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One-dimensional gravitational gas

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The Schrödinger equation for a system of N mutually gravitating particles is solved in one dimension. Solutions are constructed out of a product of Airy functions. The eigenvalues are shown to be the zeros of the derivative of the Airy function. The total energy of the system is found to be proportional to $N^{5/3}$ for a boson system.

I. INTRODUCTION–GRAVITATION IN ONE DIMENSION

In one dimension, the gravitational interaction between two particles is given by

$$V = 2\pi G m_1 m_2 |x_1 - x_2|, \tag{1}$$

where $|x_1 - x_2|$ is the distance between the two particles, and G is the gravitational constant. m_1 and m_2 are the masses of the two particles.

The physical picture of this system is that of two parallel infinite sheets, each with a mass per unit area m_1 or m_2. The sheets move along a line perpendicular to each other. This system is the gravitational analog of the electrical system of two parallel capacitor plates which are free to move along a line perpendicular to them. The gravitational sheets are allowed to penetrate each other.

The N-body problem may be considered as N self-gravitating sheets which pass through each other. This model of a self-gravitating system is quite old. It was first used in stellar dynamics to model the behavior of halo stars.[1-3] In spiral galaxies which are essentially two dimensional, there is nevertheless a galactic component of stars moving perpendicular to the galactic plane while they partake of the more general three-dimensional motion. The motion perpendicular to the galactic plane is approximated by assuming that the stars move as sheets parallel to the plane of the galaxy, oscillating to and fro about this plane.

In this study, we will consider the problem as an N-body problem in one dimension, calling the sheets as particles, moving along a line, and interpenetrating when the particles meet. In Sec. II, we will present the problem classically, and point out that no more than trivial solutions seem to be available. Sections III–VI will show the quantum solution to the problem, and in Sec. VIII, we will discuss the nonthermodynamic nature of this system.

II. A CLASSICAL SOLUTION TO THE N-BODY PROBLEM

The approach usually taken in stellar dynamics is to use the collisionless Boltzmann equation with a self-consistent field. Even for this approach of solving for the distribution function, there are no exact time-dependent solutions. There are, however, interesting numerical studies.[4,5] In this section, we will show an exact solution of the Liouville equation for the case of homologous collapse and expansion.

Let us consider an N-body system of identical particles. Put $\gamma = 2\pi G m^2$ where m is the mass of each particle. Then the Liouville equation for this system is

$$\frac{\partial f}{\partial t} + \sum_j \frac{p_j}{m}\frac{\partial f}{\partial x_j} + \gamma \sum_{i<j} \epsilon(x_i - x_j)\left(\frac{\partial}{\partial p_j} - \frac{\partial}{\partial p_i}\right) f = 0, \tag{2}$$

$$\epsilon(x_i - x_j) = \begin{cases} 1 & \text{if } x_i > x_j \\ 0 & \text{if } x_i = x_j \\ -1 & \text{if } x_i < x_j, \end{cases} \tag{3}$$

where $f = f(x_1, x_2, \ldots, x_N, p_1, p_2, \ldots, p_N, t)$ is the N-particle distribution function. The p's are the momenta of the particles. For $t < (a/\gamma)^{1/2}$, the collapse time of an initially homogeneous system, an exact solution of the Liouville equation is

$$f(t) = \sum_{j=1}^{N} \delta\left(x_j - x_0 - (j-1)a + \tfrac{1}{2}\gamma(N - 2j + 1)t^2\right) \times \delta\left(p_j - \gamma(N - 2j + 1)t\right), \tag{4}$$

where a is the distance between two adjacent particles, δ is the usual Dirac δ function, and it is assumed that all the particles are at rest for $t = 0$. The solution above corresponds to homologous collapse. For $(a/\gamma)^{1/2} < t < 2(a/\gamma)^{1/2}$, the system undergoes homologous expansion. This is the simplest solution for the classical problem, and there does not seem to be known other exact time-dependent solutions.

Stationary solutions of the problem have also been studied, mostly to predict the density and probability distribution.[1,2] Some studies have gone so far as comparing the theoretical results with actual stellar density distributions and velocity distributions for some stars in our galaxy.

III. QUANTUM PROBLEM

It turns out that the quantum-mechanical problem is simpler than the classical problem. Basically, this is due to the fact that in classical mechanics, one must keep track of where the particles are when they exchange positions. In quantum mechanics, only symmetry rules are observed when particles exchange positions. The following solution is a detailed discussion of a Letter announcement.[6]

Let us now consider an infinite system of N particles with equal masses. The total potential energy is

$$V=\gamma\sum_{i<j}^{N}|x_i-x_j|. \tag{5}$$

The Schrödinger equation for this problem is

$$\frac{-\hbar^2}{2m}\sum_j\frac{d^2\Psi}{dx_j^2}+\gamma\sum_{i<j}|x_i-x_j|\Psi=E\Psi, \tag{6}$$

where Ψ is the N-particle wave function and E is the total energy.

In one-dimensional systems, the procedure of ordering particles is quite standard,[7-9] and we use this simplification. Consider the region $x_1<x_2<x_3<\cdots<x_N$, then we can write the potential energy as

$$\sum_{i<j}^{N}|x_i-x_j|=-\gamma\sum_{j=1}^{N}(N-2j+1)x_j. \tag{7}$$

This simplification was used by Baxter[10] in studying the one-dimensional plasma.

Let us put

$$\Psi=\prod_j\psi_j(x_j),\quad E=\sum_j\epsilon_j,\quad F_j=\gamma(N-2j+1).$$

Then we get the separated equation

$$\frac{\hbar^2}{2m}\frac{d^2\psi_j}{dx_j^2}+(F_jx_j+\epsilon_j)\psi_j=0. \tag{8}$$

Equations of the above type were solved by Titchmarsh.[11] Instead of using Titchmarsh's notation, we will show the solution using a method described by Goldman and Krivchenko[12] for a related but different problem. We restrict our solution to the center-of-mass coordinate system.

We first show the solution for $x>0$. Let us suppress all j indices. Let

$$\psi(x)=\frac{1}{(2\pi\hbar)^{1/2}}\int dp\,\varphi(p)e^{ipx/\hbar}, \tag{9}$$

then the Schrödinger equation in momentum space becomes

$$(p^2-2m\epsilon)\varphi(p)=-2i\hbar mF\frac{\partial}{\partial p}\varphi(p), \tag{10}$$

so that

$$\varphi(p)=a\exp\left[i\left(\frac{p^3}{6mF\hbar}-\frac{\epsilon p}{\hbar F}\right)\right], \tag{11}$$

where a is a normalization constant. To find a, we normalize the wave function by setting

$$\int dp\,\varphi(p,\epsilon)\varphi^*(p,\epsilon')=\tfrac{1}{2}\delta(\epsilon-\epsilon'), \tag{12}$$

which gives $|a|^2=(2|F|\hbar)^{-1}$. The factor 2 comes from the fact that we are just solving the problem for the half space $x>0$. The case when $F=0$ corresponds to a free particle and is discussed later.

In coordinate space, we have

$$\psi(x)=\frac{2}{(2|F|\hbar)^{1/2}}\int_0^\infty dp\cos\left[\frac{p^3}{6mF\hbar}-\left(\frac{\epsilon}{F\hbar}+\frac{x}{\hbar}\right)p\right], \tag{13}$$

for $x>0$.

For $x<0$,

$$\psi(x)=\frac{2}{(2|F|\hbar)^{1/2}}\int_0^\infty dp\cos\left[\frac{p^3}{6mF\hbar}-\left(\frac{\epsilon}{F\hbar}-\frac{x}{\hbar}p\right)\right]. \tag{14}$$

The nature of the solution depends on the signs of x and F. By inspection, we find that the solution may be put in the form

$$\psi(x)=\frac{2}{(2|F|\hbar)^{1/2}}\times\int_0^\infty dp\cos\left[\frac{p^3}{6m|F|\hbar}-\left(\frac{\epsilon}{|F|\hbar}\pm\frac{x}{\hbar}\operatorname{sgn}F\right)p\right], \tag{15}$$

where the + sign is used for $x>0$, and the − sign for $x<0$.

Now we use the definition

$$(3a)^{-1/3}\pi\mathrm{Ai}(-(3a)^{-1/3}x)=\int_0^\infty\cos(at^3-xt)\,dt, \tag{16}$$

which allows us to write the solution in final form. Putting the j index again, we have

$$\psi(x_j)=[(2m|F_j|\hbar^{-2})^{1/3}/(2|F_j|)^{1/2}]\times\mathrm{Ai}(-(2m|F_j|\hbar^{-2})^{1/3}(\epsilon_j/|F_j|\pm x\operatorname{sgn}F_j)). \tag{17}$$

The solutions for the two regions $x>0$ and $x<0$ must be matched at $x=0$. Matching logarithmic derivatives, we find that

$$\mathrm{Ai}'(-(2m|F_j|\hbar^{-2})^{1/3}\epsilon_j/|F_j|) = -\mathrm{Ai}'(-(2m|F_j|\hbar^{-2})^{1/3}\epsilon_j/|F_j|), \quad (18)$$

so that the zeros of $\mathrm{Ai}(-z)$ determine the eigenvalues ϵ_j. Thus,

$$\epsilon_j = (\gamma^2\hbar^2/2m)^{1/3}|N-2j+1|^{2/3}b_i, \quad (19)$$

where the b_i are the zeros of the function $\mathrm{Ai}'(-z)$. The first few zeros of this function are tabulated by Abramowitz and Stegun.[13]

The total energy of the system depends on the symmetry chosen for the wave functions. For bosons, the particles could have the same quantum number, which in this case is a zero of $\mathrm{Ai}'(-z)$. For fermions, no two particles can have the same quantum number. This will be discussed in Secs. IV and V.

The solution shown here is valid for even N. For odd N, Eq. (8) simplifies to a free-particle Schrödinger equation for $j=\frac{1}{2}(N+1)$. Physically, this means that there are an equal number of particles to the left or right of the middle particle. The force on any one particle is just the number of particles to its right minus the number of particles to its left. This does not mean that the middle particle could have any energy and therefore make the energy spectrum continuous. The middle particle must still be quantized because of the symmetry requirement that any permutation of the particles must still result in the same total energy. An example is shown in Sec. IV.

IV. EXAMPLE–THE THREE-BODY PROBLEM

To illustrate the solution, we let $N=3$, and consider the wave functions for this case. We have for for the ordered region $x_1<x_2<x_3$

$$\psi_1(x_1) = \tfrac{1}{2}(4m\gamma\hbar^{-2})^{1/3}\mathrm{Ai}(-(4m\gamma\hbar^{-2})^{1/3}(\tfrac{1}{2}\epsilon_1 \pm x_1)),$$

$$\psi_2(x_2) = c_1 e^{ikx_2/\hbar} + c_2 e^{-ikx_2/\hbar}, \quad (20)$$

$$\psi_3(x_3) = \tfrac{1}{2}(4m\gamma\hbar^{-2})^{1/3}\mathrm{Ai}(-(4m\gamma\hbar^{-2})^{1/3}(\tfrac{1}{2}\epsilon_3 \mp x_3)),$$

where $k^2/2m=\epsilon_2$, $\epsilon_i=(\gamma^2\hbar^2/m)b_i$, and $E=\epsilon_1+\epsilon_2+\epsilon_3$.

For a boson system, the total energy is

$$E = \left(\frac{2\gamma^2\hbar^2}{m}\right)^{1/3}\sum_{i=1}^{3} b_i \quad (21)$$

and the ground-state energy is $E_0=(2\gamma^2\hbar^2/m)(1.0188)$.

The complete wave function should include all permutations of the particles, and we get

$$\Psi(x_1,x_2,x_3) = \frac{1}{\sqrt{3!}}\sum_P \psi_1(x_1)\psi_2(x_2)\psi_3(x_3), \quad (22)$$

where $\sum_P$ represents the sum over all permutations.

For the boson case in this example, we are unable to calculate the normalization constant for the middle particle. This difficulty persists for any odd number of particles. This technical problem does not exist for even N.

In the fermion case, no two particles can have the same energy. The simplest way of viewing the low-lying energy levels of a three-particle system is to consider the first three zeros of $\mathrm{Ai}'(-z)$. Then the ground-state energy of the fermion system is $E_0(2\gamma^2\hbar^2/m)(b_1+b_2+b_3)$ where the b's are the first three zeros.

The wave functions are found using the Slater determinant.[14] Thus

$$\Psi(x_1,x_2,x_3) = \frac{1}{\sqrt{3!}}\sum_P \delta_P\,\psi_1(x_1)\psi_2(x_2)\psi_3(x_3), \quad (23)$$

where δ_P is positive or negative depending on whether the permutation of the particles is even or odd.

V. GROUND STATE FOR N-BOSON SYSTEM

Using the first zero of $\mathrm{Ai}'(-z)$, we have the ground state of the N-boson system:

$$E_0 = \left(\frac{\gamma^2\hbar^2}{2m}\right)^{1/3}(1.0188)\sum_{j=1}^{N}|N-2j+1|^{2/3}, \quad (24)$$

which for large N may be approximated by $E_0 = 0.49(\gamma^2\hbar^2/m)N^{5/3}$. It is interesting to compare this with a heuristic estimate based on a calculation by Levy-Leblond.[15]

The energy of an N-particle system is approximated by $E=(Np^2/2m)+\frac{1}{2}\gamma a N(N-1)$ where p is the average momentum of the particles and a is the average distance between the particles. Now from Heisenberg's uncertainty principle, $pa\simeq\hbar$ and so E is approximately $N(\hbar^2/2ma^2)+\frac{1}{2}\gamma a N^2$. Minimizing this expression with respect to a, we find that the ground state is $(\gamma^2\hbar^2/2m)^{1/3}N^{5/3}[\frac{3}{2}(\frac{1}{2})^{1/3}]$, which is quite close to the exact result.

VI. GROUND STATE OF N-FERMION SYSTEM

There does not seem to exist an asymptotic expression for the zeros of $\mathrm{Ai}'(-z)$; so we merely write down the ground-state energy of the fermion system in the following form:

$$E_0 = \left(\frac{\gamma^2\hbar^2}{2m}\right)^{1/3}\sum_{j=1}^{N}|N-2j+1|^{2/3}b_j, \quad (25)$$

where it is understood that the b's are all different zeros.

VIII. COMMENTS

The result of the calculations show that the total energy of the gravitational gas is not proportional to N. The system is not extensive. This means that the system can not be considered thermodynamic. In the past, calculations were made using the pressure ensemble partition function.[16] In fact, other ensemble calculations could be made to show that at some value of N and a confining length L, the pressure will become zero and even negative. This is interpreted to mean gravitational collapse. As our quantum calculation shows, the system does not really collapse to a point. However, the N-body system behaves like a giant atom which becomes more compact as more particles are added. Our result here highlights the inapplicability of ordinary ensemble calculations to deduce the thermodynamics of a gravitational system. A straightforward thermodynamic interpretation is suspect anyway because of the nature of the gravitational potential.[17]

To study the physical characteristics of the system, one must go back to the wave functions. It would be interesting, for example, to go to the classical limit of the density distribution, and compare the results with suggested approximate solutions of the collisionless Boltzmann equation with self-consistent fields. We hope to study this numerically in the future.

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