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Statistical properties of a many-mode laser*

F. T. Arecchi† and A. M. Ricca
Centro Informazioni Studi Esperienze, P.O. Box 3986, 20100 Milano, Italy
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Using a generalized version of the Volterra-Wiener functional technique, we study a model for a many-mode laser action in which the coupling among modes takes place through the intensities. It is shown that the coupled Langevin equations for the field amplitudes can be solved avoiding the infinities at the mode thresholds. In particular, mean photon numbers, variances, and correlation and cross-correlation functions of the modes can be computed. In order actually to carry out the computations, it is necessary to evaluate the solution of a set of coupled deterministic first-order differential equations and to diagonalize a matrix whose dimension is given by the number of modes envisaged. As an application, the two-mode case is treated in detail and, besides a derivation of the stationary properties, in part already known, the transient photon statistics, during the buildup of the two modes, is thoroughly studied. The advantage of this approach is due to its simplicity and to the possibility of dealing with a large number of interacting modes.

I. INTRODUCTION

In a previous paper[1] (from hereon referred to as I) we introduced a generalization of the Volterra-Wiener method of functional expansion and showed its power in nonlinear stochastic processes. As an example, we applied such a method to the statistical description of a single-mode laser, deducing in a simple way results already known either in the stationary[2-4] or in the transient regime,[5] and furthermore giving the expression of the photon number transient correlation function.

In this paper, we consider a many-mode laser model, where the mode-mode coupling is due to intensity interactions and give for it both stationary and transient statistical behavior. Some of the results here reported were already given in a series of papers.[6-8] However, as in I, we give further results on the transient correlation and cross correlation of photon number, which have a physical appeal and suggest simple experiments.

We have chosen the most elementary form of mode-mode coupling because our aim, more than giving a detailed treatment of the laser problem, was to show the power of the functional expansion method in problems with many degrees of freedom. This suggests the possibility of exploring space-time dependent cooperative phenomena as, e.g., turbulence, chemical instabilities, and critical points of phase transitions.

In Sec. II we briefly summarize the many-mode laser field equations and deduce the corresponding photon numbers equations. In Sec. III we give the solution for the photon numbers equations in the steady state. In Sec. IV we show how to compute mean photon numbers of the modes and in Sec. V how to compute photon numbers, correlation functions, and variances in the steady state. Section VI deals with the photon statistics during the buildup of the modes. In Sec. VII, as an application, we give results for the two-mode case both in the steady state and in the transient regime.

II. BASIC EQUATIONS

We start from the following equations of motion for the field amplitudes a_k of a many-mode laser[9,10]:

$$\dot{a}_k - \tilde{\gamma}_k a_k + \sum_{k'} \tilde{\beta}_{kk'} \, |a_{k'}|^2 a_k = \Gamma_k \ . \tag{2.1}$$

Here k and k' are mode labels, and $\tilde{\gamma}_k$ represents the difference between linear gain and linear loss pertaining to mode k. $\tilde{\beta}_{kk'}$ is a complex matrix which characterizes the interaction among the modes and the self-interaction of each mode. Γ_k is a complex forcing term accounting for the stochastic noise sources. Besides we assume that $\Gamma_k(t)$ is a stationary Gaussian noise with zero average and δ-correlated in time. That is

$$\langle \Gamma_k(t) \rangle = 0, \quad \langle \Gamma_k^*(t)\, \Gamma_{k'}(t') \rangle = Q_k \delta_{kk'} \delta(t - t') \ . \tag{2.2}$$

In principle, Eq. (2.1) should also contain all other possible trilinear terms of the type $a^*_{k'''} a_{k''} a_{k'}$. However, we suppose here that the interactions among the modes take place via the intensities of the fields only. This is perhaps a crude approximation, but in this way the mathematical treatment simplifies to a great extent.

By considering the complex conjugate of Eq. (2.1) and by adding to it Eq. (2.1), we have

$$\tfrac{1}{2}\dot{n}_k - (\gamma_k - G_k)\, n_k + \beta_{kk} n_k^2 = c_k(t) \ , \tag{2.3}$$

where n_k is defined as $a_k^* a_k$ and represents the number of photons pertaining to the k mode. γ_k,

G_k, $\beta_{kk'}$, and c_k are defined as follows:

$$\gamma_k = \mathrm{Re}\,\tilde{\gamma}_k, \quad G_k = \sum_{k' \neq k} \beta_{kk'} n_{k'}, \qquad \beta_{kk'} = \mathrm{Re}\,\tilde{\beta}_{kk'}, \quad c_k = \tfrac{1}{2}(a_k^* \Gamma_k + \Gamma_k^* a_k). \tag{2.4}$$

We may look at Eq. (2.3) as the photon number equation of motion in a one-mode theory (I) in which the gain γ_k has been replaced by $\gamma_k - G_k$. This modified gain, in turn, depends on the population of all the other modes.

In Sec. III we solve Eq. (2.3) with a functional technique which is the generalization of the one discussed in I for the single-mode laser. It is noteworthy that equations of the type [Eq. (2.3)] are also encountered in the theory of cooperative effects in chemical reactions.[11] In that case n_k represents the concentration of the kth chemical component, while c_k accounts for the spontaneous production of new molecules.

III. SOLUTION OF THE PHOTON-NUMBER EQUATIONS IN THE STEADY STATE

Equation (2.3) can be solved once the term $\frac{1}{2}(a_k^* \Gamma_k + \Gamma_k^* a_k)$ is known. To this aim, as for the single-mode case, we expand a_k in a Volterra series as

$$a_k \simeq a_k^0 + \sum_l \int_0^t dt_1 \left.\frac{\delta a_k}{\delta \Gamma_l}\right|_{\Gamma_l = c_l^0} \{\Gamma_l - c_l^0\} + \sum_l \int_0^t dt_1 \left.\frac{\delta a_k}{\delta \Gamma_l^*}\right|_{\Gamma_l = c_l^0} \{\Gamma_l^* - c_l^0\}, \tag{3.1}$$

retaining only the zeroth and the first-order terms. We chose a_k^0 as

$$a_k^0 = \alpha_k^0 h(t) + B_k h(t) \cos\omega t, \tag{3.2}$$

where α_k^0 is the asymptotic solution for $t \to +\infty$ of Eq. (2.1) with $\Gamma_k = 0$. B_k is a real parameter to be defined later. However, as we shall see in Secs. IV and V, the final results do not depend on α_k^0 and B_k, which therefore need not to be actually computed. In I the physical meaning of the choice (3.2) for a_k^0 is explained thoroughly. Here it suffices to say that the meaning of the term $B_k \cos\omega t$ is that of simulating in a deterministic way the effect of the noise Γ_k. In Appendix A it is shown how the functional derivatives $(\delta a_k / \delta \Gamma_l)|_{\Gamma_l = c_l^0}$, $(\delta a_k / \delta \Gamma_l^*)|_{\Gamma_l = c_l^0}$ can be evaluated.

Equation (2.3) can be viewed as a stochastic equation driven by a noise source term $c_k(t)$. To solve Eq. (2.3) it is necessary to know the statistical properties of $c_k(t)$. In Appendix B it is shown that

$$\langle c_k(t) \rangle \simeq \tfrac{1}{2} Q_k, \qquad \langle c_k(t)\, c_{k'}(t') \rangle \simeq \tfrac{1}{4} Q_k Q_{k'} + \tfrac{1}{2} \langle n_k \rangle Q_k \delta_{kk'} \delta(t - t'). \tag{3.3}$$

This result seems rather odd because the correlation functions of the noise source term $c_k(t)$ are expressed in terms of the average value of the photon number $\langle n_k \rangle$ in the steady state yet to be evaluated. However, as we shall see later [see Eq. (4.3)] we express n_k as a function of $\langle n_k \rangle$ itself. As a consequence we evaluate $\langle n_k \rangle$ straightforwardly [see Eq. (4.4)]. In other words here we suppose to know $\langle n_k \rangle$ and then we deduce its actual value by a self-consistent procedure.

Next we put

$$n_k = n_k^0 + \Delta n_k. \tag{3.4}$$

We choose n_k^0 to satisfy the equation

$$\tfrac{1}{2}\dot{n}_k^0 - \gamma_k n_k^0 + n_k^0 \sum_{k'} \beta_{kk'} n_{k'}^0 = \tfrac{1}{2} Q_k, \tag{3.5}$$

with the initial conditions $n_k^0(t) = N_k$, N_k being the number of photons of the k mode at $t = 0$. As a consequence Δn_k must be the solution of the following set of equations:

$$\tfrac{1}{2}\Delta\dot{n}_k - \left(\gamma_k - \sum_{k'} \beta_{kk'} n_{k'}^0 - n_k^0 \beta_{kk}\right)\Delta n_k + n_k^0 \sum_{k' \neq k} \beta_{kk'} \Delta n_{k'} + \Delta n_k \sum_{k'} \beta_{kk'} \Delta n_{k'} = \Delta c_k, \tag{3.6}$$

where $\Delta n_k(0) = 0$ and $\Delta c_k(t) = c_k(t) - \frac{1}{2} Q_k$. As a consequence of Eq. (3.3), we have

$$\langle \Delta c_k(t) \rangle \simeq 0, \qquad \langle \Delta c_k(t)\, \Delta c_{k'}(t') \rangle = \tfrac{1}{2} Q_k \langle n_k \rangle \delta_{kk'} \delta(t - t'). \tag{3.7}$$

Equations (3.5) and (3.6) can be interpreted as follows: $n_k^0(t)$ represents the "deterministic evolution" of the photon number for the k mode corresponding to a certain initial condition, $\Delta n_k(t)$ represents a jitter motion due to the noise term $\Delta c_k(t)$. The asymptotic value $n_k^0(+\infty)$ can be looked upon as the zeroth-order approximation to the steady-state photon number. In addition it represents a generalization of the "quasilinear approximation" of the single-mode laser case.[12] For what concerns the two-mode case the functions n_k^0 have been already introduced by Grossmann and Richter[6,7] recurring to the Landau theory of phase transitions. Unfortunately it is not possible to compute $n_k^0(t)$ analytically. It is possible only to solve Eq. (3.5) by the computer or, whenever the modes are loosely coupled by means of an approximate scheme. This happens when β has small off-diagonal elements. Once $n_k^0(t)$ has been obtained it is possible to define the coefficients B_k of Eq. (3.2) as

$$\tfrac{1}{2} B_k^2 = n_k^0 - |\alpha_k^0|^2, \quad \text{if } n_k^0 \geq |\alpha_k^0|^2; \qquad B_k = 0, \quad \text{if } n_k^0 < |\alpha_k^0|^2. \tag{3.8}$$

This can be seen as a generalization of what was done in I for the single-mode case.

Now we turn to Eq. (3.6) which can be formally solved by means of the Volterra expansion as

$$\Delta n_k(t) = \sum_{k'} \int_0^t dt_1 G^{(1)}_{kk'}(t,t_1)\Delta c_{k'}(t_1) + \frac{1}{2!}\sum_{k'}\sum_{k''} \int dt_1 \int dt_2 G^{2}_{kk'k''}(t,t_1,t_2) \times \Delta c_{k'}(t_1)\Delta c_{k''}(t_2) + \cdots, \tag{3.9}$$

where for simplicity we have put

$$G^{(1)}_{kk'}(t,t_1) = \frac{\delta \Delta n_k(t)}{\delta \Delta c_{k'}(t_1)}\bigg|_{\Delta c=0}, \qquad G^{(2)}_{kk'k''}(t,t_1,t_2) = \frac{\delta^2 \Delta n_k(t)}{\delta \Delta c_{k''}(t_2)\,\delta\Delta c_{k'}(t_1)}\bigg|_{\Delta c=0}. \tag{3.10}$$

In Appendix C we compute $G^{(1)}$ and $G^{(2)}$ under the assumption that Eq. (3.5) has no limit cycle. In this case, for times long compared with the onset time of the mode, we have

$$G^{(1)}_{kk'}(t,t_1) \underset{t\to+\infty}{\simeq} 2h(t-t_1)\sum_q B^{-1}_{kq}B_{qk'}e^{-\lambda_k(t-t_1)}$$

$$G^{(2)}_{kk'k''}(t,t_1,t_2) \underset{t\to+\infty}{\simeq} -\sum_p\sum_q \beta_{pq}\int_0^t d\tau\, G^{(1)}_{kp}(t,\tau)\; G^{(1)}_{pk'}(\tau,t_1)\,G^{(1)}_{qk''}(\tau,t_2) - \sum_p\sum_q \beta_{pq}\int_0^t d\tau\, G^{(1)}_{kp}(t,\tau)\; G^{(1)}_{pk''}(\tau,t_2)\,G^{(1)}_{qk'}(\tau,t_1), \tag{3.11}$$

where $h(t)$ is the Heaviside step function and λ_k are the eigenvalues of the matrix A defined as

$$A_{kl} = -2\delta_{kl}\left(\gamma_k - \sum_{k'}\beta_{kk'}n^0_{k'}\right) + 2n^0_k\beta_{kl} \tag{3.12}$$

and B^{-1} is the matrix of the eigenvectors of A. Here it is assumed that the real part of each λ_k is positive. If this is not the case, the system is not stable and the mode picture loses its meaning.

IV. MEAN PHOTON NUMBERS IN THE STEADY STATE

In this section we compute $\langle n_k\rangle$ in the steady state. By virtue of Eqs. (3.4), (3.9), and (3.7) we have, long after the onset

$$\langle n_k\rangle \underset{t\to+\infty}{\simeq} n^0_k(+\infty) + \lim_{t\to+\infty}\frac{1}{4}\sum_{k'} Q_{k'}\langle n_{k'}\rangle \times \int_0^t dt_1 G^{(2)}_{kk'k'}(t,t_1,t_1). \tag{4.1}$$

Then one should substitute in Eq. (4.1) the expression of $G^{(2)}$. As a consequence $\langle n_k\rangle$ would be represented by Eq. (4.1) as a summation over six indices. Even if the modes envisaged are few, the evaluation of $\langle n_k\rangle$ in this way requires lengthy calculations. Thus, it is better to work out an approximate expression for $\langle n_k\rangle$. To do this we neglect the effect of the coupling among the fluctuations Δn_k of the modes in evaluating $G^{(1)}$. This implies that the off-diagonal elements of A are neglected. Therefore every mode is viewed as coupled to the other modes via average intensities only. This can be looked upon as a "mean-field" approach. We have

$$G^{(1)}_{kk'}(t,\tau) \underset{t\to+\infty}{\simeq} 2\delta_{kk'}h(t-\tau)\,e^{-A_{kk}(t-\tau)}. \tag{4.2}$$

As a consequence we can easily compute $G^{(2)}$, put it into Eq. (4.1) and after straightforward calculations have

$$\langle n_k\rangle \underset{t\to+\infty}{\simeq} n^0_k(+\infty) - 2\langle n_k\rangle Q_k\beta_{kk}/A^2_{kk}. \tag{4.3}$$

Hence

$$\langle n_k\rangle \underset{t\to+\infty}{\simeq} \frac{n^0_k(+\infty)}{1+2Q_k\beta_{kk}/A^2_{kk}}. \tag{4.4}$$

In this way, once the zeroth-order approximation $n^0_k(+\infty)$ has been computed, the average value $\langle n_k\rangle$ can be easily obtained, by virtue of Eq. (4.4) up to the second order in the perturbative expansion. We observe that the quantities α^0_k, B_k, $\delta a_k/\delta\Gamma_l|_{\Gamma_l=c^0_l}$, and $\delta a_k/\delta\Gamma^*_l|_{\Gamma_l=c^0_l}$ need not be computed actually. This happens because they do not enter the expressions of $G^{(1)}$ and $G^{(2)}$ and hence $\langle n_k\rangle$. The same applies for the photon numbers correlation functions (see Sec. V) in the steady state.

It is seen that for what concerns mean photon numbers, the computation labor is due only to the evaluation of n^0_k. However, n^0_k can be easily computed, for a number of modes up to 30 for instance, by a digital computer. Therefore our method is particularly suitable to deal with the many-mode case.

V. STEADY-STATE PHOTON NUMBER CORRELATION FUNCTIONS AND VARIANCES

In this section we are concerned with the evaluation of photon number correlation functions $\langle n_k(t)\, n_{k'}(t+\tau)\rangle$ in the steady state, i.e., when t goes to infinity. Once we know the correlation functions, it is possible to compute the photon number variances σ_k^2 of the modes as well as the overall photon number variance σ^2. To do this, by using Eq. (3.4), we have in the steady state

$$\langle n_k(t)\, n_{k'}(t+\tau)\rangle \underset{t\to+\infty}{\simeq} n_{k'}^0(+\infty)\, n_k^0(+\infty) + n_k^0(+\infty)\langle \Delta n_{k'}(+\infty)\rangle + n_{k'}^0(+\infty)\langle \Delta n_k(+\infty)\rangle + \lim_{t\to+\infty}\langle \Delta n_k(t)\,\Delta n_{k'}(t+\tau)\rangle \,. \tag{5.1}$$

Next $\Delta n_k(t)$ could be simply obtained approximately as

$$\Delta n_k(t) \simeq \sum_{k'} \int_0^t dt_1 G_{kk'}^{(1)}(t,t_1)\,\Delta c_{k'}(t_1) \tag{5.2}$$

according to Eq. (3.9) neglecting terms of order higher than the first. However, this choice is not always satisfactory because while the average value $\langle \Delta n_k(t)\rangle$ as obtainable from Eq. (5.2) is zero, the zeroth-order approximation $n_k^0(+\infty)$ is slightly different from the average value $\langle n_k(t)\rangle$. Therefore this approximation is not entirely self-consistent. To increase the degree of self-consistency of the procedure we can do as follows. First we write $n_k(t)$ as

$$n_k(t) = \langle n_k\rangle + \Delta\tilde{n}_k(t)\,, \tag{5.3}$$

where $\langle n_k\rangle$ is computed according to Eqs. (4.1) or (4.4) and $\Delta\tilde{n}_k(t)$ is chosen so that Eq. (2.3) be satisfied. In this way the zeroth-order approximation $\langle n_k\rangle$ to $n_k(t)$ is chosen to coincide with the average value $\langle n_k\rangle$ computed with the accuracy of a second-order calculation. In the steady state the motion equation of $\Delta\tilde{n}_k(t)$ is found to be

$$\tfrac{1}{2}\Delta\dot{\tilde{n}}_k - \Big(\gamma_k - \sum_{k'}\beta_{kk'}\langle n_{k'}\rangle - \beta_{kk}\langle n_k\rangle\Big)\Delta\tilde{n}_k + \langle n_k\rangle \sum_{k'\neq k}\beta_{kk'}\Delta\tilde{n}_{k'} + \Delta\tilde{n}_k \sum_{k'}\beta_{kk'}\Delta\tilde{n}_{k'} = \epsilon_k + \Delta c_k(t)\,, \tag{5.4}$$

where ϵ_k is defined as

$$\epsilon_k = \gamma_k\langle n_k\rangle - \langle n_k\rangle \sum_{k'}\beta_{kk'}\langle n_{k'}\rangle + \tfrac{1}{2}Q_k\,. \tag{5.5}$$

We can observe that Eq. (5.4) is equal to Eq. (3.6) once we change n_k^0 into $\langle n_k\rangle$. We can also note that ϵ_k must be a small quantity since if we replace $\langle n_k\rangle$ with n_k^0 in Eq. (5.5), ϵ_k turns out to be exactly zero [see Eq. (3.5) on the steady state, i.e., when $\dot{n}_k^0 = 0$]. Equation (5.4) could be solved once again using the Volterra functional expansion. Dropping terms of order higher than the first in the expansion, we have

$$\Delta\tilde{n}_k \simeq \sum_{k'}\int_0^t dt_1 \tilde{G}_{kk'}^{(1)}(t,t_1)\left[\epsilon_{k'} + \Delta c_{k'}(t_1)\right], \tag{5.6}$$

where $G_{kk'}^{(1)}(t,t_1)$ is the kernel as given by Eq. (3.11) with the replacement $n_k^0 \to \langle n_k\rangle$. From Eqs. (5.6) and (3.11) it results

$$\Delta\tilde{n}_k(t) \simeq \Big(2\sum_{k'}\epsilon_{k'}\sum_q \frac{B_{kq}^{-1}B_{qk'}}{\lambda_k}\Big) + \sum_{k'}\int_0^t dt_1 \tilde{G}_{kk'}^{(1)}(t,t_1)\,\Delta c_{k'}(t_1)\,. \tag{5.7}$$

The term in large parentheses is a correction to the mean value $\langle n_k\rangle$ as obtained before. The other terms account for the fluctuations. We now turn to the evaluation of the correlation function $\langle n_k(t)\, n_{k'}(t+\tau)\rangle$ in the steady state. It results from Eqs. (5.3) and (5.7):

$$\langle n_k(t)\, n_{k'}(t+\tau)\rangle \underset{t\to+\infty}{\simeq} \langle\tilde{n}_k\rangle\langle\tilde{n}_{k'}\rangle + \lim_{t\to+\infty}\sum_q\sum_{q'}\int dt_1\int dt_2 \tilde{G}_{kq}^{(1)}(t,t_1)\,\tilde{G}_{k'q'}^{(1)}(t+\tau)\,\langle \Delta c_q(t_1)\,\Delta c_{q'}(t_2)\rangle\,, \tag{5.8}$$

with

$$\langle\tilde{n}_k\rangle = \langle n_k\rangle + 2\sum_{k'}\epsilon_{k'}\sum_q \frac{B_{kq}^{-1}B_{qk'}}{\lambda_k}\,. \tag{5.9}$$

Recalling Eqs. (3.7) and (3.11), we have

$$\langle n_k(t)\, n_{k'}(t+\tau)\rangle \underset{t\to+\infty}{\simeq} \langle\tilde{n}_k\rangle\langle\tilde{n}_{k'}\rangle + \sum_{l'} E_{l'}(k,k')\, e^{-\lambda_{l'}\tau}\,, \tag{5.10}$$

where

$$E_{l'}(k,k') = -2\sum_q Q_q\langle n_q\rangle B_{k'l'}^{-1}B_{l'q}\sum_l \frac{B_{kl}^{-1}B_{lq}}{\lambda_l + \lambda_{l'}}\,. \tag{5.11}$$

The correlation functions among the fluctuations are approximately equal to weighted sums of exponentially decaying terms. For the two-mode case this has been already pointed out elsewhere recurring to the Landau theory of phase transitions[6-8] or to the Fokker-Planck equation.[13] From Eq. (5.10) we obtain the photon number variance σ_k^2 pertaining to the mode k as

$$\sigma_k^2 = \sum_{l'} E_{l'}(k,k') . \tag{5.12}$$

As a consequence the overall radiation variance σ^2, defined as

$$\sigma^2 = \langle N_0^2 \rangle - \langle N_0 \rangle^2 , \tag{5.13}$$

where

$$N_0 = \sum_k n_k , \tag{5.14}$$

is simply expressed as

$$\sigma^2 = \sum_k \sigma_k^2 + \sum_{\substack{\text{all pairs}\\ k \neq k'}} (\langle\langle n_k n_{k'} \rangle\rangle - \langle n_k \rangle \langle n_{k'} \rangle) . \tag{5.15}$$

Analogously the overall photon number correlation function results as

$$\langle N_0(t) N_0(t+\tau) \rangle\rangle \underset{t\to+\infty}{\sim} \sum_{k'} \sum_k \langle \bar{n}_k \rangle \langle \bar{n}_{k'} \rangle + \sum_l e^{-\lambda_l \tau} \sum_k \sum_{k'} E_l(k,k') . \tag{5.16}$$

VI. TRANSIENT STATISTICS

In this section we discuss the statistics of the laser radiation during the transient with a procedure which proved to be successful for the single-mode case I. During the transient the system "remembers" the initial condition. Therefore, to obtain the observable quantities as $\langle\langle n_k(t) \rangle\rangle$, $\langle\langle n_k(t) n_{k'}(t+\tau) \rangle\rangle, \ldots$, we must carry on the averaging operation also with respect to the initial photon numbers $N_1, N_2, \ldots$. It has been shown elsewhere[5] for a single-mode laser that during the transient, the effect of the noise can be neglected provided that the laser operates well above threshold. We presume that the same property applies to the many-mode case. Therefore in the following we perform the averaging operation with respect to the initial photon numbers $N_1, N_2, \ldots$, only, thus neglecting the noise.

We define $n_k(t,\underline{N})$ as the solution of system [Eq. (2.3)] with $c_k = 0$. Here $\underline{N}$ is the vector representation of the initial values. Though $n_k(t,\underline{N})$ can be evaluated easily by computer, it is impossible to compute mean values as mean photon numbers, variances, and correlation functions because of the great number of initial condition which must be considered to perform statistical averages.

To overcome this difficulty we proceed as follows. First we expand $n_k(t,\underline{N})$ in a truncated Taylor series as

$$n_k(t,\underline{N}) \simeq n_k(t,\langle \underline{N} \rangle) + \sum_q (N_q - \langle N_q \rangle)\, \alpha_{kq}(t,\langle \underline{N} \rangle) + \frac{1}{2} \sum_q \sum_{q'} (N_q - \langle N_q \rangle)(N_{q'} - \langle N_{q'} \rangle) \times \eta_{kqq'}(t,\langle \underline{N} \rangle) , \tag{6.1}$$

where $\langle \underline{N} \rangle$ is the vector representation of the mean photon numbers before the gain switch and α_{kq} and $\eta_{kqq'}$ are given by

$$\alpha_{kq}(t,\langle \underline{N} \rangle) = \left.\frac{\partial n_k(t,\underline{N})}{\partial N_q}\right|_{\underline{N}=\langle \underline{N} \rangle} ,$$
$$\eta_{kqq'}(t,\langle \underline{N} \rangle) = \left.\frac{\partial^2 n_k(t,\underline{N})}{\partial N_q\, \partial N_{q'}}\right|_{\underline{N}=\langle \underline{N} \rangle} . \tag{6.2}$$

As a consequence of the Bose Einstein character of the photon statistics before the switch of the laser gain we can easily compute $\langle\langle n_k(t) \rangle\rangle$, $\langle\langle n_k^2(t) \rangle\rangle$, $\langle\langle n_k(t) n_{k'}(t+\tau) \rangle\rangle$, etc. As a result we have

$$\langle\langle n_k(t,\underline{N}) \rangle\rangle \simeq n_k(t,\langle \underline{N} \rangle) + \frac{1}{2} \sum_q \langle N_q \rangle^2 \eta_{kqq}(t,\langle \underline{N} \rangle), \tag{6.3}$$
$$K_{kk'}(t,\tau) \simeq \sum_q \langle N_q \rangle^2 \alpha_{kq}(t,\langle \underline{N} \rangle)\, \alpha_{k'q}(t+\tau,\langle \underline{N} \rangle) ,$$

where $K_{kk'}(t,\tau)$, is the correlation function defined as

$$K_{kk'}(t,\tau) = \langle\langle n_k(t,\underline{N}) n_{k'}(t+\tau,\underline{N}) \rangle\rangle - \langle\langle n_k(t,\underline{N}) \rangle\rangle \langle\langle n_{k'}(t+\tau,\underline{N}) \rangle\rangle . \tag{6.4}$$

As a result of the second part of Eq. (6.3) putting $k' = k$ and $\tau = 0$ we obtain the transient photon variances σ_k^2 as

$$\sigma_k^2 = K_{kk}(t,0) . \tag{6.5}$$

Analogously for what concerns the overall photon number in the cavity $N_0(t,N)$, defined as $\sum_q n_q(t,\underline{N})$, we have

$$\langle N_0(t,N) \rangle\rangle \simeq \sum_q n_q(t,\langle \underline{N} \rangle) + \frac{1}{2} \sum_q \sum_{q'} \langle N_{q'} \rangle^2 \eta_{qq'q'}(t,\langle \underline{N} \rangle) , \tag{6.6}$$
$$K(t,\tau) \simeq \sum_q \sum_{q'} K_{qq'}(t,\tau) ,$$

where $K(t,\tau)$ is the correlation function for the overall radiation defined as

$$K(t,\tau) = \langle N_0(t,\underline{N}) N_0(t+\tau,\underline{N}) \rangle\rangle - \langle N_0(t,\underline{N}) \rangle\rangle \langle N_0(t,t+\tau,\underline{N}) \rangle\rangle . \tag{6.7}$$

By virtue of Eqs. (6.5)–(6.7) the overall photon number variance $\sigma_0^2(t)$ is given by

$$\sigma_0^2(t) \simeq \sum_k \sigma_k^2(t) + \sum_{q\neq q'} K_{qq'}(t) . \tag{6.8}$$

From a numerical point of view it is possible to compute both $\alpha_{kk'}$ and $\eta_{kk'k''}$ by solving system [Eq. (2.3)], with Γ_k and therefore c_k set equal to zero, in correspondence to different initial conditions. Then $\alpha_{kk'}$ and $\eta_{kk'k''}$ are computed according to Eq. (6.2) as incremental ratios numerically. As a consequence, for what concerns the transient statistics the computation labor is due only to the solution of a system of differential equations in correspondence to a number of different initial conditions which is equal to twice the number of modes envisaged plus one.

VII. APPLICATION TO THE TWO-MODE CASE

In this section we specialize the method exposed in the past sections for the two-mode case. Here we obtain the mean photon numbers of the two modes, the correlation functions, and the variances both in the steady state and in the transient regime. For simplicity we consider here the case in which

$$Q_1 = Q_2 = Q, \quad \beta_{11} = \beta_{22} = \beta, \quad \beta_{12} = \beta_{21} = \xi\beta , \tag{7.1}$$

where ξ can be looked upon as the one coupling parameter. We introduce the following dimensionless quantities:

$$I_\lambda = 2(\beta/Q)^{1/2} n_\lambda, \quad p = 2\gamma_1/(\beta Q)^{1/2},$$
$$t' = \tfrac{1}{2}(\beta Q)^{1/2} t, \quad \delta = [2/(\beta Q)^{1/2}](\gamma_1 - \gamma_2), \quad \lambda = 1, 2 . \tag{7.2}$$

p is the pump parameter and δ represents the difference between the dimensionless gains of the two modes. δ also represents the threshold, on the pump parameter axis, of the second mode if the interaction were absent. From now on δ is referred to as the *bare* threshold in distinction to the *renormalized* threshold, defined in the following. To see first qualitatively how the photons are shared by the two modes, we look for the steady-state solution of Eq. (2.3) for $\Gamma = 0$. In doing this, we neglect the effects of the noise. Using Eq. (7.1) and the dimensionless quantities of Eq. (7.2), we have three different cases:

(i) If $p < 0$ and $p - \delta < 0$:

$$I_1 = I_2 = I_1 + I_2 = 0 . \tag{7.3}$$

No photons in the cavity.

(ii) If $p > 0$ and $p < \delta/(1-\xi)$, then

$$I_1 = p, \quad I_2 = 0, \quad I_1 + I_2 = p . \tag{7.4}$$

Only the first mode oscillates even if p is greater than the bare threshold δ of the second mode.

(iii) If $p > 0$, $p > \delta/(1-\xi)$, then

$$I_1 = \frac{p(1-\xi)+\xi\delta}{1-\xi^2}, \quad I_2 = \frac{p(1-\xi)-\delta}{1-\xi^2},$$
$$I_1 + I_2 = \frac{2p(1-\xi) - \delta(1-\xi)}{1-\xi^2} . \tag{7.5}$$

Both modes oscillate. It is seen that the quantity $\tilde{\delta}$ defined as

$$\tilde{\delta} = \delta/(1-\xi) \tag{7.6}$$

plays the role of the effective or renormalized threshold for the second mode.

Obviously, to have better physical insight, the effect of the noise must be taken into account properly according to the analysis of Secs. IV–VI. In the following we discuss some typical results. Figure 1 shows the steady-state behavior of a two-mode laser whose coupling parameter is $\frac{2}{3}$ and whose bare threshold δ for the second mode is ten. Our results confirm analogous results obtained elsewhere[6–8] with the Landau method. Figure 1(a) shows the normalized mean photon numbers $\langle I_\lambda \rangle$ of the modes $(\lambda = 1, 2)$ and of the overall radiation $(\lambda = 0)$ versus the pump parameter p. It is seen that according to Eq. (7.6) as an effect of the coupling the effective threshold of the second mode is 30 while the bare threshold is 10. Figure 1(b) shows the effective intensity fluctuation linewidths $\Delta\omega_{\lambda\lambda'}$ of the modes $(\lambda, \lambda' = 1, 2)$ and of the overall radiation $(\lambda = \lambda' = 0)$ vs p. $\Delta\omega_{\lambda\lambda'}$ is defined so that $e^{-\Delta\omega_{\lambda\lambda'}\tau}$ fits the τ-dependent part [see Eqs. (5.10) and (5.16)] of $\langle n_\lambda(t) n_{\lambda'}(t+\tau)\rangle$ (for λ, $\lambda' = 1, 2$) and of $\langle N_0(t) N_0(t+\tau)\rangle$ for $\lambda = \lambda' = 0$. We see that at the renormalized thresholds of the two modes the phenomenon of slowing-down of the fluctuations occurs. Much above the second threshold, the slope of $\Delta\omega_{00}$ vs p is 2 and it coincides with the slope of a single-mode intensity fluctuation linewidth versus p much above threshold.

Figure 1(c) shows the equal time intensity correlation functions

$$K_{\lambda\lambda'} = \langle I_\lambda I_{\lambda'} \rangle - \langle I_\lambda \rangle \langle I_{\lambda'} \rangle$$

of the two modes $(\lambda, \lambda' = 1, 2)$ and of the overall radiation $(\lambda = \lambda' = 0)$ versus the pump parameter p. It is seen that at the renormalized thresholds the correlation functions undergo a pronounced change. Our numerical calculations show that $K_{12} = K_{21}$ with a very good accuracy and this complies with general symmetry requirements. Owing to the negative value of K_{12} the photon number variance K_{00} for the overall radiation is smaller than the sum $K_{11} + K_{22}$. That is, there is an interference effect. This is essentially due to the fact that ξ is positive. However, one may suppose that

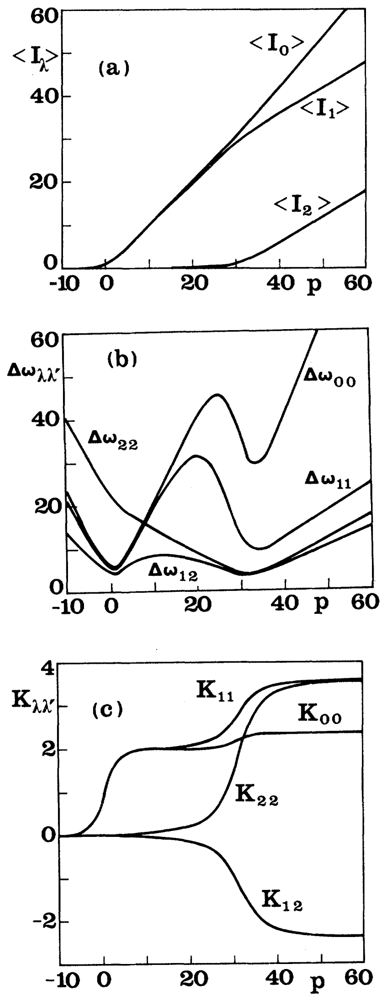

FIG. 1. Steady-state behavior of a two-mode laser whose coupling parameter ξ is $\frac{2}{3}$ and whose bare threshold of the second mode is 10. Figure 1(a) shows the normalized mean photon numbers $\langle I_\lambda \rangle$ of the modes ($\lambda = 1, 2$) and of the overall radiation ($\lambda = 0$) vs the pump parameter p. Figure 1(b) shows the intensity fluctuation linewidth $\Delta\omega_{\lambda\lambda'}$ vs p. Figure 1(c) shows the equal time intensity correlation function $K_{\lambda\lambda'}$ vs p.

under special circumstances ξ is negative. This is the case when an absorber is placed within the cavity. In that case if a mode is excited, the transparency of the absorber is enhanced. Therefore the loss of the second mode is lowered. This phenomenon could be accounted for by a negative coupling parameter ξ between the modes. Recently the case of an atomic three-level medium with two adjacent allowed transitions almost equal in frequency has been studied.[14] In such a case two cavity modes can be fed by different transitions. The presence of a common level between the two transitions can give rise to intensity coupling effects with a negative ξ.

To see what happens we have studied the case in which $\xi = -\frac{1}{2}$ and $\delta = 30$. Figure 2(a) shows the normalized photon numbers $\langle I_\lambda \rangle$ vs p. It is seen [according to Eq. (7.6)] that in this case the renormalized threshold of the second mode is reduced by a factor $\frac{2}{3}$ with respect to the bare one. Therefore, as expected, we have cooperation between the modes instead of competition. The first mode helps the second to oscillate. Figure 2(b) shows the effective intensity fluctuation linewidths versus p. Here again we see the slowing down at the two renormalized thresholds. Figure 2(c) shows the equal time intensity correlation functions $K_{\lambda\lambda'}$ vs p. We see in this case that K_{00} is greater than $K_{11} + K_{22}$ owing to the fact that K_{12} is positive. Again we have, with a very good accuracy, that $K_{12} = K_{21}$. Thus far, the two-mode analysis has given results already obtainable by the Landau method, even though the application to the negative coupling had not been shown previously. It is more interesting, however, to investigate

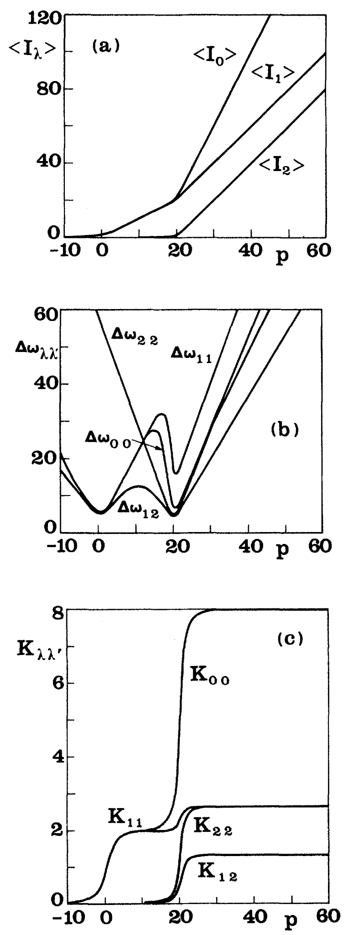

FIG. 2. Same of Fig. 1 with $\xi = -\frac{1}{2}$ and $\delta = 30$.

the transient behavior of two modes. We confine ourselves to the case of high gains for both modes. Further, to simplify the computations to the maximum extent we have disregarded $\eta_{kqq'}$ in Eq. (6.1). This corresponds to the same approximation used for the single-mode case I which gave a good agreement with experiments. In this way to obtain α_{kq} we must solve Eq. (2.3) with $c_k = 0$ in correspondence to three different initial conditions. In the following we present some typical results.

Figure 3 shows the transient mean photon number $\langle I_\lambda \rangle$ of the two modes $\lambda = 1, 2$ and of the over-

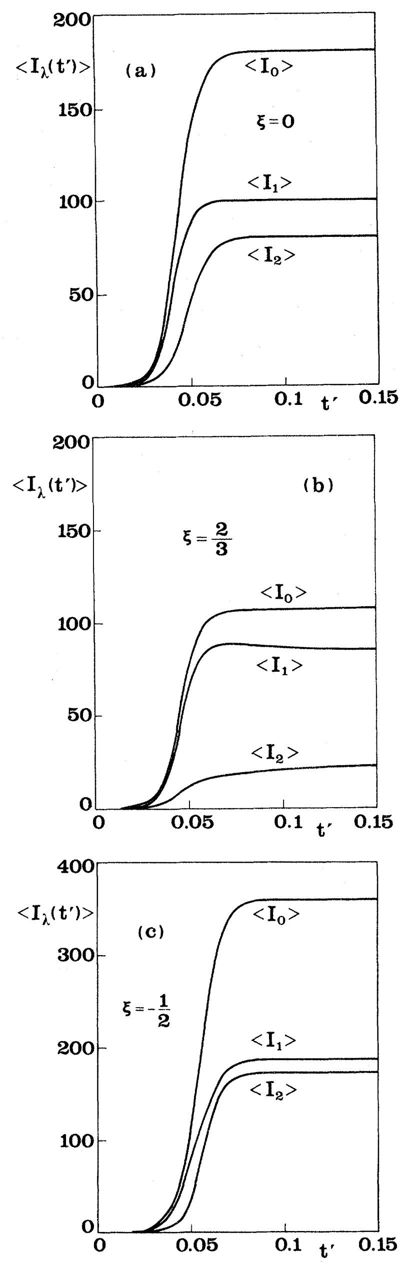

FIG. 3. Transient mean photon number $\langle I_\lambda \rangle$ of the two modes ($\lambda = 1, 2$) and of the overall radiation ($\lambda = 0$) for $p = 100$, $\delta = 20$, $\langle N_1 \rangle = \langle N_2 \rangle = 40$ in correspondence of the following values for the coupling parameter ξ: Fig. 3(a), $\xi = 0$; Fig. 3(b), $\xi = \frac{2}{3}$; Fig. 3(c), $\xi = -\frac{1}{2}$.

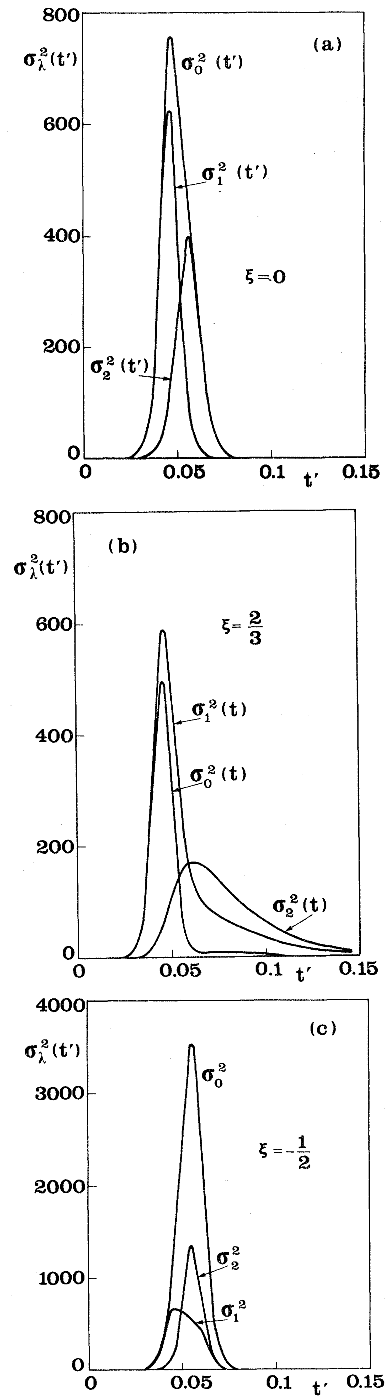

FIG. 4. Transient behavior of the photon number variances $\sigma^2(t)$ of the two modes ($\lambda = 1, 2$) and of the overall radiation ($\lambda = 0$) in correspondence to the three cases of Fig. 3.

all population ($\lambda = 0$) for $p = 100$, $\delta = 20$ in correspondence to three different values of the coupling parameter: Fig. 3(a), $\xi = 0$; Fig. 3(b), $\xi = \frac{2}{3}$; Fig. 3(c), $\xi = -\frac{1}{2}$. When ξ is zero the two modes do not interact, when $\xi = \frac{2}{3}$ the two modes compete and the second mode with a lower gain is depressed in comparison with the uncoupled mode case. The opposite happens when ξ is negative and as a result the two modes help each other to oscillate.

Figure 4 shows the transient behavior of the photon number variances $\sigma_\lambda^2(t) = \langle I_\lambda(t)^2\rangle - \langle I_\lambda(t)\rangle^2$ of the two modes ($\lambda = 1, 2$) and of the overall radiation ($\lambda = 0$) in the same three cases of Fig. 3. We see that owing to the coupling there is a strong interference effect for $\sigma_0^2(t)$. Namely, with reference to the case when $\xi = 0$ where $\sigma_0^2(t)$ is just [see Eq. (6.8)] the sum of $\sigma_1^2(t) + \sigma_2^2(t)$, we see that when ξ is positive, $\sigma_0^2(t)$ not only is less than $\sigma_1^2(t) + \sigma_2^2(t)$ but is smaller than $\sigma_1^2(t)$. The opposite happens when ξ is negative. In this case $\sigma_0^2(t)$ is higher than $\sigma_1^2(t) + \sigma_2^2(t)$ roughly by a factor of 2 at the peak. For what concerns the photon number variances of the two modes, if ξ is positive, the first mode lowers the peak of the variance of the second mode which in addition is flattened. If ξ is negative, the variance of the first mode is depressed. We have mentioned before for the two-mode case in the steady state that, owing to symmetry requirements, $K_{12} = K_{21}$. However, this does not apply to the transient regime. Figure 5 shows $K_{12}(t', \tau')$ and $K_{21}(t', \tau')$ vs t' in correspondence to four different time delays τ' ($\tau' = 0$; 0.01; 0.02; 0.03).

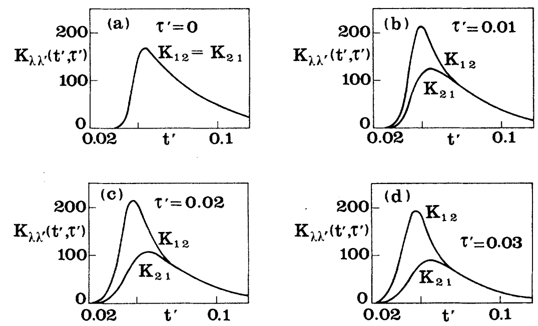

FIG. 5. Evolution of the correlation function $K_{12}(t', \tau')$ and $K_{21}(t', \tau')$ vs t' in correspondence to four different time delays τ': 0, 0.01, 0.02, 0.03.

Apart from the trivial case $\tau' = 0$ for which $K_{12} = K_{21}$ always holds, for $\tau' \neq 0$ we see that the condition $K_{12} = K_{21}$ is fulfilled only at the end of the transient. This is an interesting case of transient break up of the time-reversal symmetry, which is well established for equilibrium cases.[15]

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APPENDIX A: EVALUATION OF THE FUNCTIONAL FIELD DERIVATIVES $\delta a_k/\delta\Gamma_j|_{\Gamma = c0}$ AND $\delta a_k/\delta\Gamma_j^*|_{\Gamma = c0}$

By deriving functionally both sides of Eqs. (2.1) with respect to $\Gamma_j(t_1)$ and $\Gamma_j^*(t_1)$ we have

$$\begin{aligned}
&\frac{\partial}{\partial t}K_{kj} - \tilde{\gamma}_k K_{kj} + \sum_{k'} \tilde{\beta}_{kk'}(K_{k'j}\, a_{k'}^{0*} a_k^0 + a_{k'}^0 P_{k'j}^* a_k^0 + |a_{k'}^0|^2 K_{kj}) = \delta_{kj}\,\delta(t - t_1),\\
&\frac{\partial}{\partial t}P_{kj} - \tilde{\gamma}_k P_{kj} + \sum_{k'} \tilde{\beta}_{kk'}(P_{k'j} a_{k'}^{0*} a_k^0 + a_{k'}^0 K_{k'j}^* a_k^0 + |a_{k'}^0|^2 P_{kj}) = 0\,,\\
&\frac{\partial}{\partial t}P_{kj}^* - \tilde{\gamma}_k^* P_{kj}^* + \sum_{k'} \tilde{\beta}_{kk'}(K_{k'j} a_{k'}^{0*} a_k^{0*} + a_{k'}^0 P_{k'j}^* a_k^{0*} + |a_{k'}^0|^2 P_{kj}^*) = 0,\\
&\frac{\partial}{\partial t}K_{kj}^* - \tilde{\gamma}_k^* K_{kj}^* + \sum_{k'} \tilde{\beta}_{kk'}^*(P_{k'j} a_{k'}^{0*} a_k^{0*} + a_{k'}^0 K_{k'j}^* a_k^{0*} + |a_{k'}^0|^2 K_{kj}^*) = \delta_{kj}\,\delta(t - t_1)\,,
\end{aligned} \tag{A1}$$

where

$$K_{kj} \equiv \frac{\delta a_k}{\delta \Gamma_j}\bigg|_{\Gamma = c^0}, \quad P_{kj} \equiv \frac{\delta a_k}{\delta \Gamma_j^*}\bigg|_{\Gamma = c^0}. \tag{A2}$$

As a consequence of Eq. (A1) we have

$$\begin{aligned}
&K_{kj}(t, t) = K_{kj}^*(t, t) = \delta_{kj}\,,\\
&P_{kj}(t, t) = P_{kj}^*(t, t) = 0\,.
\end{aligned} \tag{A3}$$

Equations (A1) are a system of linear differential equations in the unknown functions K_{kj}, P_{kj}, K_{kj}^*, and P_{kj}^* which can be solved since the functions a_k^0 and a_k^{0*} are assumed to be known. However, if we confine ourselves to a mean-field theory, in the sense of Sec. IV, the functions K, P, K^*, and P^* need not be computed.

APPENDIX B: EVALUATION OF $\langle c_k(t)\rangle$ AND $\langle c_k(t)\, c_{k'}(t')\rangle$

From Eq. (3.1) and its complex conjugate, in the notation of Appendix A, we have

$$a_k(t) \simeq \tilde{r}_k^0(t) + \sum_l \int_0^t dt_1 K_{kl}(t,t_1)\,\Gamma_l(t_1) + \sum_l \int_0^t dt_1 P_{kl}(t,t_1)\,\Gamma_l^*(t_1)\,,$$

$$a_k^*(t) \simeq \tilde{r}_k^{0*}(t) + \sum_l \int_0^t dt_1 K_{kl}^*(t,t_1)\,\Gamma_l^*(t_1) + \sum_l \int_0^t dt_1 P_{kl}^*(t,t_1)\,\Gamma_l(t_1)\,, \tag{B1}$$

where r_k^0 is defined as

$$r_k^0 = a_k^0 - \sum_l \int_0^t dt_1 K_{kl}(t,t_1)\, c_l^0(t_1) - \sum_l \int_0^t dt_1 P_{kl}(t,t_1)\, c_l^{0*}(t_1)\,. \tag{B2}$$

Recalling Eqs. (2.2) and (A3) we easily obtain

$$\langle c_k(t)\rangle\!\rangle = \tfrac{1}{2}\langle a_k^*(t)\,\Gamma_k(t) + \Gamma_k^*(t)\,a_k(t)\rangle\!\rangle = \tfrac{1}{2}Q_k\,. \tag{B3}$$

For what concerns the correlation functions $\langle c_k(t_1)\,c_{k'}(t_2)\rangle\!\rangle$, we have

$$\langle c_k(t_1)\,c_k(t_2)\rangle\!\rangle = \tfrac{1}{4}\langle a_k^* a_q^*\,\Gamma_k\,\Gamma_q + a_k^* a_q\,\Gamma_k\,\Gamma_q^* + a_k a_q^*\,\Gamma_k^*\,\Gamma_q + a_k a_q\,\Gamma_k^*\,\Gamma_q^*\rangle\,. \tag{B4}$$

By virtue of Eq. (B1) after lengthy calculations, we have

$$\langle c_k(t_1)\,c_q(t_2)\rangle\!\rangle = \frac{Q_k Q_q}{4} + \frac{Q_k}{2}\delta_{kq}\delta(t_2-t_1)\left(|r_k^0|^2 + \sum_m Q_m \int_0^t d\tau_1[\,|K_{km}(t,\tau_1)|^2 + |P_{km}(t_1,\tau_1)|^2]\right). \tag{B5}$$

Now we observe that by virtue of Eq. (B1), Eq. (B5) can be written

$$\langle c_k(t_1)\,c_q(t_2)\rangle\!\rangle \simeq \tfrac{1}{4}Q_k\,Q_q + \tfrac{1}{2}Q_k\delta_{kq}\delta(t_1-t_2) \times \langle a_k^*(t_1)\,a_k(t_1)\rangle\!\rangle\,. \tag{B6}$$

Hence

$$\langle c_k(t_1)\,c_q(t_2)\rangle\!\rangle \simeq \tfrac{1}{4}Q_k\,Q_q + \tfrac{1}{2}\langle n_k\rangle\,Q_k\delta_{kq}\delta(t_1-t_2)\,. \tag{B7}$$

The symbol $\simeq$ indicates that the relation (B7) within the correlation functions $\langle c_k(t_1)\,c_q(t_2)\rangle\!\rangle$ and $\langle n_k\rangle$ has been established in the form [Eq. (B7)] by means of the approximate expressions (B1).

APPENDIX C: EVALUATION OF $G^{(1)}$ AND $G^{(2)}$

Taking the functional derivatives of both sides of Eq. (3.6) with respect to $\Delta c_{k'}(t_1)$ and further putting $\Delta c = 0$, we have

$$\frac{1}{2}\frac{\partial}{\partial t}G_{kk'}^{(1)}(t,t_1) - \left(\gamma_k - \sum_q \beta_{kq}\,n_q^0\right)G_{kk'}^{(1)}(t,t_1) + n_k^0\sum_q \beta_{kq}\,G_{qk'}^{(1)}(t,t_1) = \delta_{kk'}\delta(t-t_1)\,. \tag{C1}$$

Next we set

$$G_{kk'}^{(1)}(t,t_1) = h(t-t_1)\,\phi_{kk'}(t,t_1)\,,\qquad \phi_{kk'}(t_1,t_1) = 2\delta_{kk'}\,. \tag{C2}$$

Equation (C1) written in terms of ϕ becomes

$$\frac{\partial\phi}{\partial t} = 2A\phi\,, \tag{C3}$$

where A is defined according to Eq. (3.12). If Eq. (3.5) has no limit cycle, A is a constant matrix and the solution of (C3) is simply

$$\phi(t,t_1) = 2B^{-1}e^{-[BAB^{-1}(t-t_1)]}B\,, \tag{C4}$$

where B is an arbitrary nonsingular matrix. It is convenient to choose B in order to diagonalize the matrix BAB^{-1}. Therefore B^{-1} must be chosen as the matrix of column eigenvectors of A. Hence, we have

$$G_{qq'}^{(1)}(t,t_1) = 2h(t-t_1)\sum_k B_{qk}^{-1}B_{kq'}e^{-\lambda_k(t-t_1)}\,, \tag{C5}$$

where λ_k are the eigenvalues of A. The above result is meaningful whenever $\mathrm{Re}\,\lambda_k$ is positive. To obtain $G^{(2)}$ we derivate functionally both sides of Eq. (3.6) with respect to $\Delta c_{k'}(t_1)$ and $\Delta c_{k''}(t_2)$ and further put $\Delta c = 0$. We have

$$\frac{1}{2}\frac{\partial}{\partial t}G_{kk'k''}^{(2)}(t,t_1,t_2) - \left(\gamma_k - \sum_q \beta_{kq}\,n_q^0\right)G_{kk'k''}^{(2)}(t,t_1,t_2) + n_k^0\sum_q \beta_{kq}\,G_{qk'k''}^{(2)}(t,t_1,t_2)$$
$$= -G_{kk'}^{(1)}(t,t_1)\sum_q \beta_{kq}\,G_{qk''}^{(1)}(t,t_2) - G_{kk''}^{(1)}(t,t_2)\sum_q \beta_{kq}\,G_{qk'}^{(1)}(t,t_1)\,. \tag{C6}$$

If we indicate by $\rho_{kk'k''}(t,t_1,t_2)$ the right-hand side of Eq. (C6), by virtue of Eq. (C1), $G^{(2)}$ can be expressed as

$$G^{(2)}_{kk'k''}(t, t_1, t_2) = \sum_q \int_0^t d\tau\, G^{(1)}_{kq}(t, \tau)\, \rho_{qk'k''}(\tau, t_1, t_2)\,. \tag{C7}$$

From which we have

$$\begin{aligned} G^{(2)}_{kk'k''}(t, t_1, t_2) = &-\sum_q \sum_{q'} \beta_{q'q} \int_0^t d\tau\, G^{(1)}_{kq'}(t, \tau)\; G^{(1)}_{q'k'}(\tau, t_1)\, G^{(1)}_{qk''}(\tau, t_2) \\ &-\sum_q \sum_{q'} \beta_{q'q} \int_0^t d\tau\, G^{(1)}_{kq'}(t, \tau)\; G^{(1)}_{q'k''}(\tau, t_2)\, G^{(1)}_{qk'}(\tau, t_1)\,. \end{aligned} \tag{C8}$$

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