Wigner function as the expectation value of a parity operator

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It is pointed out that the Wigner function $f(r,p)$ is $2/h$ times the expectation value of the parity operator that performs reflections about the phase-space point r,p. Thus $f(r,p)$ is proportional to the overlap of the wave function ψ with its mirror image about r,p; this is clearly a measure of how much ψ is *centered* about r,p, and the Wigner distribution function now appears physically more meaningful and natural than it did previously.

In 1932, Wigner[1] associated with the quantum wave function $\psi(r)$ a phase-space quasiprobability distribution function

$$f(r,p)=\frac{2}{h}\int ds\, e^{-2ips/\hbar}\psi(r-s)^*\psi(r+s)\,, \tag{1}$$

or, in terms of the momentum representative $\tilde{\psi}(p)=h^{-1/2}\int dr\, e^{-ipr/\hbar}\psi(r)$,

$$f(r,p)=\frac{2}{h}\int dk\, e^{-2ikr/\hbar}\tilde{\psi}(p+k)^*\tilde{\psi}(p-k)\,. \tag{2}$$

This Wigner "representation" has proved useful for studying the passage from quantum to classical mechanics and establishing quantum corrections to classical results, and generally it enhances understanding by favoring the use of classical intuition in quantum problems.[2]

At first sight the constructions (1) and (2) seem rather *ad hoc* and devoid of any deep physical or mathematical significance. A somewhat more meaningful expression for $f(r,p)$ was provided by Moyal,[3] namely

$$f(r,p)=h^{-2}\int dk\int ds\, e^{-i(kr+sp)/\hbar}\,\langle\psi|e^{i(k\hat{R}+s\hat{P})/\hbar}|\psi\rangle\,, \tag{3}$$

where $\hat{R}$ and $\hat{P}$ are the position and momentum operators, respectively, satisfying $[\hat{R},\hat{P}]=i\hbar$. The form (3) is conspicuous to statisticians: $\langle\psi|e^{i(k\hat{R}+s\hat{P})}|\psi\rangle$ appears as a "characteristic function," being the expectation of the operator that corresponds to the function $e^{i(kr+sp)}$ in Weyl's rule of association.[4]

Here we wish to point out that $f(r,p)$ has a much more direct physical meaning, in that it is the expectation value of the parity operator about the phase-space point r,p.

To show this, let us first rewrite

$$f(r,p)=(2/h)\langle\psi|\Pi_{rp}|\psi\rangle\,, \tag{4}$$

where the operator Π_{rp} has the following three equivalent expressions in view of (1)–(3):

$$\Pi_{rp}=\int ds\, e^{-2ips/\hbar}|r-s\rangle\langle r+s|\,, \tag{5}$$

$$=\int dk\, e^{-2ikr/\hbar}|p+k\rangle\langle p-k|\,, \tag{5'}$$

$$=\frac{1}{2h}\int dk\int ds\,\exp\{(i/\hbar)[k(\hat{R}-r)+s(\hat{P}-p)]\}\,, \tag{5''}$$

where $|r\rangle$ and $|p\rangle$ are eigenstates of $\hat{R}$ and $\hat{P}$, respectively. Let us now consider the special case $r=0$, $p=0$, and denote $\Pi_{r=0,p=0}\equiv\Pi$; we have

$$\Pi=\int dr\,|-r\rangle\langle r|\,, \tag{6}$$

$$=\int dp\,|p\rangle\langle -p|\,, \tag{6'}$$

$$=\frac{1}{2h}\int dk\int ds\, e^{i(k\hat{R}+s\hat{P})/\hbar}\,. \tag{6''}$$

From (6) or (6′) it is immediately apparent that Π is the parity operator (about the origin): it changes $\psi(r)$ into $\psi(-r)$ and $\tilde{\psi}(p)$ into $\tilde{\psi}(-p)$, or equivalently (note that $\Pi^{-1}=\Pi$),

$$\Pi\hat{R}\Pi=-\hat{R}\,,\quad \Pi\hat{P}\Pi=-\hat{P}\,. \tag{7}$$

We now observe that Π_{rp} may be obtained from Π by a unitary transformation

$$\Pi_{rp}=D(r,p)\Pi D(r,p)^{-1}\,; \tag{8}$$

here

$$D(r,p)\equiv e^{i(p\hat{R}-r\hat{P})/\hbar} \tag{9}$$

is a phase-space displacement operator, introduced by Glauber[5] in connection with a different, though related, type of phase-space representation of quantum mechanics, the coherent-state representation. We have the actions

$$\begin{aligned} D(r,p)^{-1}\hat{R}D(r,p)&=\hat{R}+r\,,\\ D(r,p)^{-1}\hat{P}D(r,p)&=\hat{P}+p\,, \end{aligned} \tag{10}$$

and more generally

$$D(r,p)^{-1}F(\hat{R},\hat{P})D(r,p)=F(\hat{R}+r,\hat{P}+p)$$

$[F(\hat{R},\hat{P})$ being a power series in $\hat{R}$ and $\hat{P}]$, whence Eq. (8), in view of (5″) and (6″) [and noting that $D(r,p)^{-1}=D(-r,-p)$].

Using (7), (8), and (10), we readily verify that

$$\Pi_{rp}(\hat{R}-r)\Pi_{rp}=-(\hat{R}-r)\,,\qquad \Pi_{rp}(\hat{P}-p)\Pi_{rp}=-(\hat{P}-p)\,; \tag{11}$$

that is, Π_{rp} reflects about the phase-space point r,p and is thus the parity operator about that point. Note that

$$(\Pi_{rp})^2=1\,. \tag{12}$$

The Wigner function, Eq. (4), is thus $2/h$ times the expectation value of the parity operator about r,p. Alternatively, $f(r,p)$ is proportional to the overlap of ψ with its mirror image about r,p, which is clearly a measure of how much ψ is "centered" about r,p.

Let us now discuss some simple implications of the preceding considerations.

We first observe that Π_{rp} has eigenvalues ± 1 [in view of (12)], and its eigenfunctions $\phi^{\pm}_{rp}$, satisfying

$$\Pi_{rp}|\phi^{\pm}_{rp}\rangle=\pm|\phi^{\pm}_{rp}\rangle\,, \tag{13}$$

are functions that are either symmetric or antisymmetric about r,p. They may be obtained by displacing in phase-space functions of the same symmetry about the origin, i.e.,

$$|\phi^{\pm}_{rp}\rangle=D(r,p)|\phi^{\pm}\rangle\,, \tag{14}$$

where ϕ^+ and ϕ^- satisfy $\Pi|\phi^{\pm}\rangle=\pm|\phi^{\pm}\rangle$, or equivalently $\phi^{\pm}(-r)=\pm\phi^{\pm}(r)$, $\tilde{\phi}^{\pm}(-p)=\pm\tilde{\phi}^{\pm}(p)$.

Let us define projectors P^+_{rp} and P^-_{rp} on the spaces of functions symmetric and functions antisymmetric about r,p, respectively:

$$P^{\pm}_{rp}\equiv\tfrac{1}{2}(1\pm\Pi_{rp}) =D(r,p)P^{\pm}D(r,p)^{-1}\,, \tag{15}$$

where $P^{\pm}=\frac{1}{2}(1\pm\Pi)$ projects on the space of functions symmetric (antisymmetric) about the origin. We have

$$(P^{\pm}_{rp})^2=P^{\pm}_{rp}\,, \tag{16}$$

$$P^+_{rp}+P^-_{rp}=1\,, \tag{17}$$

$$P^+_{rp}-P^-_{rp}=\Pi_{rp}\,. \tag{18}$$

Let us now separate ψ into components symmetric and antisymmetric about r,p:

$$\psi=\psi^+_{rp}+\psi^-_{rp}\,, \tag{19}$$

where

$$|\psi^{\pm}_{rp}\rangle\equiv P^{\pm}_{rp}|\psi\rangle\,. \tag{20}$$

By (16) we have

$$\langle\psi|P^{\pm}_{rp}|\psi\rangle=\langle\psi^{\pm}_{rp}|\psi^{\pm}_{rp}\rangle\equiv\|\psi^{\pm}_{rp}\|^2\,. \tag{21}$$

Then by (4), (18), and (21),

$$f(r,p)=(2/h)(\|\psi^+_{rp}\|^2-\|\psi^-_{rp}\|^2)\,. \tag{22}$$

That is, the Wigner function equals $2/h$ times the difference of the squared norms of the symmetric and antisymmetric (about r,p) parts of ψ. By (17) and (21) we further have

$$\langle\psi|\psi\rangle=1=\|\psi^+_{rp}\|^2+\|\psi^-_{rp}\|^2\,. \tag{23}$$

This implies $\|\psi^{\pm}_{rp}\|\leq 1$, implying in turn, in view of (22), that $f(r,p)$ is bounded by the values $-2/h$ and $2/h$:

$$-2/h\leq f(r,p)\leq 2/h\,. \tag{24}$$

This result was previously obtained by means of Schwarz's inequality.[6] We can now be much more specific: the lower equality in (24) is realized if and only if ψ is antisymmetric about r,p, i.e., of the form (14) (−sign), and the upper equality if and only if ψ is antisymmetric about r,p. One may, in fact, construct ψ such that the corresponding $f(r,p)$ equal any preassigned value f inside the interval $[-2/h,2/h]$. Indeed, given any two *normalized* functions ϕ^+ and ϕ^-, respectively symmetric and antisymmetric about the origin, set

$$|\psi\rangle=D(r,p)(c_+|\phi^+\rangle+c_-|\phi^-\rangle)\,. \tag{25}$$

We then have $\langle\psi|\Pi_{rp}|\psi\rangle=c_+^2-c_-^2$ and $\langle\psi|\psi\rangle=c_+^2+c_-^2$. We thus simply require that c_+ and c_- satisfy $(2/h)(c_+^2-c_-^2)=f$ and $c_+^2+c_-^2=1$.

[1]E. P. Wigner, Phys. Rev. 40, 749 (1932).

[2]See, for instance, R. Balescu, *Equilibrium and Non-Equilibrium Statistical Mechanics* (Wiley, New York, 1975); S. R. de Groot and L. G. Suttorp, *Foundations of Electrodynamics* (North-Holland, Amsterdam, 1972); E. A. Remler, Ann. Phys. (N.Y.) 95, 455 (1975); B. Leaf, J. Math. Phys. 9, 65, 769 (1968).

[3]J. E. Moyal, Proc. Cambr. Phil. Soc. 45, 99 (1949).

[4]See, e.g., L. Cohen, in *Contemporary Research in the Foundations and Philosophy of Quantum Mechanics*, edited by C. A. Hooker (Reidel, New York, 1973).

[5]R. J. Glauber, Phys. Rev. 131, 2766 (1963), Eqs. (3.10) and (3.11).

[6]See, e.g., S. R. de Groot and L. G. Suttorp, *Foundations of Electrodynamics* (North-Holland, Amsterdam, 1972).