PHYSICAL REVIEW A VOLUME 15, NUMBER 2 FEBRUARY 1977

Plasma mode of weakly and strongly coupled one-component plasmas

Marc Baus*

Chimie-Physique II,† Université Libre de Bruxelles, Campus Plaine CP. 231, B-1050 Bruxelles, Belgium

(Received 14 May 1976)

Starting from a novel exact expression for the dielectric constant of a classical one-component plasma, we study the important modifications of the plasma oscillation modes which appear when one progressively increases the plasma expansion parameter λ. For small λ values the evaluation of the plasma mode requires a non-Markovian theory. We find good agreement between our results, recent computer findings, and known theoretical expressions. For instance, the dispersion of the plasma frequency is shown to take on values slightly *above* the mean-field prediction. For large λ values the plasma mode is shown to be expressible in terms of hydrodynamical quantities. Here the dispersion of the plasma frequency is shown to be *negative*, while the damping of the plasma mode is given by the strong-coupling limit of the longitudinal viscosity from which the non-Markovian contributions to the bulk viscosity are to be deleted.

I. INTRODUCTION

Since the first observation, half a century ago, of the high-frequency electrostatic plasma oscillations by Penning[1] and the subsequent theoretical work of Tonks and Langmuir,[2] the *plasma mode* has attracted a lot of interest. In a celebrated paper[3] Landau indicated that even in the collisionless limit this mode would be damped, and he calculated the finite-wavelength corrections to the plasma frequency, introducing thereby the famous Landau damping correcting an earlier result of Vlassov.[4] Since then the particular properties of Coulomb systems have been extensively studied.[5] From the kinetic theoretical point of view most investigations have been concerned with the infinite-wavelength zero-frequency weak-coupling collision operator of Balescu-Guernsey-Lenard[6] (BGL). More recently a number of finite-frequency and/or finite-wavelength phenomena have been considered within various approximations,[7] especially in connection with the high-frequency electrical conductivity of an electron-ion plasma.[8]

In this paper we will consider a classical one-component plasma (OCP) and study its plasma mode in the two limiting regions of weak and strong coupling. In a previous paper,[9] hereafter referred to as I, we have obtained from first principles the exact expressions of the five long-wavelength modes of the OCP which result from the conservation of particle number, momentum, and energy. These modes, although not identical to them, are the exact equivalents of the hydrodynamical modes of uncharged-particle systems. Both sets of modes reflect the symmetry properties, or better restore a broken symmetry, of the system and are valid at arbitrary coupling or density. The Coulomb singularity slightly modifies the heat mode while it shifts the sound modes of the neutral particle system into the high-frequency plasma modes. As shown in I, there are also important modifications in the hydrodynamic correlation functions. The most striking result, however, is the fact that because of the appearance of a finite frequency in this problem, the plasma frequency ω_p, the exact expression of the plasma modes can not be written in terms of hydrodynamical concepts. This is quite surprising, in view of the popular derivation of the plasma mode from the linearized hydrodynamical equations.[10] To explore this question we will derive various limiting results from our exact expression and compare them with the recent computer work of Hansen *et al.*[11] and with some known theoretical results. Throughout we will concentrate ourselves on the literature related to the OCP and note that the extension of our results to the two-component plasma is not a trivial matter.[12]

The physical considerations which emerge from our microscopic calculations in the subsequent sections can be summarized as follows: Our starting point is a novel exact expression of the dielectric constant $\epsilon(\vec{k}, z)$ which naturally splits into a static (frequency independent) and a dynamic (frequency dependent) contribution. In the dynamic contribution the result of the conservation laws can be built in from the start, a very useful property in the region of small wave vectors $\vec{k}$.

In Sec. II we check the general properties of $\epsilon(\vec{k}, z)$, while the expression of the small-k plasma mode derived in I is easily recovered. In the limit of weak coupling $\lambda \ll 1$, $\lambda = k_D^3/n$ being the plasma parameter of a OCP of density n and Debye wave vector k_D, the dispersion and damping of the plasma mode are shown to be strongly non-Markovian through their dependence on the plasma frequency ω_p. This can be understood if we argue in terms of a kinetic equation, for instance, the exact equation derived in I. This equation is easily seen to be controlled by the relative order of magnitude of the Vlassov mean-field term and the collision term. The mean-field term of the OCP is singular for small k and of order $\omega_p(k_D/k)$, while we estimate the collision term as roughly of order $\omega_c = \lambda\omega_p$, ω_c being the collision frequency. For long wavelengths ($k \ll k_D$) the weakly coupled OCP ($\lambda \ll 1$) is controlled by the Vlassov mean-field term.

The small collisional corrections to the dispersion and damping of the plasma mode can be calculated only with the aid of a non-Markovian z and $\vec{k}$ dependent collision operator. This is done in Sec. III to lowest order in λ with the aid of the finite-z and -$\vec{k}$ extension of the linearized BGL operator. Both the dispersion and the damping of the plasma mode are shown to be shifted above their mean-field values, in agreement with the computer results of Ref. 11(a) and with the theoretical expression of the damping rate obtained by Dubois and Gilinsky.[13] In a number of asides we also point out the errors involved in calculating the collisional contribution to the dispersion with the aid of sum rules or when using Markovian approximations for the collision operator. The long-wavelength plasma mode of a weakly coupled OCP remains thus definitively non-Markovian and nonhydrodynamical. This will no longer be true in the opposite limit of strong coupling ($\lambda \gg 1$). Indeed, here we have a possibility for the kinetic equation to become collision dominated. For the collision term to dominate the Vlassov term we should have roughly $\omega_p(k_D/k) \ll \omega_c = \lambda\omega_p$ or $k_D \ll \lambda k$, which for long-wavelengths ($k \ll k_D$) is seen to imply strong coupling ($\lambda \gg 1$).

In contradistinction with neutral-particle systems, where the kinetic equation is always collision dominated for small k, hydrodynamics will eventually emerge as an approximate property of the OCP which can become exact only in the limit of infinite coupling. This question is taken up in Sec. IV. In the strong-coupling limit we will clearly not be able to write expressions as explicit as those of Sec. III. We exploit, however, the fact that for strong coupling we can expand our expressions for small ω_p/ω_c values, which is technically equivalent to taking the low-frequency Markovian limit. We then show that the dispersion of the plasma frequency is given by the strong-coupling limit of the expression one would obtain from a straightforward use of the linearized hydrodynamic equations, such as was done in Refs. 10 and 11(b). Similarly, the damping of the plasma mode is given by the strong-coupling limit of the hydrodynamic expression from which we have, moreover, to *delete* the genuine non-Markovian contributions to the bulk viscosity. In agreement with the computer results of Hansen *et al.*[11] we find that the dispersion of the plasma mode is negative in the hydrodynamic region $k_D \ll \lambda k$, whereas our last remark might explain the small values which have been observed for the bulk viscosity. A question which is left unanswered, however, concerns the explicit calculation of the strong-coupling limiting values involved. Finally, our conclusions are given in Sec. V.

II. PLASMA MODE

The plasma oscillation mode is set up in a system of charged particles as the response to a small perturbation in the charge density.[5] The properties of this response are fully described by the frequency- and wave-vector-dependent dielectric constant $\epsilon(\vec{k}, z)$ or the related electrical susceptibility $\chi(\vec{k}, z)$:

$$\frac{1}{\epsilon(\vec{k}, z)} = 1 + V_{\vec{k}}\,\chi(\vec{k}, z)\,, \quad V_{\vec{k}} = 4\pi e^2/k^2\,, \tag{2.1}$$

where $V_{\vec{k}}$ is the Fourier transform of the Coulomb potential acting between the particles of charge e building up our OCP. The susceptibility χ can also be identified with the density-density response function, which is simply related to the density-density correlation function of I, G_{nn}, by the fluctuation-dissipation theorem. If n is the number density and β the inverse temperature this theorem reads

$$\chi(\vec{k}, t) = n\beta\partial_t G_{nn}(\vec{k}, t)\,, \quad \beta^{-1} = \kappa T\,, \tag{2.2}$$

where for simplicity we have switched back the Laplace transforms to the time domain. In I, whose notation is largely standard and which we adopt here, it was shown how the exact kinetic equation[9] (I 2.16), obeyed by the two-point correlation function $S(1,2;t)$, say, $zS(z) - \Sigma(z)S(z) = S_0$ (skipping the details[9]), can be transformed into a system of equations (I 2.19) for the hydrodynamic correlation functions G_{ij}, say, $zG_{ij}(z) - \sum_{j'} \Omega_{ij'}(z)G_{j'j}(z) = G^0_{ij}$. We now solve this system (I 3.2) for the density-density correlation function $G_{nn}(\vec{k}, z)$, use Eq. (2.2), and substitute the result into Eq. (2.1). This then yields a novel exact expression for the dielectric constant of Eq. (2.1):

$$\epsilon(\vec{k}, z) = \left(1 + \frac{\omega_p^2}{\Delta_p(\vec{k}, z)}\right)^{-1} \equiv \frac{\Delta_p(\vec{k}, z)}{\omega_p^2 + \Delta_p(\vec{k}, z)} \equiv 1 - \frac{\omega_p^2}{\omega_p^2 + \Delta_p(\vec{k}, z)}\,, \tag{2.3}$$

where we have introduced the fundamental quantity Δ_p, defined by

$$\Delta_p(\vec{k}, z) = z[z - \Omega(\vec{k}, z)] - \omega_p^2(\vec{k})\,, \tag{2.4}$$

which naturally splits into a static frequency-independent term $\omega_p^2(\vec{k})$ and a dynamic frequency-dependent contribution $z[z-\Omega(\vec{k}z)]$. Indeed, $\omega_p^2(\vec{k})$ is given in terms of the equilibrium direct correlation function $c(\vec{k})$ related to the static structure factor through the Ornstein-Zernike relation (I 2.15). Explicitly we have

$$\omega_p^2(\vec{k}) = \omega_p^2(k^2/k_D^2)[1-c(\vec{k})]\,, \tag{2.5}$$

where ω_p is the plasma frequency ($\omega_p^2 = 4\pi e^2 n/m$) and k_D the Debye wave vector ($k_D^2 = 4\pi e^2 n\beta$) of a OCP of particles of charge e, mass m, number density n, and equilibrium temperature $\kappa T = \beta^{-1}$. As shown in I, $c(\vec{k})$ is singular for small k in such a way that $\omega_p(k=0)$ [Eq. (2.5)] equals the plasma frequency ω_p. Because of the dynamic contribution to $\Delta_p(\vec{k},z)$ we can however not identify $\omega_p(\vec{k})$ with the finite-k plasma frequency. The dynamic contribution to Δ_p is determined by $\Omega(\vec{k},z)$, which is defined in terms of the matrix elements $\Omega_{ij}(\vec{k},z)$, already introduced in I, through

$$\Omega(\vec{k},z) = \Omega_{ll}(\vec{k},z) + \Omega_{l\epsilon}(k,z)[z-\Omega_{\epsilon\epsilon}(\vec{k},z)]^{-1}\Omega_{\epsilon l}(\vec{k},z)\,, \tag{2.6}$$

where, as in I, l and ϵ denote, respectively, the longitudinal momentum and energy state. The $\Omega_{ij}(\vec{k},z)$ matrix elements can be further analyzed in terms of the collision term of the kinetic equation of I as follows:

$$\Omega_{ij}(\vec{k},z) = \langle i|[\overline{\Sigma}^0(\vec{k}) + \overline{\Sigma}^s(\vec{k}) + \overline{\Sigma}^c(\vec{k},z)]|j\rangle + \langle i|[\overline{\Sigma}^0(\vec{k}) + \overline{\Sigma}^c(\vec{k},z)]\overline{Q}\{z - \overline{Q}[\overline{\Sigma}^0(\vec{k}) + \overline{\Sigma}^c(\vec{k},z)]\overline{Q}\}^{-1}\overline{Q}[\overline{\Sigma}^0(\vec{k}) + \overline{\Sigma}^c(\vec{k},z)]|j\rangle\,, \tag{2.7}$$

where the scalar product in momentum space has been defined precisely in (I 2.18), while the expressions of the free-flow term, $\overline{\Sigma}^0(\vec{k})$, the mean-field term, $\overline{\Sigma}^s(\vec{k})$, and the nonlocal and non-Markovian collision term, $\overline{\Sigma}^c(\vec{k},z)$, have been given in (I 2.17) and will not be repeated here. We also recall that the projection operators $\overline{Q}$ appearing in (2.7) prevent $\Omega_{ij}(\vec{k},z)$ from becoming singular for small $\vec{k}$ and z values, while the conservation laws dictate their small-$\vec{k}$ behavior. For instance, momentum conservation implies that Eq. (2.6) can be rewritten $\Omega(\vec{k},z) = k^2 D(\vec{k},z)$, where $D(\vec{k}=0,z)$ is a finite quantity.

From Eq. (2.3) we see that the dispersion equation $\epsilon(\vec{k},z)=0$ can be given the equivalent form $\Delta_p(\vec{k},z)=0$, displaying the physical meaning of Δ_p. The dispersion equation of the plasma mode, $\Delta_p(\vec{k},z(\vec{k}))=0$, reduces, according to Eq. (2.4), for small k to

$$z = \pm\,\omega_p(k) + \tfrac{1}{2}k^2 D(k,z)\,. \tag{2.8}$$

The solutions of Eq. (2.8), $z=z(\vec{k})$, will be given the following standard form:

$$z_\pm(k) = \pm\,\omega_p[1+\tfrac{1}{2}k^2\gamma_p(k)] - \tfrac{1}{2}ik^2\Gamma_p(k)\,. \tag{2.9}$$

Comparing Eqs. (2.8) and (2.9) we obtain for the dispersion coefficient (γ_p) and damping rate (Γ_p) of the plasma mode the following general expressions:

$$\gamma_p(\vec{k}) = [\omega_p^2(\vec{k}) - \omega_p^2]/\omega_p^2 k^2 + \mathrm{Re}[D(\vec{k},\omega_p)/\omega_p]\,, \tag{2.10a}$$

$$\Gamma_p(\vec{k}) = -\,\mathrm{Im}D(\vec{k},\omega_p)\,, \tag{2.10b}$$

where we took into account the fact that the general symmetry properties of the $\Omega_{ij}(\vec{k},z)$ imply that $\mathrm{Re}D(\vec{k},\pm\omega_p) = \pm\,\mathrm{Re}D(\vec{k},\omega_p)$, while $\mathrm{Im}D(\vec{k},\pm\omega_p) = \mathrm{Im}D(\vec{k},\omega_p)$. For later use we also introduce dimensionless quantities $\overline{\gamma}_p$, $\overline{\Gamma}_p$, and $\overline{D}$ according to $\overline{\gamma}_p = k_D^2\gamma_p$, $\overline{\Gamma}_p = k_D^2\Gamma_p/\omega_p$, $\overline{D}(\vec{k},z) = k_D^2 D(\vec{k},z)/\omega_p$, and such that (2.9) becomes

$$z_\pm/\omega_p = \pm[1+\tfrac{1}{2}(k^2/k_D^2)\overline{\gamma}_p] - \tfrac{1}{2}i(k^2/k_D^2)\overline{\Gamma}_p\,.$$

The expression we have obtained for the dielectric constant [Eqs. (2.3) and (2.4)] is exact. Its superiority over the standard Kubo relation stems from the fact that $\epsilon(\vec{k},z)$ has been expressed here entirely in terms of the familiar one- or two-body concepts of kinetic theory instead of the N-body operators entering the Kubo formula. Hence a lot of information on the symmetry properties and conservation laws of the system has been built in into our expression of $\epsilon(\vec{k},z)$. Moreover, the only matrix element Ω_{ij} which depends on the singular mean field, namely, $\Omega_{ln}(\vec{k},z)$ [see (2.7) and (I 2.20)], has been separated completely from the dynamic contribution $\Omega(\vec{k},z)$ and lumped together into the single static quantity $\omega_p^2(\vec{k})$, which is finite as $k\to 0$ and determined completely by the equilibrium binary correlations as seen from (2.4) and (2.5). Finally, the plasma mode defined through Eqs. (2.9) and (2.10) becomes identical for small k to the result of I [see (I 3.17)].

Before closing this section let us, for the sake of completeness, check in Eq. (2.3) the two limiting values of $\epsilon(\vec{k},z)$ which establish the existence of the plasma frequency and of the screening length. From Eq. (2.4) we see immediately that $\Delta_p(\vec{k}=0,z) = z^2 - \omega_p^2$, and hence we obtain from (2.3) the well-known infinite-wavelength value of $\epsilon(\vec{k},z)$:

$$\epsilon(\vec{k}=0,z) = 1 - \omega_p^2/z^2\,, \tag{2.11}$$

quite trivial result which implies, however, the existence of two plasma-oscillation modes with frequencies $z=\pm\omega_p$, in agreement with (2.9). Moreover, as the collision term $\overline{\Sigma}^c(\vec{k},z)$ vanishes as z^{-1} for large z [see below, Eq. (3.5)], we deduce from Eqs. (2.3) and (2.4) that (2.11) yields also the first term of the high-frequency expansion of $\epsilon(\vec{k},z)$ at finite k:

$$\lim_{z\to\infty}\epsilon(\vec{k}z)=1-\omega_p^2/z^2+\mathcal{O}(z^{-4}). \tag{2.12}$$

Finally, since the $\Omega_{ij}(\vec{k},z)$ are finite at $z=0$, the static dielectric constant $\epsilon(\vec{k},0)$ is immediately obtained from Eqs. (2.3) and (2.4) as

$$\epsilon(\vec{k},z=0)=1+\omega_p^2/[\omega_p^2(k)-\omega_p^2]. \tag{2.13}$$

For small k we can use the expansion (I 3.7b) in Eq. (2.5):

$$\omega_p^2(k)=\omega_p^2[1+k^2/k_s^2+\mathcal{O}(k^4)], \tag{2.14}$$

and hence for (2.13) we obtain

$$\epsilon(\vec{k},z=0)=1+k_s^2/[k^2+\mathcal{O}(k^4)]\simeq 1+k_s^2/k^2, \tag{2.15}$$

identifying k_s as the inverse screening length. We can also write $k_s^2=k_D^2\chi_T/\chi_T^0$, where χ_T is the isothermal compressibility of the OCP while $\chi_T^0=\beta/n$ is its ideal-gas value. Equation (2.15) is then often referred to as the compressibility sum rule.[5] In recent numerical calculations[11,14] it was found that the inverse compressibility χ_T^{-1} becomes negative when the plasma parameter λ exceeds a critical value $\lambda>\lambda_c\simeq 36\pi$. Here, $\lambda=k_D^3/n$ denotes the inverse of the number of particles in a Debye cube. Related, often used plasma parameters are $\epsilon=\lambda/4\pi$ (as in Ref. 14) and $\Gamma=3^{-1/3}(\lambda/4\pi)^{2/3}$ (as in Ref. 11). Equation (2.15) indicates then that when $\lambda>\lambda_c$ the effective potential $V_{\vec{k}}/\epsilon(\vec{k},0)$, as well as the pair correlations, changes sign for $k<k_c=k_D|\chi_T/\chi_T^0|^{1/2}$, while a pair of poles appear on the real axis at $k=\pm k_c$ (here $k_s^2=-k_c^2$ for $\lambda>\lambda_c$, while $k_s^2=k_c^2$ for $\lambda<\lambda_c$). We can thus expect spatial oscillations to occur[15] with the characteristic wave vector k_c instead of the spatial screening occurring for $\lambda<\lambda_c$. Moreover, thermodynamic stability requires that $1-c(k)\geq 0$ for all k, because the static density fluctuations[9] $[1-c(k)]^{-1}$ have to remain positive. Hence for $\lambda>\lambda_c$ the OCP will become thermodynamically unstable against local density fluctuations with $k>k_c$, because, as shown in (I 3.7b), $1-c(k)=k_D^2/k^2+\chi^0/\chi_T+\mathcal{O}(k^2)$.

This is a necessary but not a sufficient condition for a phase transition to a system with short-range order to appear. The real crystallization has been shown in Ref. 11 to occur only at $\lambda=\lambda_s\gg\lambda_c$, with $\lambda_s/4\pi\simeq 3.245$.

The properties of the "critical point" λ_c and its relation to the special features of the OCP model (inert neutralizing background) still have to be analyzed further.[14] Now that we have checked $\epsilon(\vec{k},z)$ of Eq. (2.3) for its general properties, we will concentrate on the study of the plasma mode defined by Eqs. (2.9) and (2.10).

III. WEAK-COUPLING LIMIT

Let us start by considering the more familiar limit of small plasma parameters, $\lambda\ll 1$. Since $\lambda=k_D^3/n$ depends on the charge, density, and temperature of the OCP according to $\lambda=(4\pi)^{3/2}e^3n^{1/2}(\kappa T)^{-3/2}$, whereas $\lambda^{2/3}$ appears as a dimensionless coupling constant ($\sim e^2$), we will briefly characterize the plasmas satisfying the condition $\lambda\ll 1$ as weakly coupled. The dependence of $\epsilon(\vec{k},z)$ on λ is twofold, through the equilibrium properties entering the static term $\omega_p^2(k)$ and through the collision term $\overline{\Sigma}^c(\vec{k},z)$ appearing in the dynamic contribution $\Omega(\vec{k},z)$. For small λ the equilibrium properties can be taken from the literature, while $\Omega(\vec{k},z)$ can be expanded with respect to $\overline{\Sigma}^c(\vec{k},z)$. Indeed, from the general expression (I 2.17c) it is apparent (see also Sec. IV) that we have $\overline{\Sigma}^c(\vec{k},z)=\omega_c\hat{\overline{\Sigma}}^c(\vec{k},z;\lambda)$, where $\omega_c=\lambda\omega_p$ is a collision frequency, so that for small λ we can consider the collision term $\overline{\Sigma}^c$ itself as small.

A. Zero coupling

For completeness we first consider the extreme case of zero coupling by simply dropping $\overline{\Sigma}^c$ from $\Omega(\vec{k},z)$. It is fair to say that in this limit our expressions (2.6) and (2.7) are unnecessarily complicated because they are "irreducible" with respect to the conserved states, which play no particular role in the collisionless limit. Using Eq. (2.2) we can, however, easily transform Eq. (2.3) back to its "reducible" form, with the well-known result

$$\lim_{\lambda\to 0}\epsilon(\vec{k},z)\equiv\epsilon^0(k,z)=1-\frac{k_D^2}{k^2}\int d\vec{p}\,\frac{\vec{k}\cdot\vec{v}}{z-\vec{k}\cdot\vec{v}}\varphi(\vec{p}), \tag{3.1}$$

where $\varphi(\vec{p})$ is the Maxwellian normalized to unity. To obtain (3.1) we have used the known result[5] $\lim_{\lambda\to 0}c(\vec{k})=c^0(\vec{k})=-k_D^2/k^2$. Equation (3.1), which is usually called the random-phase or Vlassov mean-field approximation, can also be rewritten $\epsilon^0(\vec{k},z)=1-V_{\vec{k}}\chi^0(\vec{k},z)$, where $\chi^0(\vec{k},z)$ is the free-particle susceptibility. Comparing this expression with (2.1) shows that if we would have considered the charge e^2 small, instead of λ, we would have obtained instead of (3.1) $\epsilon(\vec{k},z)=[1+V_{\vec{k}}\chi^0(\vec{k},z)]^{-1}$. In the long-wavelength limit the weakly damped zeros of $\epsilon^0(\vec{k},z)$ define a pair of plasma oscillations whose celebrated expression was first obtained by Landau.[3] Using the general form (2.9) we can rewrite this result, using dimensionless variables, as

$$\lim_{k\to 0}\lim_{\lambda\to 0}\bar{\gamma}_p(k) = 3\,, \tag{3.2a}$$

$$\lim_{k\to 0}\lim_{\lambda\to 0}\bar{\Gamma}_p(k) = \left(\frac{\pi}{2}\right)^{1/2}\left(\frac{k_D}{k}\right)^5 \exp\left[-\frac{1}{2}\left(\frac{k_D}{k}\right)^2 - \frac{3}{2}\right] \simeq 0\,. \tag{3.2b}$$

These values will be often referred to as the mean-field values of the plasma mode. Note that for the Landau damping (3.2b) the order of the limits has some importance. Indeed, keeping $\lambda \neq 0$ the conservation laws force the damping rate to be analytic in k at least to order k^2 and hence $\lim_{\lambda\to 0}\lim_{k\to 0}\bar{\Gamma}_p(k)$ is strictly zero. If we first let $\lambda \to 0$, we introduce a singularity in the propagator $\{z - \bar{Q}[\bar{\Sigma}^0(k) + \bar{\Sigma}^c(\vec{k}z)]\bar{Q}\}^{-1}$ appearing in (2.7) and obtain the small Landau damping of (3.2b). In view of this nonuniformity in (k,λ) space it makes little sense to superpose, as is often done in the literature, the Landau damping to the collisional damping.

B. Small but finite coupling

From now on we consider $\lambda \neq 0$, in which case the plasma modes are analytic in k, at least to order k^2, and their general expression (2.10) can be rewritten in dimensionless form

$$\bar{\gamma}_p(\vec{k}=0) = k_D^2/k_s^2 + \mathrm{Re}\bar{D}(\vec{k}=0,\omega_p)\,, \tag{3.3a}$$

$$\bar{\Gamma}_p(\vec{k}=0) = -\,\mathrm{Im}\bar{D}(\vec{k}=0,\omega_p)\,, \tag{3.3b}$$

where we have taken (2.14) into account. We are now ready to evaluate (3.3) for small but finite coupling $(0<\lambda\ll 1)$. Considering $\bar{\Sigma}^c(k,z)$ as small in (2.6) and (2.7) we obtain for (2.6) after a number of compensations

$$\Omega(\vec{k},z) = \left(\frac{4}{3}+\frac{2}{3}\right)\frac{k^2}{k_D^2}\frac{\omega_p^2}{z} + \langle l|\bar{\Sigma}^c(\vec{k},z)|l\rangle + \frac{\langle l|\bar{\Sigma}^0(\vec{k})\bar{\Sigma}^c(\vec{k},z)|l\rangle}{z} + \frac{\langle l|\bar{\Sigma}^c(\vec{k},z)\bar{\Sigma}^0(\vec{k})|l\rangle}{z} + \frac{\langle l|\bar{\Sigma}^0(\vec{k})\bar{\Sigma}^c(\vec{k},z)\bar{\Sigma}^0(\vec{k})|l\rangle}{z^2} + \mathcal{O}((\bar{\Sigma}^c)^2,k^4)\,, \tag{3.4}$$

where we have retained only those terms contributing to first order in both $\bar{\Sigma}^c$ and k^2. It is interesting to observe that (3.4) also contains the high-frequency expansion of $\Omega(\vec{k},z)$ exact up to terms of order z^{-2}. Indeed, the expansion parameter $\bar{\Sigma}^c/z$ is small both for weak coupling and for high frequencies. This then opens the possibility of evaluating (3.4) and hence the plasma mode with the aid of a high-frequency sum-rule analysis. This possibility was explored in a recent paper by Ichimaru *et al.*[16] starting from the dynamic structure factor $S(\vec{k},z) = nG_{nn}(\vec{k},z)$ instead of $\Omega(\vec{k},z)$. We now proceed to recover their result.

1. High-frequency analysis

From (I 2.17c) the high-frequency expansion of $\bar{\Sigma}^c(\vec{k},z)$ is obtained as

$$\begin{aligned}\Sigma^c(\vec{k},z;\vec{p},\vec{p}')n\varphi(\vec{p}') &= \sum_{n=1}^{\infty} z^{-n}\Sigma_n(\vec{k};\vec{p},\vec{p}')n\varphi(\vec{p}')\\ &= \sum_{n=0}^{\infty}\left(\delta f(\vec{k},\vec{p})\,\Big|\,L\frac{Q}{z}\left(\frac{QLQ}{z}\right)^n QL\,\Big|\,\delta f(\vec{k},\vec{p}')\right),\end{aligned} \tag{3.5}$$

where only the first frequency moment $\Sigma_1(\vec{k})$ will be needed here:

$$\Sigma_1(\vec{k};\vec{p},\vec{p}')n\varphi(\vec{p}') = \vec{\partial}\cdot\vec{D}\cdot\vec{\partial}'\delta(\vec{p}-\vec{p}')n\varphi(\vec{p}') + \vec{\partial}\cdot\vec{A}(\vec{k})\cdot\vec{\partial}'\varphi(\vec{p})n\varphi(\vec{p}')\,. \tag{3.6}$$

In Eq. (3.6) we have put

$$\vec{D} = \frac{n}{\beta}\int d\vec{r}\,V(\vec{r})\vec{\nabla}\vec{\nabla}h(\vec{r})\,, \tag{3.7a}$$

$$\vec{A}(\vec{k}) = -\,\vec{D} + \vec{k}\vec{k}\left(c(k) + \frac{k_D^2}{k^2}\right)\beta^{-2} - \frac{n}{\beta}\int d\vec{r}(e^{-i\vec{k}\cdot\vec{r}} - 1)h(\vec{r})\vec{\nabla}\vec{\nabla}V(\vec{r})\,, \tag{3.7b}$$

with $V(\vec{r}) = e^2/r$ the Coulomb potential, while $h(\vec{r})$ is the inverse Fourier transform of the pair correlation function $h(\vec{k})$ related to $c(\vec{k})$ by the Ornstein-Zernike relation $1+h(\vec{k}) = [1-c(\vec{k})]^{-1}$. The other symbols are quite standard ($\vec{\nabla} = \partial/\partial\vec{r}$, $\vec{\partial} = \partial/\partial\vec{p}$, etc.). The difference between (3.6) and (3.7) and the results published in the literature[17] are entirely due to the Coulomb singularity. Note, however, that momentum conservation for Σ_1 implies that $\vec{A}(\vec{k}) + \vec{D}$ vanishes with $\vec{k}$, a property shared by (3.7) but not by another published result.[18] For weak coupling we have from Eqs. (3.4) and (3.5)

$$\Omega(\vec{k},z) = 2\frac{k^2}{k_D^2}\frac{\omega_p^2}{z} + \frac{\langle l|\Sigma_1(\vec{k})|l\rangle}{z} + \mathcal{O}(z^{-3},k^4)\,, \tag{3.8}$$

where the z^{-2} contributions vanish for symmetry reasons. For small k we can evaluate (3.8) from (3.6) and (3.7), with the result

$$\Omega(\vec{k},z) = 2\frac{k^2}{k_D^2}\frac{\omega_p^2}{z} + \frac{k^2}{k_D^2}\frac{\omega_p^2}{z}\left(1 - \frac{k_D^2}{k_s^2} + \frac{4E_c}{15}\right) + \mathcal{O}(z^{-3},k^4)\,, \tag{3.9}$$

where $E_c = \frac{1}{2}k_D^2\int_0^\infty dr\,rh(r)$ is the correlation energy divided by the kinetic energy density $n\beta^{-1}$. Substitution of (3.9) into (3.3) yields

$$\bar{\gamma}_p(\vec{k}=0) = 3 + \tfrac{4}{15}E_c\,, \tag{3.10a}$$

$$\bar{\Gamma}_p(k=0)=0\,, \tag{3.10b}$$

which is identical to the result obtained to this order in Ref. 16 by a different method. For weak coupling we obtain[16] from (3.10a)

$$\bar{\gamma}_p(\vec{k}=0)=3-\lambda/30\pi\,, \tag{3.11}$$

a result which contradicts the value of $\bar{\gamma}_p$ (to be calculated in Sec. III B 4) and the computer results of Ref. 11, both of which indicate a positive value for $\bar{\gamma}_p-3$ for small λ. It has been shown in Ref. 16 that when the next-higher-order frequency moment is taken into account the sign of $\bar{\gamma}_p-3$ is reversed and is thus in agreement with the computer results. This, however, involves the introduction of terms of higher order in λ, whereas we show in Sec. III B 4 that $\bar{\gamma}_p-3$ has the correct sign already to first order in λ when the collisions are treated properly.

The difficulty with the high-frequency expansion (3.5) can be understood as follows: Such an expansion is clearly designed to yield good results for very large z values, a fact which explains the absence of dissipation and hence (3.10b). Here we have however a problem with a large frequency ω_p built into it. If we write the high-frequency expansion as $\sum_n a_n(\omega_p/z)^n$, then it is obvious that this series will remain meaningful, at least asymptotically, for $|z|\gg\omega_p$. Here we are, according to (3.3), evaluating this series at $z=\omega_p$ and hence we have to care about $\sum_n a_n$ itself. One easily convinces oneself, however, that many frequency moments, Σ_n, contribute to $\sum_n a_n$ to a given order in λ, because the expansion (3.5) is with respect to the full Liouville operator L and not just its potential part. Truncation of the high-frequency series with respect to the moments involved or with respect to λ may thus give quite different results.

2. *Finite-frequency analysis*

To turn around the above difficulty we will have to take into account properly the *finite*-frequency effects ($z=\omega_p$) which appear in the definition of the plasma mode (3.3). To this end we introduce to the collision term the approximation $\bar{\Sigma}^c(\vec{k},z)\simeq\bar{\Sigma}^c_D(\vec{k},z)$, with $\bar{\Sigma}^c_D$ defined by

$$\begin{aligned}\Sigma^c_D(\vec{k},t;\vec{p}_1,\vec{p}_2)=-i\int\frac{d\vec{l}}{8\pi^3}\int d\vec{p}_{1'}\,d\vec{p}_{2'}\,V_{\vec{l}}\,\vec{l}\cdot\vec{\partial}_1[&S(\vec{k}-\vec{l},t;\vec{p}_1,\vec{p}_{2'})S(\vec{l},t;\vec{p}_{1'},\vec{p}_2)V_{\vec{l}-\vec{k}}(\vec{l}-\vec{k})\cdot\vec{\partial}_2\\&-S(\vec{k}-\vec{l},t;\vec{p}_1,\vec{p}_2)S(\vec{l},t;\vec{p}_{1'},\vec{p}_{2'})V_{\vec{l}}\,\vec{l}\cdot\vec{\partial}_2][n\varphi(\vec{p}_2)]^{-1}\,,\end{aligned} \tag{3.12}$$

where $S(\vec{k},t;\vec{p},\vec{p}\,')$ is the exact one-particle propagator, or phase-space correlation function, introduced in (I 2.7). Equation (3.12) consists of all disconnected contributions to $\bar{\Sigma}^c$. The approximation $\bar{\Sigma}^c\simeq\bar{\Sigma}^c_D$ thus neglects the connected contributions to $\bar{\Sigma}^c$, these being important whenever close collision processes are involved. We will come back elsewhere to the properties of $\bar{\Sigma}^c_D$ [Eq. (3.12)] which was also at the basis of our study of the long-time behavior of the nonlocal shear viscosity of the OCP.[19] The main interest of $\bar{\Sigma}^c_D$ stems from the fact that the use of the local ($k\to0$) Markovian ($z\to0$) limit of the Laplace transform of (3.12), say, $\bar{\Sigma}^c_D(\vec{k}=0,z=0)$, would become equivalent in the limit of weak coupling ($\lambda\ll1$) to the linearized BGL theory.[5,6] This can be shown explicitly by approximating the one-particle propagators $S(\vec{k},t;\vec{p},\vec{p}\,')$ appearing in (3.12) by their lowest-order Vlassov approximation. Here, however, we are interested in keeping z and $\vec{k}$ finite. We then compute (3.4) with the aid of (3.12) and obtain after some algebra the following expression for $\bar{D}(\vec{0},z),\Omega(\vec{k},z)=(k^2/k_D^2)\omega_p\bar{D}(\vec{k},z)$:

$$\bar{D}(\vec{0},z)=2\frac{\omega_p}{z}-i\frac{\lambda}{15\pi^2}\left(I_1(z)+\frac{\omega_p}{z}I_2(z)+\frac{\omega_p^2}{z^2}I_3(z)\right)\,, \tag{3.13}$$

where the inverse Laplace transforms $I_j(t)$ ($j=1,2,3$) of the $I_j(z)$ appearing in (3.13) are given by

$$I_1(t)=\frac{\omega_p}{2}\int_0^\infty\frac{dx}{x^2}\left(3G^2_{nn}(x,t)+G_{nn}(x,t)x\frac{\partial}{\partial x}G_{nn}(x,t)\right)\,, \tag{3.14a}$$

$$I_2(t)=\omega_p\int_0^\infty\frac{dx}{x}G_{nn}(x,t)[G_{nl}(-x,t)+G_{ln}(-x,t)]\,, \tag{3.14b}$$

$$\begin{aligned}I_3(t)=6\omega_p\int_0^\infty dx\,[&G_{nl}(x,t)G_{ln}(-x,t)\\&+G_{nn}(x,t)\{G_{ll}(-x,t)+\tfrac{4}{3}G_\perp(-x,t)\}]\,,\end{aligned} \tag{3.14c}$$

where the $G_{ij}(x,t)$ are the correlation functions of (I 2.19) written in terms of the dimensionless wave vector $x=k/k_D$, $G_{ij}(x,t)\equiv G_{ij}(k,t)$, and where the transverse momentum correlation function G_t of (I 3.1) has been denoted $G_\perp\equiv G_t$ in (3.14c) in order to avoid confusion with the time variable. To proceed with the evaluation of the plasma mode (3.3) we need an explicit expression for the correlation functions G_{ij} appearing in (3.14). Within the weak coupling approximation (3.4) we can clearly compute the G_{ij} from the zeroth-order

Vlassov approximation to obtain $\bar{D}$ of (3.13) correct to first order in λ. We then disregard $\bar{\Sigma}^c$ in (I 2.16) and obtain the following zeroth-order expression for the G_{ij}

$$\lim_{\lambda\to 0} G_{ij}(\vec{k},z) \equiv G^0_{ij}(\vec{k},z)$$
$$= [\bar{G}_{ij}(\vec{k},z)/\epsilon^0(\vec{k},z)]$$
$$\times\{1+\delta_{in}\delta_{jn}h^0(k)+\delta_{i\perp}\delta_{j\perp}[\epsilon^0(k,z)-1]\} \tag{3.15}$$

where the $\bar{G}_{ij}$ are the free-particle correlation functions

$$\bar{G}_{ij}(\vec{k},z) = i\langle i\,|\,[z-\bar{\Sigma}^0(\vec{k})]^{-1}\,|j\rangle\,, \tag{3.16}$$

$\epsilon^0(\vec{k},z)$ being the Vlassov approximation to the dielectric constant (2.1) as obtained in (3.1), while $h^0(\vec{k})=\lim_{\lambda\to 0} h(\vec{k})$ is the Debye-Hückel result $h^0(k)=-k_D^2/(k^2+k_D^2)$ for the binary correlations. With the aid of (3.15) and (3.16) we can evaluate (3.13) exactly to first order in λ. A difficulty might appear with the large wave-vector integration limit of some of the expressions of (3.14), because the weak-coupling approximation ($\lambda \ll 1$) does not treat correctly some of the unavoidable close-collision contributions. As we will show this is the case for the damping rate [Eq. (3.3b)] but not for the dispersion coefficient [Eq. (3.3a)]. As usual, we cut off the divergent integrals at the Landau wave vector $k_L=(e^2\beta)^{-1}$, which will introduce a logarithmic indeterminacy. A more precise evaluation would require the use of improved equilibrium correlations $h(k)$. To proceed analytically we will use a further approximation to (3.14) and (3.15).

3. *Approximation*

In order to evaluate (3.14) by simple means we will use (3.15) with static screening $\epsilon^0(\vec{k},z) \simeq \epsilon^0(\vec{k},0)$. This popular approximation does probably yield good results for the $I_j(z)$ in the region of interest for the plasma mode ($z=\omega_p$). It is probably a less good approximation in the region $z=2\omega_p$, where according to (3.14) a resonance should occur due to the coupling of two Vlassov plasma modes. This approximation then will allow us to proceed analytically. After a lot of simple algebra,[20] which we will omit, we obtain for the real and imaginary parts of the Laplace transforms of (3.14) evaluated for real z, $z=\omega+i0$,

$$\mathrm{Im}\,I_1(\omega)=\tfrac{1}{4}\pi a[-\tfrac{7}{32}+\tfrac{13}{48}a^2-3a^4+\tfrac{7}{12}a^6+\tfrac{7}{24}a^4\phi(a^2)]\,, \tag{3.17a}$$

$$\mathrm{Im}\,I_2(\omega)=\tfrac{1}{4}\pi[-\tfrac{3}{4}-\tfrac{1}{2}a^2+a^4-a^4\phi(a^2)]\,, \tag{3.17b}$$

$$\mathrm{Im}\,I_3(\omega)=\tfrac{1}{4}\pi a\{\tfrac{47}{2}-\tfrac{3}{8}a^2-\tfrac{1}{4}a^4-16(\pi/a^2)^{1/2}$$
$$+(16-16a^2+4a^4+\tfrac{1}{4}a^6)[\phi(a^2)/\sqrt{a^2}]\}\,, \tag{3.17c}$$

where a denotes a reduced frequency $a=\omega/2\omega_p$, while $\phi(a^2)$ is an auxiliary function defined in terms of the error function

$$\mathrm{erf}x=\frac{2}{\sqrt{\pi}}\int_0^x dt\,e^{-t^2}$$

through $\phi(a^2)=\sqrt{\pi a^2}e^{a^2}(1-\mathrm{erf}\sqrt{a^2})$. For the real parts of $I_j(\omega)$ we obtain

$$\mathrm{Re}\,I_1(\omega)=\tfrac{1}{4}\sqrt{\pi}\,\tfrac{7}{6}[\tfrac{1}{2}-\tfrac{1}{4}a^2+\tfrac{1}{4}a^4-\tfrac{1}{4}a^6e^{a^2}E_1(a^2)]\,, \tag{3.18a}$$

$$\mathrm{Re}\,I_2(\omega)=\tfrac{1}{4}\sqrt{\pi}a[1-a^2+a^2e^{a^2}E_1(a^2)]\,, \tag{3.18b}$$

$$\mathrm{Re}\,I_3(\omega)=\tfrac{1}{4}\sqrt{\pi}[-\tfrac{11}{2}+\tfrac{21}{2}a^2-\tfrac{3}{2}a^4$$
$$+(-4+4a^2-\tfrac{15}{2}a^4+3a^6)\,e^{a^2}E_1(a^2)$$
$$+4E_1(a^2/x^2_{\max})]\,, \tag{3.18c}$$

where $E_1(x)=\int_1^\infty dt\,e^{-xt}/t$ is the exponential integral. In this way we obtain from (3.13), (3.17), and (3.18) an analytic expression for the long-wavelength dielectric constant of the weakly coupled OCP. As a result of the weak-coupling approximation, which does not treat the close collisions exactly, there appears a divergent term in (3.18c). As usual we will cut off the divergence by using a maximum wave vector $x_{\max}=k_{\max}/k_D=\lambda^{-1}(k_{\max}/k_L)$, where $k_L=(e^2\beta)^{-1}$ is the Landau wave vector corresponding to the thermal average of the inverse distance of closest approach. A more refined treatment is necessary to determine the precise value of $k_{\max}/k_L$. Note, however, that all remaining contributions to (3.17)–(3.19) are divergence-free. This is the case, for instance, with (3.17) and hence for γ_p of (3.3a). Note further that the previously used general properties for real ω, $\mathrm{Re}D(\vec{0},\pm\omega) = \pm\,\mathrm{Re}(\vec{0},\omega)$ and $\mathrm{Im}D(\vec{0},\pm\omega)=\mathrm{Im}D(\vec{0},\omega)$ are easily checked on Eqs. (3.13), (3.17), and (3.18).

4. *Weak-coupling result*

We now return to the plasma mode (3.3) and obtain from (3.17) and (3.18), for $\omega=\omega_p$ (or $a=\frac{1}{2}$) and using the tabulated values[21] of erf$x$ and $E_1(x)$, the result

$$\lim_{\lambda\to 0}\bar{\gamma}_p(\vec{k}=\vec{0})=k_D^2/k_s^2+2+1.63\lambda/15\pi\,, \tag{3.19a}$$

$$\lim_{\lambda\to 0}\bar{\Gamma}_p(\vec{k}=\vec{0})=(\lambda/15\pi^{3/2})[E_1(1/4x^2_{\max})-1.55]\,. \tag{3.19b}$$

In (3.19a) we still have to expand $k_D^2/k_s^2\equiv\chi_T^0/\chi_T$ for weak coupling. This is easily obtained by computing the isothermal compressibility χ_T

from the Debye-Hückel equation of state, while recalling that $\chi_T^0 = \beta/n$. If, moreover, we take $k_{\max} = k_L$ in (3.19b) and keep only the dominant terms with respect to λ ($\lambda \ll 1$) we obtain, finally,

$$\lim_{\lambda\to 0} \bar{\gamma}_p(\vec{k}=\vec{0}) = 3 - \frac{\lambda}{48\pi} + 1.63\frac{\lambda}{15\pi} \equiv 3 + 0.35\frac{\lambda}{4\pi}, \quad (3.20a)$$

$$\lim_{\lambda\to 0} \bar{\Gamma}_p(k=0) = (2\lambda/15\pi^{3/2})[\ln(2\lambda^{-1}e^{-\gamma/2}) - 0.77] \equiv \tfrac{2}{15}(\lambda/\pi^{3/2})(\ln\lambda^{-1} - 0.37), \quad (3.20b)$$

where $\gamma = 0.57\ldots$ is Euler's constant.[21] From (3.20a) we see that the static contribution to $\bar{\gamma}_p$, arising from the compressibility χ_T, tends to lower $\bar{\gamma}_p$ below its mean-field value (3.2a), just as in (3.11), but is finally dominated by the collisional correction to $\bar{\gamma}_p$, so as to yield a value of $\bar{\gamma}_p$ slightly *above* its mean-field value, in agreement with the numerical finding of Hansen *et al.*[11(a)] For large values of λ we will show in Sec. IV that $\bar{\gamma}_p$ actually becomes negative, thus reversing completely the present small-λ tendency. With respect to the collisional damping $\bar{\Gamma}_p$ there exists a vast literature concerned with the electron ion plasma. For the OCP, however, the only microscopic calculation of $\bar{\Gamma}_p$ comparable to the present one is the expression obtained by Dubois and Gilinsky,[13] who also consider some of the earlier results. To dominant order their result[22] is identical to (3.20b). The numerical factor $\ln\lambda^{-1} - 0.37$ of (3.20b) is slightly different from theirs; for $k_{\max} = 10k_D$, as considered by them, we obtain a value of 1.93, whereas they obtained 2.95 including dynamic screening and treating the short-distance cutoff quantum mechanically.

As an aside, it is interesting to point out the errors involved in some often-used approximations for the collision term $\bar{\Sigma}^c(\vec{k}, z)$. The quantity of interest here is $I(\omega) = I_1(\omega) + (\omega_p/\omega)I_2(\omega) + (\omega_p^2/\omega^2)I_3(\omega)$, which according to (3.13) determines $\bar{D}(\vec{0}, \omega)$ and hence the plasma mode (3.3). If we had used an infinite-wavelength collision operator $\bar{\Sigma}^c(\vec{k}, z) \simeq \bar{\Sigma}^c(\vec{0}, z)$, we would have obtained, by invoking momentum conservation, $I(\omega) \doteq (\omega_p^2/\omega^2)I_3(\omega)$. This then would yield a 10% error for $I(\omega_p)$. If we had used a zero-frequency or Markovian collision operator, $\bar{\Sigma}^c(\vec{k}, z) \simeq \bar{\Sigma}^c(\vec{k}, 0)$, we would have obtained $I(\omega) = I_1(0) + (\omega_p/\omega)I_2(0) + (\omega_p^2/\omega^2)I_3(0)$, and from (3.17) and (3.18) we obtain $\mathrm{Im}\,I_1(0) = \mathrm{Im}\,I_3(0) = \mathrm{Re}\,I_2(0) = 0$. Finally, if we had taken the full Markovian approximation, $\bar{\Sigma}^c(\vec{k}, z) \simeq \bar{\Sigma}^c(\vec{0}, 0)$, as would have been the case if we had used the BGL operator, we would have obtained the dominant term of $\bar{\Gamma}_p$ to within 50%, but again we would have obtained the wrong sign for $\mathrm{Im}\,I_3(0)$ and hence for $\bar{\gamma}_p - 3$.

IV. STRONG-COUPLING LIMIT

Now that we have seen what the transition from zero coupling ($\lambda = 0$) to weak coupling ($\lambda \ll 1$) looks like, one might wonder whether we can say something about the general expressions of the dielectric constant (2.3) and the plasma mode (3.3) in the opposite case of strong coupling ($\lambda \gg 1$). It is easily shown that in this limit we will not be able to write expressions as explicit as those of Sec. III. Enough can be said, however, to demonstrate the important modifications which occur and to show that they are at least in qualitative agreement with the computer results of Hansen *et al.*[11]

In a nutshell, our reasoning would run as follows: Once we have used the conservation laws to extract the relevant factors, the plasma mode, defined by Eq. (2.10) or (3.3), should depend for small k only on the dimensionless variable ω_c/ω_p, where ω_c is a collision frequency $\omega_c \simeq \lambda\omega_p$. For weak coupling we would expand our expressions for small ω_c/ω_p values. This is then equivalent to a large ω_p expansion. In the opposite limit of strong coupling it is ω_p/ω_c which becomes small and we then can expect a low-frequency analysis to become valid. In the low-frequency region we then can expect to establish contact with hydrodynamical concepts and with the popular hydrodynamic derivation of the plasma mode.[10] The remarkable difference with neutral-particle systems then stems from the fact that this connection with hydrodynamics will occur here *only* for large enough coupling ($\lambda \gg 1$). Within this context the connection with the traditional Chapman-Enskog expansion can then be understood as follows: Let us estimate the collision term of a given kinetic equation, for instance, (I 2.16), by the collision frequency ω_c. The Vlassov mean-field term is singular for charged particles and of order $\omega_p(k_D/k)$, while the remaining streaming terms are of order k or z. It then follows that for the collision term to dominate the kinetic equation, and hence the Chapman-Enskog expansion to become valid, we need to satisfy the condition $\omega_p(k_D/k) \ll \omega_c \simeq \lambda\omega_p$ or $k_D \ll \lambda k$. Hence the smaller we take k the larger the coupling will have to be, and since we always need to take $k < k_D$ this implies strong coupling, $\lambda > 1$. Note that no such relation between k and λ exists for a neutral-particle system, because there the mean-field term is regular and of order k. To proceed, let us see whether these expectations are confirmed by a closer analysis of the microscopic expressions of Sec. II.

A. Strong-coupling expansion

From the general expressions (2.3) and (2.4) we see that the λ dependence of the dielectric constant $\epsilon(\vec{k},z)$ is concentrated into the static and the dynamic quantities $\omega_p^2(k)$ and $\Omega(\vec{k},z)$, defined, respectively, by Eqs. (2.5) and (2.6). Let us consider the more difficult dynamic quantity $\Omega(\vec{k},z)$. This quantity is seen from (2.6) and (2.7) to depend on λ only through the collision term $\overline{\Sigma}^c(\vec{k},z)$, as shown explicitly in Eq. (2.7). From (I 2.17c) we recall the general expression of the collision term

$$\Sigma^c(\vec{k},z;\vec{p},\vec{p}\,')=(\delta f(\vec{k},\vec{p})\,|\,LQ(z-QLQ)^{-1}QL\,|\,\delta f(\vec{k},\vec{p}\,'))\times[n\phi(\vec{p}\,')]^{-1},$$

where $\delta f(\vec{k},\vec{p})$ denotes the fluctuation of the phase-space density f evolving according to the Liouville operator L while Q projects out the one-particle states as more fully explained in I. The dependence of $\overline{\Sigma}^c$ on λ can have two origins. First, each interaction will introduce a factor λ, for instance $\delta L\sim\lambda$, where δL denotes the potential part of the Liouville operator $L=L_0+\delta L$. Second, each particle summation will introduce a factor λ^{-1}. In these estimates we are using e^2 and n^{-1} as discreteness parameters,[5] a procedure originally given by Rostoker and Rosenbluth[23] and which can be made rigorous by going over to dimensionless variables, with k_D^{-1} and ω_p^{-1} as finite space-time units, so as to preserve the plasma characteristics.[5(c)] We then have $(\delta f|LQ\sim\lambda$, because, as shown in I, only the potential part of L contributes here, $(\delta f|LQ\equiv(\delta f|\,\delta LQ$. Because of the presence of Q, however, there will be necessarily a summation over a second particle, and hence $(\delta f|LQ\sim\lambda\lambda^{-1}$. Taking into account the $(n\phi)^{-1}$ factor in $\overline{\Sigma}^c$ we arrive at the general form $\overline{\Sigma}^c=\lambda\hat{\Sigma}(\lambda)$, where the remaining λ dependence of $\hat{\Sigma}(\lambda)$ stems from the intermediate propagator $(z-QLQ)^{-1}$. Quite generally we can write $QLQ=A+\lambda B$, where A stems from the contributions of the free motion L_0 and also from those potential contributions which are compensated by particle summations. In the weak-coupling limit we neglect B and our analysis stops. In the limit of strong coupling it is, however, precisely the λB term which will become important, as it will eventually dominate the $z-A$ term. Naively we could write $(z-QLQ)^{-1}\sim(-\lambda B)^{-1}$ as $\lambda\to\infty$; however, in order to take no unnecessary risks with the complicated operator character of these expressions we will assume only that as $\lambda\to\infty$ we can neglect the finite constant z ($\sim\omega_p$) in front of QLQ. More precisely we will assume that

$$\lim_{\lambda\to\infty}\lambda^{-1}\overline{\Sigma}^c(\vec{k},z)=\lim_{\lambda\to\infty}\lambda^{-1}\overline{\Sigma}^c(\vec{k},0)\ , \tag{4.1}$$

where we have multiplied $\overline{\Sigma}^c$ with λ^{-1} to extract the overall λ factor $\overline{\Sigma}^c(\lambda)=\lambda\hat{\Sigma}(\lambda)$. The advantage of (4.1) is that $\overline{\Sigma}^c$ is a well-defined two-body quantity. Equation (4.1) then states that the large-λ behavior of $\overline{\Sigma}^c(\vec{k},z)$ is, to dominant order in λ^{-1}, the same as the large-λ behavior of its Markovian limit $\overline{\Sigma}^c(\vec{k},0)$, a quantity which is generally assumed to exist. As yet we are not able to evaluate the limiting value involved in (4.1), but already, as such, Eq. (4.1) will allow us to draw a number of interesting conclusions.

The above considerations do not constitute a proof. They instead indicate the plausibility of our arguments. These arguments could have been given (or hidden) in a diagrammatical language or some other formal device, but we think we have summarized their essence. Taking for granted the large-λ analysis of $\overline{\Sigma}^c$ contained in Eq. (4.1), we now proceed with the λ analysis of $\Omega(\vec{k},z)$. From Eqs. (2.6) and (2.7) we see that $\overline{\Sigma}^c$ enters the second term in the right-hand side of Eq. (2.7) through the propagator $\overline{Q}[z-\overline{Q}\overline{\Sigma}(\vec{k},z)\overline{Q}]^{-1}\overline{Q}$. To second order in k we, in fact, need consider its value only for vanishing k, $\overline{Q}[z-\overline{Q}\overline{\Sigma}^c(\vec{0},z)\overline{Q}]^{-1}\overline{Q}$. For large λ we assume $\overline{\Sigma}^c$ itself becomes large and hence we can replace this propagator by $-\overline{Q}[\overline{Q}\overline{\Sigma}^c(\vec{0},z)\overline{Q}]^{-1}\overline{Q}$, an operator which is known to control the transport coefficients[9,17] and which exists at $z=0$ whenever those coefficients exist. This result, together with (4.1), allows us to write $\Omega_{ij}(\vec{k},z)\simeq\Omega_{ij}(\vec{k},0)$ for large λ. We also recall from I that momentum conservation implies $\Omega_{ll}(\vec{k},z)=-ik^2D_l(\vec{k},z)$, $\Omega_{l\epsilon}(\vec{k},z)=kD_{l\epsilon}(\vec{k},z)$, and $\Omega_{\epsilon l}(\vec{k},z)=kD_{\epsilon l}(\vec{k},z)$, while because of energy conservation we have $\Omega_{\epsilon\epsilon}(\vec{k},z)=-ik^2D_\epsilon(\vec{k},z)+zB_\epsilon(\vec{k},z)$. Finally, we can thus write $\Omega(\vec{k},z)$ of Eq. (2.6) for small k and large λ as

$$\lim_{\lambda\to\infty}\Omega(\vec{k},z)=\lim_{\lambda\to\infty}\{-ik^2D_l(\vec{0},0)+k^2D_{l\epsilon}(\vec{0},0)D_{\epsilon l}(\vec{0},0)/z[1-B_\epsilon(\vec{0},0)]+\mathcal{O}(k^4)\}. \tag{4.2}$$

This is our basic strong-coupling result. It can be further simplified, but in order to establish easy contact with known quantities we keep (4.2) as such.

B. Relation to hydrodynamics

The connection between (4.2) and the hydrodynamical concepts can be established as follows: From (I 3.11) we obtain for the second term on the right-hand side of (4.2)

$$D_{l\epsilon}(\vec{0},0)D_{\epsilon l}(\vec{0},0)/[1-B_\epsilon(\vec{0},0)]=\overline{c}^2-c^2\equiv c^2(c_P/c_V-1)\ , \tag{4.3}$$

where c and $\bar{c}$ denote, respectively, the isothermal and isentropic sound speeds, whereas c_P/c_V is the specific heat ratio. The first term on the right-hand side of (4.2) can be identified as follows: From (I 3.15) we obtain the longitudinal viscosity ϕ, consisting of a shear contribution (η) and a bulk viscosity (ξ), $\phi=\frac{4}{3}\eta+\xi$, as

$$\frac{\phi}{nm}=D_l(\vec{0},0)+\frac{d}{dz}\frac{D_l(\vec{0},z)\Omega_{\epsilon\epsilon}(\vec{0},z)+iD_{l\epsilon}(\vec{0},z)D_{\epsilon l}(\vec{0},z)}{1-B_\epsilon(\vec{0},z)}\bigg|_{z=0}. \tag{4.4}$$

As well known,[24] the bulk viscosity can be split into a Markovian (ξ_M) and a non-Markovian (ξ_{NM}) contribution, the latter being called the nonstationarity contribution in Ref. 24. The second term of Eq. (4.4) corresponds clearly to the non-Markovian corrections to the kinetic equation and hence determines ξ_{NM}/nm. The first term of (4.2) is thus determined as

$$D_l(\vec{0},0)=\phi_M/nm\equiv(\tfrac{4}{3}\eta+\xi_M)/nm\,, \tag{4.5}$$

ϕ_M being the Markovian part of the longitudinal viscosity ϕ. From (4.2), (4.3), and (3.3a) we obtain

$$\lim_{\lambda\to\infty}\bar{\gamma}_p(\vec{k}=0)=\lim_{\lambda\to\infty}\left[\frac{k_D^2}{k_s^2}+\frac{c^2}{v_0^2}\left(\frac{c_P}{c_V}-1\right)\right]. \tag{4.6}$$

If we also remember that $k_D^2/k_s^2\equiv c^2/v_0^2\equiv\chi_T^0/\chi_T$ we can simplify (4.6) and write our final result as

$$\lim_{\lambda\to\infty}\bar{\gamma}_p(\vec{k}=\vec{0})=\lim_{\lambda\to\infty}[(c_P/c_V)\chi_T^0/\chi_T], \tag{4.7a}$$

$$\lim_{\lambda\to\infty}\bar{\Gamma}_p(\vec{k}=0)=\frac{k_D^2}{\omega_p}\lim_{\lambda\to\infty}\left(\frac{\frac{4}{3}\eta}{nm}+\frac{\xi_M}{nm}\right), \tag{4.7b}$$

whereas for the dielectric constant we would obtain from (2.3), (2.4), and (4.2)–(4.5) the simple result

$$\epsilon(\vec{k},z)=1-\frac{\omega_p^2}{z^2-k^2\bar{c}^2+zik^2\phi_M/nm},\quad k\ll k_D,\quad 1\ll\lambda. \tag{4.8}$$

As already stated we are not able to compute explicitly the limiting values ($\lambda\gg1$) involved in Eqs. (4.7) and (4.8). As such these results are, however, of interest and at least in qualitative agreement with the computer results.[11] Indeed, it was shown in Ref. 11 that for large λ, c_P/c_V tends to unity while χ_T^0/χ_T becomes negative for $\lambda>\lambda_c\simeq36\pi$, so that the limiting value appearing in Eq. (4.7a) is actually negative, in agreement with the observed negative dispersion[11(a)] of the plasma mode for large λ. Moreover, Eq. (4.7b) also offers a hint to the understanding of the computer finding indicating a small bulk viscosity contribution to the longitudinal viscosity ϕ compared to the shear viscosity contribution $\frac{4}{3}\eta$. Indeed, the longitudinal viscosity ϕ was calculated in Ref. 11(b) from the longitudinal correlation function $G_{ll}(\vec{k},z)$ through "generalized hydrodynamics." In I we have shown that as $k\to0$ the longitudinal correlation function $G_{ll}(\vec{k},z)$ is dominated by the plasma mode, while Eq. (4.7b) indicates that in the hydrodynamic region $\lambda\to\infty$ only the Markovian part of the bulk viscosity ξ_M, instead of the complete bulk viscosity $\xi=\xi_M+\xi_{\mathrm{NM}}$, contributes to the damping of the plasma mode. This result, added to the observed[11(b)] increase of η for large λ, might well explain the smallness of the observed bulk viscosity contribution to ϕ.

There are two remarkable differences between our results (4.7) and (4.8) and the hydrodynamic expression one obtains from the linearized hydrodynamic equations with the mean electric field added.[10] Indeed, one obtains Eq. (4.7) but with the full bulk viscosity in (4.7b) instead of ξ_M and without the restriction to large λ values ($\lambda\gg1$). As a consequence many authors have extrapolated the result (4.7) back to the weak-coupling region, in which case (4.7a) would yield, for example, $\bar{\gamma}_p=\frac{5}{3}$, a result which differs both from the weak-coupling result (3.20a) and from the "hydrodynamic" or strong-coupling result (4.7a) predicting a negative value for $\bar{\gamma}_p$. These discrepancies can, however, easily be understood within the present theory. In fact, adding the Vlassov term to the standard (neutral fluid) linearized hydrodynamical equations amounts to first assuming the hydrodynamical equations to be valid and then adding the term which precisely prevents them from being generally valid. Indeed, starting from first principles we have shown in I how the mean electric field, because of its Coulomb singularity, shifts some of the static ($z=0$) transport coefficients to their finite-frequency value ($z=\omega_p$). This Coulomb singularity also prevents the collision term from dominating automatically the kinetic equation for small k values and hence prevents the Chapman-Enskog expansion from being generally valid. Said differently, the hydrodynamic description of the OCP does not automatically become valid as $k\to0$, in contradistinction to systems with a regular interaction potential [$V(k=0)<\infty$]. This will happen only when the coupling is strong enough for the collision term to dominate the Vlassov singularity. This, then presumably implies $\lambda>k_D/k$. The hydrodynamic results (4.7) and (4.8) appear, then, as the first or dominant terms of a strong-coupling expansion of the exact results of Sec. II. Indeed, in the strong-coupling limit the finite-frequency or non-Markovian transport coefficients should, according to (4.1), be well approximated by their low-frequency or Markovian approximation, showing the internal consistency of our arguments.

V. CONCLUSIONS

Starting from first principles we have derived an exact expression (2.3) for the dielectric constant $\epsilon(\vec{k}, z)$ of a classical one-component plasma with pure Coulomb interactions. The advantage of this expression is to naturally incorporate the symmetry properties and conservation laws of the system while nicely separating the static from the dynamic contributions. The two oppositely propagating plasma modes we obtain from $\epsilon(\vec{k}, z)$ reduce for long wavelengths to those found in I, where we also obtained the exact expressions of the heat and shear modes of the OCP. These five long-wavelength modes taken together are very general properties of the OCP, equivalent to the five hydrodynamical modes of uncharged-particle systems. They merely reflect the conservation of particle number, momentum, and energy. In contradistinction with the sound modes of neutral-particle systems, the plasma modes of the OCP cannot be readily expressed in terms of the hydrodynamical transport coefficients and hence they cannot be properly termed hydrodynamical modes. This discrepancy can, as shown in I, entirely be ascribed to the introduction by the singular Coulomb potential of the finite frequency ω_p. Using the present theory we have followed the plasma mode as one varies the coupling constant $\lambda = k_D^3/n$, the only parameter still at our disposal. We have found qualitative agreement with the computer results of Hansen *et al.*[11] over the entire range of λ values covered. For strong coupling ($\lambda \gg 1$) we recover a hydrodynamical expression for the plasma mode which differs significantly, however, from the widely used expression obtained from the linearized hydrodynamical equations. That for strong coupling hydrodynamics can again come into play seems reasonable in view of the fact that in this limit the kinetic equation can again become collision dominated if the collision term is large enough to dominate, for given k, the singular Vlassov term. The region of validity of the hydrodynamic treatment should be, roughly speaking, restricted to those values of λ and k satisfying the condition $1 \ll k_D/k \ll \lambda$.

The extension of the present results to a real two-component plasma, instead of the OCP considered here, will allow one to inquire also for experimental, instead of computer, evidence. However, such an extension is not completely trivial, and is planned for the future.[12]

ACKNOWLEDGMENTS

I would like to thank J. P. Hansen for suggesting this problem, R. Balescu and P. Résibois for going through our manuscript, and I acknowledge some interesting conversations with G. Nicolis, J. W. Turner, and J. Wallenborn.

---

*Chercheur Qualifié du Fonds National Belge de la Recherche Scientifique.

†Association Euratom–Etat Belge.

[1]F. M. Penning, Physica (Utr.) 6, 241 (1926).

[2]L. Tonks and I. Langmuir, Phys. Rev. 33, 195 (1929).

[3]L. D. Landau, J. Phys. USSR 10, 25 (1946); also see, e.g., *L. D. Landau*, edited by D. Ter Haar (Pergamon, Press, New York, 1969), p. 96, Vol. 2.

[4]A. Vlassov, Zh. Eksp. Teor. Fiz. 8, 291 (1938) [English translation available in Sov. Phys.-Usp. 10, 721 (1968)].

[5]Excellent reviews of the background material are given by (a) D. Pines and Ph. Nozières [*The Theory of Quantum Liquids* (Benjamin, New York, 1966), Vol. 1], (b) P. C. Martin [in *Many-Body Physics*, edited by C. De Witt and R. Balian (Gordon and Breach, New York, 1968)], and (c) S. Ichimaru [*Basic Principles of Plasma Physics* (Benjamin, Reading, Mass. 1973)].

[6]See, e.g., D. Montgomery and D. A. Tidman, *Plasma Kinetic Theory* (McGraw-Hill, New York, 1964).

[7]C. S. Wu, Phys. Rev. 138, A51 (1965); 140, A118 (1965), and references cited therein.

[8]As a basic reference there is the paper by C. Oberman, A. Ron, and J. Dawson [Phys. Fluids 5, 1515 (1962)].

[9]M. Baus, Physica (Utr.) 79A, 377 (1975). This paper is refered to as I. (I2.16) means Eq. (2.16) of I.

[10]For the general background material and a modern approach see Chaps. 12 and 13 of the recent book by R. Balescu [*Equilibrium and Non-Equilibrium Statistical Mechanics* (Wiley, New York, 1975)].

[11](a) J. P. Hansen, I. R. McDonald, and E. L. Pollock, Phys. Rev. A 11, 1025 (1975); (b) P. Vieillefosse and J. P. Hansen, Phys. Rev. A 12, 1106 (1975), and references cited therein.

[12]For a preliminary view see M. Baus, in *Abstracts of the Contributed Papers, International Union of Pure and Applied Physics Conference on Statistical Physics*, edited by N. Menyhard, (Akad. Kiado, Budapest, 1975).

[13]D. F. Dubois and V. Gilinsky, Phys. Rev. 135, A1519 (1964), and references cited therein.

[14]H. Totsuji and S. Ichimaru, Prog. Theor. Phys. 52, 42 (1974).

[15]Theoretical studies of the onset of short-range order in the pair correlations have been given by F. del Rio and H. E. De Witt [Phys. Fluids 12, 791 (1969)] and Ph. Choquard and R. R. Sari [Phys. Lett. 40A, 109 (1972)] and completed by C. Deutsch, Y. Furutani, and M. M. Gombert [Phys. Rev. A 13, 2244 (1976)].

[16]S. Ichimaru, H. Tosuji, T. Tange, and D. Pines, Prog. Theor. Phys. 54, 1077 (1975).

[17]D. Forster, Phys. Rev. A 9, 943 (1974); D. Forster and P. C. Martin [Phys. Rev. A 2, 1575 (1970)] have written a very informative paper on this subject.

[18]E. J. Linnebur and J. J. Duderstadt, Phys. Fluids 16, 665 (1973).

[19]M. Baus and J. Wallenborn, Phys. Lett. 55A, 90 (1975), and J. Stat. Phys. (to be published).

[20]In the course of this algebra we have used

$$\int_0^\infty \frac{dx}{x}\frac{e^{-x^2}}{a^2+x^2}\int_0^x dt\, e^{t^2}=\frac{\pi^{3/2}}{4a^2}\left[1-e^{a^2}(1-\operatorname{erf}\sqrt{a^2})\right],$$

a result given by J. W. Turner (private communication) and one which I could not find in available tables of integrals.

[21]M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions* (Dover, New York, 1965).

[22]The dominant $\lambda \ln\lambda^{-1}$ term of (3.20b) also agrees with an unpublished result of H. Totsuji announced in Eq. (4.4) of Ref. 16.

[23]N. Rostoker and M. N. Rosenbluth, Phys. Fluids 3, 1 (1960).

[24]G. Nicolis and G. Severne, J. Chem. Phys. 44, 1477 (1966).