Eq. (7). In the actual computations Eq. (12a) was integrated in each interval

$$j\tau\le t\le(j+1)\tau \quad (j=0,1,\ldots)$$

by means of a standard four-step Runge-Kutta method, the values of S_j being determined recursively from Eqs. (12b) and (12c). As for the distribution of energy among the field normal modes, notice (Appendix B) that the interacting (odd) modes obey the equation

$$\ddot{a}_n+\omega_n^2 a_n=2(\pi/l)^{1/2}\sigma\dot{z}. \tag{17}$$

From this, letting

$$\xi_n=\dot{a}_n+i\omega_n a_n, \tag{18}$$

one easily finds

$$\xi_n(t)=e^{i\omega_n t}\left[\int_0^t 2\left(\frac{\pi}{l}\right)^{1/2}\sigma\dot{z}e^{-i\omega_n s}ds+\xi_n(0)\right]. \tag{19}$$

The time-average energy of the nth normal mode is, therefore,

$$\langle E_n\rangle=\lim_{T\to\infty}\frac{1}{2T}\int_0^T|\xi_n(t)|^2dt. \tag{20}$$

Once $\dot{z}$ has been computed as explained above one can compute a certain number of average normal-mode energies by numerical integration of Eq. (17).

As is shown in Appendix C, for the linear case

($\alpha = 0$) the time-average normal-mode energies can be analytically evaluated. This circumstance provided us a supplementary useful tool to check the accuracy of our numerical computations. We numerically integrated our system for different values of α and $\beta = 2\pi\sigma^2/c$ and for different initial excitations, while some of the parameters have been held fixed throughout all the computations; namely, we choose[18]

$$l = \pi, \quad m = 1, \quad c = 1, \quad \omega_0 = 5 .$$

The numercial integration was carried on until stabilization of time-average quantities was reached.[19]

One should make sure that the energy redistribution produced by the nonlinearity alone does not involve a time scale far exceeding that involved in the linear case. A useful test would be the initial excitation of a normal mode of the linear system. One might use to this end the Fourier coefficients of an eigenstate $w_{\omega*}$ (Appendix B) to find $A^0(x, t)$ from Eq. (7). A more convenient procedure consists in the excitation of the $(2n+1)-st$ unperturbed normal mode, which differs (in energy) from $w_{\omega^*_{2n+1}}$ by $o(1/n^2)$. A typical result is shown in Fig. 1 which represents the time-average energy of the mechanical oscillator. The upper part of the figure corresponds to the initial excitation of the fifth unperturbed normal mode while the lower part corresponds to the initial excitation of the 99th unperturbed normal mode. In the latter case the difference in energy from the 99th perturbed normal mode is ~10^{-3} times the total energy involved. Comparison of the two transients in Fig. 1 leads to think that the two time-scales are of the same order of magnitude.

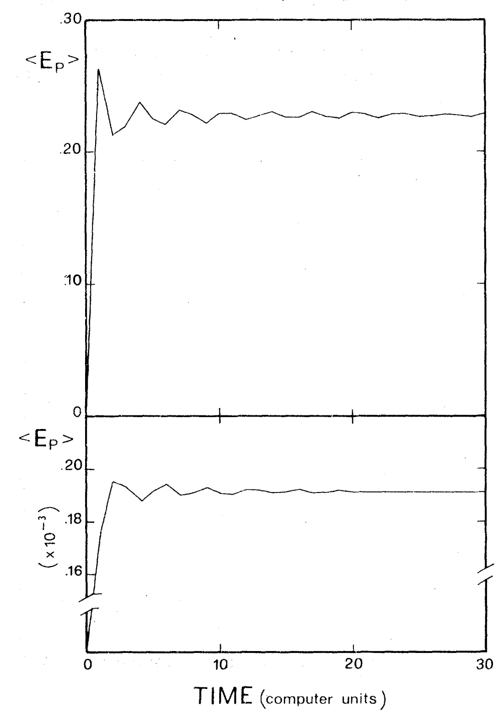

FIG. 1. Time-average energy of the mechanical oscillator as a function of time. Here, $\beta = 1$, $\alpha = 10$, and total energy $E = 1$. The upper part of the graph corresponds to the initial excitation of the fifth unperturbed normal mode. The lower part corresponds to the initial excitation of the 99th unperturbed normal mode.

In each run we computed the time-average energy of a limited number of normal modes, and typical results are shown in Figs. 2–5. Figure 2 refers to the linear case for three different values of β. Initially, the total energy is given to the charged plane. Note that only a small number of normal modes close to the proper frequency of the mechanical oscillator are significantly excited. This number, however, as well as the energy transferred from the charged plane to the field, increases with the charge of the oscillator.

The curves of Fig. 2 were obtained both by integration of the equations of motion and by use of

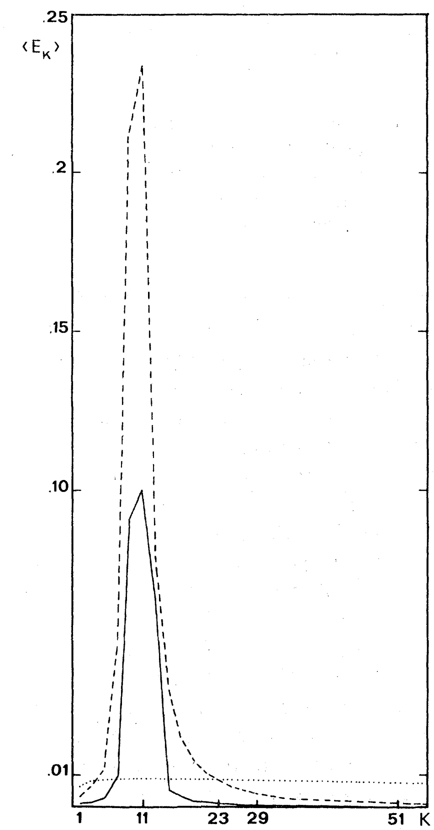

FIG. 2. Typical linear case ($\alpha = 0$). Here, the total energy is $E = 1$ and initially only the mechanical oscillator is excited. Full line, $\beta = 0.1$; dashed line, $\beta = 1$; dotted line, $\beta = 70$. The energy transfer to the field grows with β.

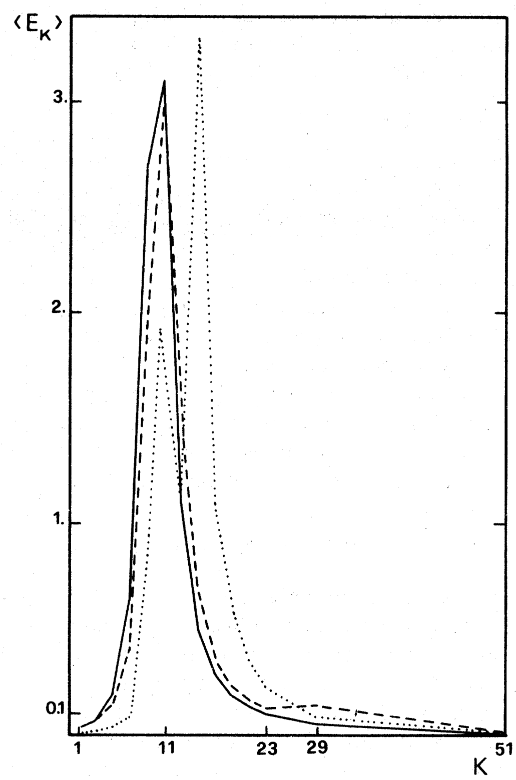

FIG. 3. Few typical nonlinear cases with $\beta = 1$. $\alpha = 1$, full line; $\alpha = 10$, dashed line; $\alpha = 100$, dotted line. The total energy $E = 12.5$ was initially given to the mechanical oscillator. In the three cases the average mechanical energy was 2.52, 2.86, and 1.4, respectively.

formula (C2). The agreement of the results confirms the accuracy of our numerical computations.

No dramatic changes appear when the nonlinearity is introduced. The only relevant fact, as is shown in Fig. 3, is that when α increases a growing number of modes to the right of ω_0 are significantly excited. The explanation of this fact is given in Sec. VII.

Figures 4 and 5 refer to the case when one normal mode is initially excited besides the mechanical oscillator. Note that the exchange of energy is larger when the frequency of the initially excited mode is close to the proper frequency of the mechanical oscillator. It is also apparent that by increasing α or β a relevant exchange of energy takes place among an increasing number of modes. Nevertheless, in no case is the exchange of energy large enough to destroy the dependence on the initial conditions.

V. STOCHASTIC CASE

For the sake of completeness we summarize here some results which will be presented in full

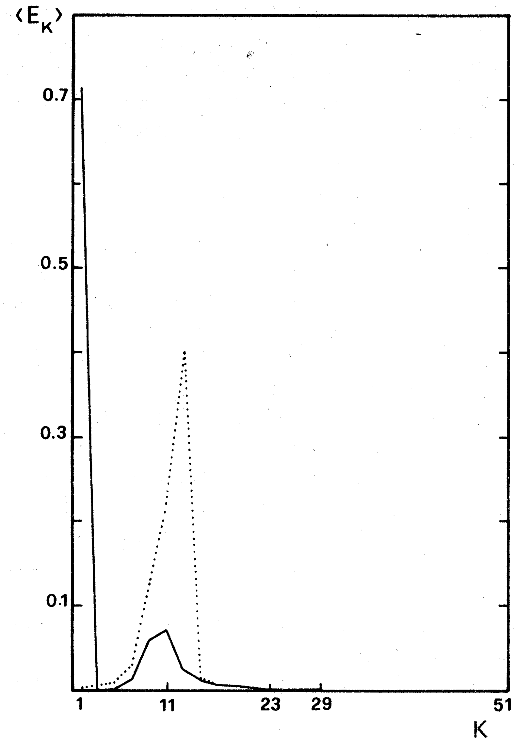

FIG. 4. Nonlinear case with $\beta = 1$, $\alpha = 10$, and total energy $E = 1$. The mechanical oscillator was given initially an energy of 0.3 units. Full line: mode 1 initially excited; dashed line: mode 13 initially excited. The average mechanical energies were 0.07 and 0.02, respectively.

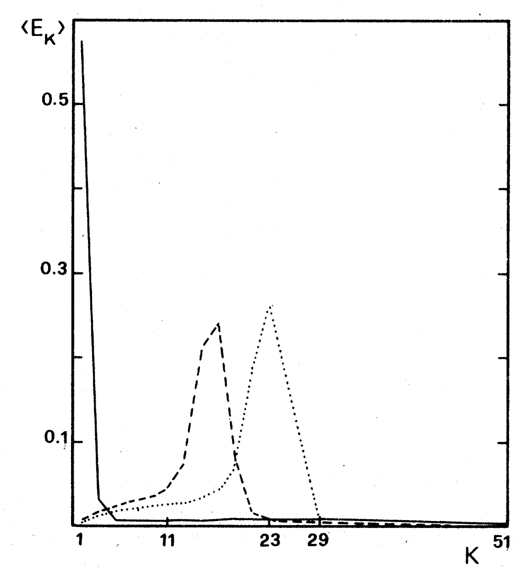

FIG. 5. Nonlinear case with $\beta = 200$, $\alpha = 200$, and total energy $E = 1$. Initial mechanical energy was 0.3. The full, dashed, and dotted lines correspond to initial excitation of modes 1, 17, and 23, respectively. In the three cases the average mechanical energies were 0.13, 0.02, and 0.01, respectively.

detail elsewhere.[18] It is apparent that the mechanical oscillator behaves too regularly to bring about any statistical effect. It is interesting therefore to investigate the opposite case, when a random force affects the oscillator. We choose a random force

$$F(\dot{z}, t) = -\gamma m \dot{z} + (2\gamma m \beta^{-1})\eta(t), \tag{21}$$

where γ is a friction coefficient, $\beta^{-1} = KT$, T being the absolute temperature of a gas whose interaction with the oscillator is supposed to account for the random force, and $\eta(t)$ is Gaussian white noise. When the electromagnetic force is removed from (2), one recovers the familiar Langevin equation.

The stochastic model so obtained exhibits a number of rather unphysical features, e.g., the assumption that the statistics of the gas undergoes no change under the interaction with the oscillator, and the presence of equally represented frequencies, no matter how high, in the power spectrum of $\eta(t)$.

These weaknesses cannot be neglected as they crucially affect the final results. A Fourier expansion performed on Eqs. (1) and (2) yields the system

$$\dot{c}_{2n+1} = \omega_{2n+1} c_{2n+2},$$
$$\dot{c}_{2n+2} = -\omega_{2n+1} c_{2n+1} + 2(2/ml)^{1/2}\pi\sigma c_0, \quad n = 1, 2, \ldots,$$
$$\dot{c}_0 = -\gamma c_0 - \left(\frac{2}{ml}\right)^{1/2} \sigma \sum_0^\infty c_{2n+2} + \theta\eta(t), \tag{22}$$

where c_{2n+1} is $(2lm^{-1})^{1/2}\omega_{2n+1}$ times the Fourier coefficient of $A(x)$ corresponding to the eigenfunction $(2l)^{-1/2}\exp(i\omega_{2n+1}x/c)$, c_{2n+2} is defined by the first equation, and $c_0 = \dot{z}$. The linear system (22) can be diagonalized by a linear transformation, similar to the one of Appendix B from the variables c_n to new variables K_n; the resulting equations are

$$\dot{K}_n = \Omega_n K_n + \lambda_n \eta(t), \quad n = 0, 1, 2, \ldots, \tag{23}$$

where the λ_n's are coefficients arising from the diagonalizing transformation, and Ω_n are the eigenvalues. The system (23) can now be solved by

$$K_n(t) = h_n \exp(\Omega_n t) + \lambda_n \int_0^t \exp[\Omega_n(t-u)]\, dW(u), \tag{24}$$

where h_n are the nonrandom initial conditions, and $W(t)$ is the stochastic Wiener's process [whose formal derivative is the white-noise process $\eta(t)$]. Finally, from Eq. (24) one can compute the time-average energy $\langle E_{2n+1}\rangle$ of the odd interacting modes. One finds

$$\langle E_{2n+1}\rangle = 2KT + O(\sigma^2). \tag{25}$$

The seemingly paradoxical result that $\langle E_{2n+1}\rangle$ does not vanish for $\sigma = 0$ is accounted for by the fact that the time averages tend to their limits for $T \to \infty$ nonuniformly with respect to σ. Moreover, averaging over a finite time interval t yields

$$\lim_{n\to\infty} \langle E_{2n+1}(t)\rangle = 0,$$

however large t may be. Again this is due to nonuniformity of the limit for $t \to \infty$ with respect to n.[19] This circumstance rules out effective statistical equilibrium.

Indeed, relaxation of $\langle E_{2n+1}\rangle$ to its limit, as well as the decay of its dependence on h_{2n+1}, is scaled by relaxation times $1/\mathrm{Re}\,\Omega = 0(n^2)$. Therefore one cannot find a time t large enough for the time average up to time t of *all* the E_{2n+1}'s to approximate their limiting values within a prescribed range of accuracy, or their dependence on the initial conditions to have reduced under a prescribed amount.

VI. DISCUSSION OF THE RESULTS

Numerical experiments indicate that the dependence of the spectrum on initial conditions is weakened but not destroyed by an increase in one or more of the parameters: charge, anharmonicity, energy. These results can be explained, qualitatively at least, in a relatively simple way.

In the linear (harmonic) case, the spectrum is peaked in correspondence to the normal frequency closest to the proper frequency of the mechanical oscillator. In other words the oscillator gives up appreciable energy only to those normal (unperturbed) modes whose frequencies are close to its own.

In order to understand this behavior in the anharmonic case let us notice that an oscillator of energy E obeying the equation

$$\ddot{x} = -\omega^2 x - \alpha x^3 \tag{26}$$

will have a period given by

$$T = \int_0^{2\pi} (\omega^2 + 4E\alpha\omega^{-2}\cos^4\varphi)^{-1/2} d\varphi. \tag{27}$$

The elliptic integral (27) can be approximated by

$$T \simeq \pi\sqrt{2}\,(\tfrac{1}{4}\omega^4 + E\alpha)^{-1/4}. \tag{28}$$

On the other hand, when the oscillator interacts with the field, formula (28), where E is now understood as total energy, will presumably provide an upper bound for the range of frequencies swept by the oscillator in the course of its motion.

It is presumably in this range that effective energy interchange between the electromagnetic and the mechanical degrees of freedom occurs. Therefore, in view of formula (28), one expects the highest frequencies of the cavity to be only slightly affected by the charged oscillator both in the linear and the nonlinear case.

Should this interpretation be correct, then any increase in the product $E\alpha$ will not alter the basic qualitative feature of the phenomena.

When a stochastic perturbation is added, an ergodic behavior arises. However, the limiting values of the time averages are not uniformly approached; therefore no actual macroscopic observation will detect a significant excitation of the highest normal modes. This fact is not a variance with our numerical results since ergodicity is only obtained at the cost of introducing unphysical elements in the model. To summarize, when finite energies are involved one cannot expect any tendency towards statistical equilibrium. One possible picture to explain our results can be borrowed from the behavior of quasi-integrable finite systems[20]: namely, only those modes whose initial energy is beyond some critical threshold will effectively exchange energy. In our case, therefore, only a finite number of modes will take part in the energy diffusion process.

On the other hand, when an infinite amount of energy is available to the system a tendency towards equipartition of energy appears, even though a real equilibrium state is never reached.

VII. CONCLUDING REMARKS

Finally, we want to speculate somewhat about the bearing of the above results on the general problem of equilibrium between charges and field.

It was seen that the main feature giving rise to the resonant and therefore nonergodic behavior of the model is that there is just one mechanical degree of freedom whose frequency band superimposes to a limited number of field frequencies.

Now note that, quite generally, interaction among field normal modes only takes place indirectly via interaction with one or more mechanical degrees of freedom. One is therefore led to conjecture that the bandwidth of electromagnetic frequencies effectively taking part in the energy sharing will depend on the bandwidth of frequencies occurring in the mechanical motion and will therefore remain limited as far as the motion remains nonsingular.

In view of the above considerations, we are inclined to think that our results are more a consequence of classical electrodynamics than of the very special structure of our model.

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APPENDIX A: SOLUTION OF EQUATION (8)

Equation (8) can be solved by means of Laplace transforms. If $\tilde{g}(x,s)$ indicates the L transform of a function $g(x,t)$, we have

$$\tilde{A}(x,s) = \tilde{A}^0(x,s) + 2\pi\sigma\tilde{\dot{z}}(s)\tilde{H}(cs,|x|)\,, \qquad \text{(A1)}$$

where $s=\eta+i\omega$ is the complex parameter of the L transform.

From Eq. (2) one obtains

$$\dot{z}(t) = v_0 - \frac{\sigma}{mc}[A(0,t) - f(0)] - \omega_0^2\int_0^t z(\tau)\,d\tau\,, \qquad \text{(A2)}$$

where $f(0)=A(0,0)$. It follows that

$$\begin{aligned}\tilde{\dot{z}}(s) &= s\tilde{z}(s) - z(0)\\ &= v_0/s - (\sigma/mc)[\tilde{A}(0,s) - f(0)/s] - \omega_0^2\tilde{z}(s)/s\,.\end{aligned} \qquad \text{(A3)}$$

The L transform of $H(c\tau,|x|)$ is

$$\tilde{H}(cs,|x|) = \frac{1}{s\cosh(sl/c)}\sinh\!\left(\frac{s}{c}(l-|x|)\right). \qquad \text{(A4)}$$

If we put $x=0$ in Eq. (A1), we obtain

$$\tilde{A}(0,s) = \left[\tilde{A}^0(0,s) + \frac{2\pi\sigma}{s^2}\left(v_0 + \frac{\sigma}{mc}f(0) - \omega_0^2\tilde{z}(s)\right)\tanh\frac{sl}{c}\right]\left[1 + \frac{2\pi\sigma^2}{mcs}\tanh\frac{sl}{c}\right]^{-1}. \qquad \text{(A5)}$$

Substituting (A3)–(A5) into (A1),

$$\tilde{A}(x,s) = \tilde{A}^0(x,s) + 2\pi\sigma\left[\frac{1}{s}\left(v_0 + \frac{\sigma}{mc}f(0) - \omega_0^2\tilde{z}(s)\right) - \frac{\sigma}{mc}\tilde{A}^0(0,s)\right]\frac{\sinh[(s/c)(l-|x|)]}{s\cosh(sl/c) + (2\pi\sigma^2/mc)\sinh(sl/c)} \qquad \text{(A6)}$$

Substituting (A5) into (A3) one obtains

$$\tilde{z}(s) = \left[z_0\left(s + \frac{2\pi\sigma^2}{mc}\tanh\frac{sl}{c}\right) + v_0 + \frac{\sigma}{mc}f(0) - \frac{\sigma s}{mc}\tilde{A}^0(0,s)\right]\frac{1}{s^2 + \omega_0^2 + (2\pi\sigma^2/mc)s\tanh(sl/c)} \qquad \text{(A7)}$$

Finally, (A6) becomes

$$\tilde{A}(x,s)=\tilde{A}^0(x,s)+2\pi\sigma\left(v_0+\frac{\sigma}{mc}f(0)-\frac{\omega_0^2 z_0}{s}-\frac{\sigma}{mc}\tilde{A}^0(0,s)\right)\frac{\sinh[(s/c)(l-|x|)]}{\cosh(sl/c)[s^2+\omega_0^2+(2\pi\sigma^2/mc)s\tanh(sl/c)]}\,. \quad \text{(A8)}$$

The right-hand side of Eq. (A7) has only simple poles $\pm i\omega^*_{2n+1}$ situated on the imaginary axis. They are solutions of the equation

$$\tan(\omega^* l/c)=[(\omega_0^2-\omega^{*2})/\omega^*]l/\gamma c\,, \quad \text{(A9)}$$

where

$$\gamma=2\pi l\sigma^2/mc^2\,.$$

From Eq. (A7) we obtain

$$z(t)=\sum_{\omega^*}\left[2\omega^*+\frac{\gamma c}{l}\tan\frac{\omega^* l}{c}+\frac{\gamma\omega^*}{\cos^2(\omega^* l/c)}\right]^{-1}$$
$$\times\left[2z_0\left(\omega^*+\frac{\gamma c}{l}\tan\frac{\omega^* l}{c}\right)\cos\omega^* t+2\left(v_0+\frac{\sigma}{mc}f(0)\right)\sin\omega^* t-\frac{2\sigma\omega^*}{mc}\int_0^t A^0(0,\tau)\cos\omega^*(t-\tau)\,d\tau\right]. \quad \text{(A10)}$$

From Eq. (A8) we obtain

$$A(x,t)=A^0(x,t)+2\pi\sigma\left(v_0+\frac{\sigma}{mc}f(0)\right)\sum_{\omega^*}\frac{2\sin[(\omega^*/c)(l-|x|)]}{2\omega^*+(\gamma c/l)\tan(\omega^* l/c)+\gamma\omega^*/\cos^2(\omega^* l/c)}\cos(\omega^* t)$$
$$-2\pi\sigma\omega_0^2 z_0\sum_{\omega^*}\frac{2\sin[\omega^*/c)(l-|x|)]\sin(\omega^* t)}{\omega^*\cos(\omega^* l/c)[2\omega^*+(\gamma c/l)\tan(\omega^* l/c)+\gamma\omega^*/\cos^2(\omega^* l/c)]}$$
$$-\frac{2\pi\sigma^2}{mc}\sum_{\omega^*}\frac{2\sin[(\omega^*/c)(l-|x|)]}{2\omega^*+(\gamma c/l)\tan(\omega^* l/c)+\gamma\omega^*/\cos^2(\omega^* l/c)}\int_0^t A^0(0,t)\cos[\omega^*(t-\tau)]\,d\tau\,, \quad \text{(A11)}$$

and finally,

$$A(x,t)=A^0(x,t)+2\pi\sigma\sum_{\omega^*}\frac{2\sin[(\omega^*/c(l-|x|)]}{2\omega^*+(\gamma c/l)\tan(\omega^* l/c)+\gamma\omega^*/\cos^2(\omega^* l/c)}$$
$$\times\left[\left(v_0+\frac{\sigma}{mc}f(0)\right)\cos\omega^* t-\omega_0^2 z_0\frac{1}{\omega^*}\sin\omega^* t-\frac{\sigma}{mc}\int_0^t A^0(0,\tau)\cos[\omega^*(t-\tau)]d\tau\right]. \quad \text{(A12)}$$

It may be observed that the following relation is valid:

$$\int_0^t A^0(0,\tau)\cos[\omega^*(t-\tau)]\,d\tau=\sum_{n=0}^{\infty}\frac{(-1)^n}{\omega_{2n+1}^2-\omega^{*2}}[b_{2n+1}\cos(\omega^* t)-\omega^* a_{2n+1}\sin(\omega^* t)$$
$$-b_{2n+1}\cos(\omega_{2n+1}t)+a_{2n+1}\omega_{2n+1}\sin(\omega_{2n+1}t)]\,. \quad \text{(A13)}$$

APPENDIX B

Equations (1) and (2) can be deduced from the Lagrangian

$$L=\tfrac{1}{2}m\dot{z}^2-U(z)$$
$$+\int_{-l}^{l}dx\left[\frac{1}{8\pi c^2}\left(\frac{\partial A}{\partial t}\right)^2-\frac{1}{8\pi}\left(\frac{\partial A}{\partial x}\right)^2+\frac{\sigma}{c}\delta(x)\dot{z}A\right],$$

$U(z)$ being the potential of $F(z)$. Let us perform the Fourier expansion

$$A=\alpha\sum_{-\infty}^{\infty}a_n\exp\left(\frac{i\omega_n x}{c}\right),$$

where $\alpha=(2l)^{-1/2}$. We have to assume $a_n=\bar{a}_{-n}$ (the bar denotes complex conjugacy) in order for A to be real. The boundary conditions are satisfied putting $\mathrm{Re}\,a_n=0$ for even n, $\mathrm{Im}\,a_n=0$ for odd n. The Lagrangian L, as a function of the variables a_n, $\dot{a}_n$ reads

$$\frac{1}{8\pi c^2}\sum(-1)^n\omega_n^2 a_n^2-\frac{1}{8\pi c^2}$$
$$\times\sum(-1)^n\dot{a}_n^2+\frac{\alpha\sigma}{c}\dot{z}\sum a_n+\tfrac{1}{2}m\dot{z}^2-U\,.$$

We find the kinetic moments

$$p_n=\frac{\partial L}{\partial\dot{a}_n}=-(-1)^n\frac{\dot{a}_n}{4\pi c^2}\,,\quad p=\frac{\partial L}{\partial\dot{z}}=m\dot{z}+\frac{\alpha\sigma}{c}\sum{}'a_n\,,$$

where primed sums extend over odd values of n only. The Hamiltonian is then

$$H=-2\pi c^2\sum(-1)^n p_n^2-\frac{1}{8\pi c^2}\sum(-1)^n\omega_n^2 a_n^2$$
$$+\frac{1}{2m}\left(p+\frac{\alpha\sigma}{c}\sum{}'a_n\right)^2+U(z)\,.$$

The Hamiltonian equations of the motion are

$$\dot{a}_n = -4\pi c^2(-1)^n p_n ,$$
$$\dot{z} = \frac{1}{m}\left(p + \frac{\alpha\sigma}{c}\sum{}' a_n\right), \qquad \text{(B1)}$$
$$\dot{p}_n = \frac{1}{4\pi c^2}(-1)^n \omega_n^2 a_n - [1-(-1)^n]\frac{\alpha\sigma}{c}\dot{z} ,$$
$$\dot{p} = \partial U/\partial z .$$

Normal modes with even n do not interact with the charged plane, and will henceforth be disregarded. We take $U = -\frac{1}{2}m\omega_0^2 z^2$ and go over to adimensional variables $x_n, y_n, n = 0, 1, 2, \ldots$ defined as follows:

$$y_n = \left(\frac{2\pi}{m}\right)^{1/2} p_{2n+1}, \quad x_n = \frac{\omega_n a_{2n+1}}{c^2(8\pi m)^{1/2}} ,$$
$$y_0 = p_0/\sqrt{2}\, mc, \quad x_0 = z/\omega_0 c\sqrt{2}$$

Then, system (B1) becomes

$$\dot{x}_n = \omega_{2n+1} y_n ,$$
$$\dot{x}_0 = \omega_0 y_0 + 2\omega_0\epsilon \sum_0^\infty \omega_{2n+1}^{-1} x_n , \qquad \text{(B2)}$$
$$\dot{y}_n = -\omega_{2n+1}x_n - \epsilon\omega_0^{-1}\dot{x}_0 ,$$
$$\dot{y}_0 = -\omega_0 x_0 .$$

States of finite total energy are described by real sequences

$$(x_0, x_1, \ldots, x_n, \ldots, y_0, y_1, \ldots y_n, \ldots)$$

such that $mc^2\sum(x_n^2 + y_n^2) < +\infty$, i.e., by vectors in the Hilbert space H of such sequences. This Hilbert space is the direct sum of the space H_1 of "configurational" sequences $(x_0, x_1, \ldots, x_n, \ldots)$ and of the space H_2 of "momentum" sequences $(y_0, y_1, \ldots, y_n, \ldots)$. Thus system (B2) describes the evolution of vectors in $H = H_1 \oplus H_2$.

We may look for eigenvectors of the linear system (B2), once its phase space is so enlarged, as to allow for complex values of the x_n, y_n's. Then the components of an eigenvector corresponding to the eigenvalue Ω solve the system

$$\Omega x_n = \omega_{2n+1} y_n ,$$
$$\Omega x_0 = \omega_0 y_0 + 2\omega_0\epsilon\sum x_n \omega_{2n+1}^{-1} ,$$
$$\Omega y_n = -\omega_{2n+1}x_n - \epsilon\omega_0^{-1}\Omega x_0 ,$$
$$\Omega y_0 = -\omega_0 x_0 .$$

From the third equation, we get

$$\sum x_n \omega_{2n+1}^{-1} = -\epsilon x_0 \Omega\omega_0^{-1}\sum_0^\infty \frac{1}{\Omega^2 + \omega_{2n+1}^2}$$
$$= \frac{i l c x_0}{4c\omega_0}\tan\frac{i l \Omega}{c}$$

(the sum is readily evaluated by means of the residue theorem). Upon substitution into the second equation we get the eigenvalue equation

$$i\frac{\pi\sigma^2}{mc}\tan\frac{i l\Omega}{c} = \frac{\Omega^2 + \omega_0^2}{\Omega} .$$

This equation has a sequence of imaginary roots $\pm i\omega_n^*$. To each eigenvalue Ω is associated an eigenvector of the form

$$u_\Omega \equiv \left(x_0, x_n = \frac{-\epsilon\Omega\omega_{2n+1}x_0}{\omega_0(\Omega^2 + \omega_{2n+1}^2)}, \; y_n = \frac{-\epsilon\Omega^2 x_0}{\omega_0(\Omega^2 + \omega_{2n+1}^2)}\right). \qquad \text{(B3)}$$

u_Ω may be normalized by suitably choosing x_0. It can be shown that the vectors u_Ω form a basis in H, regarding them as the result of a perturbation, scaled by σ, operated on a suitable complete orthonormal set.[18] A much more suitable basis is obtained, introducing the vectors

$$v_{\omega^*} = (1/2\omega^*)(u_{i\omega^*} + u_{-i\omega^*}),$$
$$w_{\omega^*} = (1/2i)(u_{i\omega^*} - u_{-i\omega^*}).$$

Inspection of (B3) shows that the w_{ω^*}'s alone form a basis for H_1, while the v_{ω^*}'s form a basis for H_2. Thus the equation of motion (B2) can be solved, expanding the initial configuration on the basis w_{ω^*} with the coefficients $\mu(\omega^*)$ [$\mu(\omega^*)$ coincide, apart from scale factors, with the coefficients of $(z, a_1, a_2, \ldots)$ on the basis v_{ω^*}] then the initial momenta on the basis v_{ω^*} with the coefficients $\nu(\omega^*)$ and finally solving the equations of the harmonic oscillator,

$$\dot{\mu}(\omega^*) = \nu(\omega^*), \quad \dot{\nu}(\omega^*) = -\omega^{*2}\mu(\omega^*) ,$$

with the initial conditions thus found. It is apparent that the variables $\mu(\omega^*)$ play the role of normal coordinates. Insofar as the field variables are involved, the vector v_{ω^*} describes an excitation of the type

$$A_{\omega^*}(x) \sim \sin\left(\frac{\omega^*(|x| - l)}{c}\right),$$

as may be checked by means of a Fourier transform. The frequencies ω_{2n+1}^* tend, in the limit $n\to\infty$, to the frequencies ω_{2n+1} of the free cavity. This is most easily seen by graphical solution of Eq. (14). The corresponding (normalized) eigenvectors $w_{\omega_{2n+1}^*}$ tend, in the norm of H, to the unperturbed eigenvectors $[1/(2l)^{1/2}]e^{i\omega_{2n+1}x} = e_{2n+1}$. In fact,

$$\| w_{\omega_{2n+1}^*} - e_{2n+1}\|^2 = 2(1 - \mathrm{Re}\langle e_{2n+1}| w_{\omega_{2n+1}^*}\rangle$$
$$= o(1/n^2) .$$

[We used Eq. (B3) properly normalized, to evaluate the Hilbertian scalar product.]

APPENDIX C

The energy which one finds for kth unperturbed normal mode (initially not excited) at time t is given by

$$E_k(t) = \frac{2\pi^2 c^2 \sigma^2}{\alpha^2} \left| \int_0^t \dot{z}(s) \exp(-i\omega_k s)\, ds \right|^2 . \qquad \text{(C1)}$$

We shall now evaluate the time average of $E_k(t)$ expressing z as the sum of a uniformly convergent series

$$\dot{z}(t) = \sum_{n=0}^{\infty} a_n \sin(\omega_n^* t + \alpha_n) .$$

We find

$$\langle E_k \rangle = \frac{2\pi^2 c^2 \sigma^2}{\alpha^2} \lim_{T\to\infty} \frac{1}{T} \int_0^T \left[\sum a_n \left(\frac{\cos\alpha_n - \cos[(\omega_n^* + \omega_k)t + \alpha_n]}{\omega_n^* + \omega_k} + \frac{\cos\alpha_n - \cos[(\omega_n^* - \omega_k)t + \alpha_n]}{\omega_n^* - \omega_k} \right) \right]^2 dt$$
$$+ \frac{2\pi c^2 \sigma^2}{\alpha^2} + \lim_{T\to\infty} \frac{1}{T} \int_0^T \left[\sum a_n \left(\frac{\sin[(\omega_n^* - \omega_k)t + \alpha_n] - \sin\alpha_n}{\omega_n^* - \omega_k} - \frac{\sin[(\omega_n^* + \omega_k)t + \alpha_n] - \sin\alpha_n}{\omega_n^* + \omega_k} \right) \right]^2 dt .$$

The time average can be evaluated term by term, thanks to uniform convergence, thus finding, after cumbersome calculations,

$$\langle E_k \rangle = \frac{\pi^2 c^2 \sigma^2}{\alpha^2} \sum_n a_n^2 \left(\frac{1}{(\omega_n^* - \omega_k)^2} + \frac{1}{(\omega_n^* + \omega_k)^2} + \frac{\cos^2\alpha_n - \sin^2\alpha_n}{\omega_n^{*2} - \omega_k^2} \right)$$
$$+ \frac{\pi^2 c^2 \sigma^2}{\alpha^2} \sum_{r\neq s} a_r a_s \frac{\omega_r^* \omega_s^* + \omega_k^2}{(\omega_s^{*2} - \omega_k^2)(\omega_r^{*2} - \omega_k^2)} \cos(\alpha_r - \alpha_s) + \frac{\pi^2 c^2 \sigma^2}{\alpha^2} \sum_{r\neq s} a_r a_s \frac{\omega_r^* \omega_s^* - \omega_k^2}{(\omega_s^{*2} - \omega_k^2)(\omega_r^{*2} - \omega_k^2)} \cos(\alpha_r + \alpha_s) . \qquad \text{(C2)}$$

In the derivation of (C2) an essential role is played by the assumption that

$$\omega_s^* \pm \omega_r^* \neq 2\omega_k \qquad \text{(C3)}$$

whatever k, s, r may be with $s \neq r$ and k an odd number. This is easily verified, since there is one, and only one, root of the Eq. (A9) in every interval $(\omega_{2n+1}, \omega_{2n+3})$. We call this root ω_n^* and form the differences

$$\epsilon_n = \omega_n^* - \omega_{2n+1} ,$$

which are the roots of the equation

$$-\cot \frac{\epsilon l}{c} = F(\epsilon + \omega_{2n+1}); \quad F(x) = \frac{\omega_0^2 - x^2}{x} \frac{l}{c\gamma}$$

in the interval $(-\omega_1, +\omega_1)$.

Monotonicity of $F(x)$ implies that the ϵ_n's form a monotonically decreasing sequence. Then (C3) is equivalent to

$$\epsilon_s + \epsilon_r \neq 2\omega_{k+r-s} , \qquad \text{(C4)}$$
$$\epsilon_s - \epsilon_r \neq 2\omega_{k-r-s-1} ,$$

which are true since $\epsilon_s < \omega_1$.

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