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Level-degeneracy effects in superradiance theory

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Up to now, the problem of the influence of level degeneracy on superradiance has been neglected. We describe here a first approach to this problem, in the frame of "small system" quantum model, generalized to the case of a collection of atoms with two degenerate levels of angular momenta j and j' connected by an electric or magnetic dipole transition. The differences between this case and the well-known case of two (nondegenerate) level atoms are of two types. The first type is due to the competition between different transitions sharing a common state: superradiant emission on a transition inhibits or initiates superradiance on transitions with smaller transition probability, depending upon which state, upper or lower, is the common one. This effect does not come from the equality of the different wavelengths and would also appear with the same states even if the degeneracy were removed; its qualitative influence on the features of the supperradiant pulse can be easily estimated. The second type of differences is quite specific to level degeneracy and consists in interferences between the transitions of the collective system having the same frequency and polarization. In analogy with the description of collections of two-level atoms as spin-1/2 systems, a group-theoretical description of the considered collection of atoms is found suitable; the problem of the determination of a basis of collective states and its group-theoretical implications are discussed; in particular, it is found that the isotropic character of spontaneous emission allows us to use tensorial formalism and simplifies the formal description of the problem.

I. INTRODUCTION

Superradiance studies, initiated by Dicke[1] in 1954, have been largely developed during the last few years, mainly theoretically since up to now only two papers[2,3] describe a direct observation of the phenomenon. However in the numerous theoretical papers concerning this subject, an interesting aspect has been neglected: the influence of level degeneracy. The atoms are described as two-level[4-6] or multilevel[7] systems, the levels always being supposed nondegenerate. This basic simplifying assumption is made in the quantum-mechanical treatments[4,5] as well as in the semiclassical studies[6] and for "small systems"[4] (with linear dimensions smaller than the wavelength) as well as for extended systems.[5,6]

This paper describes a first approach to the problem of the influence of the level degeneracy on superradiance; this approach is essentially formal and will be applied in a following paper to some simple particular cases. Our aim here is to point out the specific effects related to the level degeneracy which are responsible for the differences between the case of atoms with two degenerate levels and the well-known case of two (nondegenerate) level atoms. For this aim we have chosen to generalize the simplest quantum model which explains the main features of the superradiance of a collection of two (nondegenerate) level atoms, that is the "small system" model; we consider here the case of a collection of atoms with two degenerate levels of angular momenta j and j' (the transition being supposed electric or magnetic dipole). Such a model, in which the atoms are considered located on a same spot is not realistic (at least for optical wavelengths), and it is known presently that the superradiance of small samples would be strongly limited by the dipole-dipole interactions.[8] However, in spite of these limitations, it is established that the small-system model allows qualitative discussions of superradiance, at least as far as the collective phenomenon itself is concerned. The propagation effects, which are added to this phenomenon in the case of extended systems, are of course not taken into account here; however, the influence of level degeneracy on the propagation of pulses has already been studied, for the problem of self-induced transparency, by several authors.[9] We also neglect, in this first approach, the effects of inhomogeneous broadening,[10] which could attenuate in some sense the effects of level degeneracy. Finally it must be noted that, as is well known, the choice of a quantum model allows one to account for the beginning of the superradiant emission without additional assumption even if the initial population inversion is complete; moreover it allows one to evaluate the quantum fluctuations of the radiated intensity which could prevent the observation of the specific polarization effects arising from the degeneracy.

The obvious advantage of this model is its relative simplicity which allows to generalize the master equation and the expressions of the expectation values for the radiation field to the case of atoms with two degenerate levels: this is presented in Sec. II. It turns out that the master equation is more complex than in the nondegenerate case. Two different types of effects are responsible for the differences between the two cases. The first one is the competition between the different transitions sharing a common upper or lower state, which would also appear if the degeneracy was removed; it can be easily shown that the superradiance on the transition which has the greatest transition probability inhibits (initiates) the superradiance on the other transitions having the same upper (lower) state. The second effect is quite specific to the level degeneracy and consists in interference effects between the transitions of the collective system which have the same frequency and the same polarization (we shall say that such transitions correspond to a same "degenerate transition" of the collective system). Concerning the expressions of the expectation values for the radiated field, the difference with the nondegenerate case is that we have to take into account the polarization properties of the emitted light. The evaluation of these expectation values requires the solution of the master equation and consequently involves the choice of a basis of collective states. This problem, which is discussed in Sec. III, is also much more complicated than for the nondegenerate case. The analog to the SU(2) group spanned by the operators R_+, R_-, and R_3 in the case of two (nondegenerate) level atoms is shown to be here a SU$(2j+2j'+2)$ group; the transitions of the collective system appear only between states of a same irreducible representation of this group, which we may therefore call "cooperation group". The problem of the labelling of the states of an irreducible representation is not simple since one has here to choose a basis well adapted to the solution of the master equation; in the general case (that is for any j and j') this constraint leads to difficult group-theory problems, mainly because of internal degeneracies which appear in the suitable reduction of SU$(2j+2j'+2)$ to a chain of subgroups. In fact we have performed such a group-theoretical study only for the cases where the cooperation group is SU(4) that is for the different two-level four-state atoms ($j=0\to j'=1$, $j=1\to j'=0$, $j=\frac{1}{2}\to j'=\frac{1}{2}$); these results which allow some quantitative conclusions will be given in a following paper. However we show here, in the general case, that the use of the SU(2) group corresponding to the total angular momentum of the ensemble of atoms allows one to take advantage of the invariance under the rotations of the master equation and simplifies the formal description of the problem. Some difficulties remain which prevent the solution of the master equation in the general case: they are examined in Sec. IV and some qualitative conclusions are given.

II. BASIC EQUATIONS FOR COLLECTIVE SPONTANEOUS EMISSION

A. General form of the master equation

The relaxation by spontaneous emission of a quantum system can be studied (see for example Refs. 11–13) by considering the system as interacting with a "reservoir" of photons, that is with a quantized electromagnetic field of very large spectral width which remains in its ground state. Such a treatment leads, with convenient approximations,[13] to a master equation for the reduced density operator (the trace on the states of the reservoir of the total density operator). We are interested here in systems with degenerate levels, more precisely in systems which have transitions between different pairs of quantum states that are resonant with the same mode of the electromagnetic field[14]; such transitions will be said to correspond to a same "degenerate transition." For such a system (that we shall call an atom, for sake of simplicity), the general form of the master equation can be written

$$\mathring{\rho}(t)=\sum \tfrac{1}{2}\Gamma_{ik\to jl}\{[p_{ji},\rho(t)p_{kl}]+\text{H.c.}\}. \tag{1}$$

(i) The sum runs over all the two pairs of atomic states i,j[15] and k,l which are such that the energy of state i is greater than that of state j and the energy of k greater than l; the frequencies (ω_{ij} and ω_{kl}) and the polarizations ($\vec{\epsilon}_{ij}$ and $\vec{\epsilon}_{kl}$) of the two transitions $i\to j$ and $k\to l$ are equal (for the frequencies the equality is approximate; the detuning must be less than or of the same order as the width of the concerned atomic transitions).

(ii) The $\Gamma_{ik\to jl}$ (we use here the notation of Cohen-Tannoudji[13]) are coefficients of the atom's coupling with the electromagnetic field of the reservoir; if the two pairs of states i,j and k,l are not identical, the coefficient $\Gamma_{ik\to jl}$ is zero unless a single mode of the field is quasiresonant with the two transitions ($\omega_{ij}-\omega_{kl}\simeq 0$, $\vec{\epsilon}_{ij}=\vec{\epsilon}_{kl}$). The dependence on i, j, k and l of this coefficient is contained in the product of the two matrix elements of the atom-field interaction corresponding to the two transitions; if we consider electric or magnetic dipole transitions only and if we write $\vec{d}$ for the atomic dipole operator, we may therefore write[16]

$$\Gamma_{ik\to jl}\propto(j|\vec{d}\cdot\vec{\epsilon}\,|i)(l|\vec{d}\cdot\vec{\epsilon}\,|k), \tag{2}$$

with $\vec{\epsilon}=\vec{\epsilon}_{ij}=\vec{\epsilon}_{kl}$.

(iii) The p_{ij} are the atomic operators

$$p_{ij} = |i)(j| . \tag{3}$$

The master equation describing the collective spontaneous emission of an assembly of N identical atoms confined in a volume of linear dimensions smaller than the wavelengths and interacting through the common radiation field can be simply derived from Eq. (1). The collection of atoms behaves as a single quantum system whose states are products of monatomic states and the reduced density operator of this system obeys a master equation which can be written

$$\dot{\rho}(t) = \sum \tfrac{1}{2}\Gamma_{ik\to jl}\{[P_{ji}, \rho(t)P_{kl}] + \text{H.c.}\} , \tag{1$'$}$$

where the $\Gamma_{ik\to jl}$ are the same as for a single atom [the remarks (i) and (ii) remain unchanged]; the only difference with Eq. (1) is that the monatomic operators p_{ij} are replaced by the corresponding collective ones, that is

$$P_{ij} = \sum_{\alpha=1,N} p_{ij}{}^{(\alpha)} , \tag{4}$$

where the sum runs over all the atoms.

The main difference between Eq. (1′) and the corresponding master equation in the case of atoms with two nondegenerate levels comes from the presence of $\Gamma_{ik\to jl}$ with $i, k \neq j, l$, which are characteristic of two distinct atomic transitions corresponding to a same degenerate transition of the atoms. These coefficients account for the interference effects which appear in the fluorescence light when a single mode of the electromagnetic field is resonant with several atomic transitions. These interference effects will be discussed hereafter for the case of a collection of atoms with two degenerate levels j and j', and it will be shown that they must be explicitly taken into account, whatever the choice of collective basis states.

B. Master equation for a collection of atoms with two levels of angular momenta j and j'

We shall now particularize the master equation (1′) to the case of a collection of atoms having two levels characterized by their angular momentum j and j' (j concerning the upper level), the transition between the two levels being supposed electric or magnetic dipole (which implies $|j-j'| = 0, 1$). The $\Gamma_{ik\to jl}$ coefficients in Eqs. (1) or (1′) different from zero are, according to relation (2),

$$\Gamma_{m_1 m_2 \to m_1' m_2'} \propto \sum_q (j'm_1'|d_q^{(1)}|jm_1)(j'm_2'|d_q^{(1)}|jm_2) , \tag{5}$$

where $d^{(1)}$ is the monatomic dipole operator (in tensor form), $m_1' - m_1 = m_2' - m_2 = q$, q taking the values 1, 0, or −1. According to the Wigner-Eckart theorem the dependence on m of the coefficients (5) is given by a product of $3j$ symbols and they can be written

$$\Gamma_{m_1 m_2 \to m_1' m_2'} = (2j+1)\Gamma(-1)^{j'-m_1'+j'-m_2'} \times \begin{pmatrix} j' & 1 & j \\ -m_1' & q & m_1 \end{pmatrix} \begin{pmatrix} j' & 1 & j \\ -m_2' & q & m_2 \end{pmatrix} . \tag{6}$$

The constant Γ in this equation is in fact the decay constant of a state $|jm\rangle$ of the upper level for a single atom; in effect this decay constant is obtained from Eq. (1) in the form

$$\sum_q \Gamma_{mm \to m+q\, m+q} , \tag{7}$$

which is, according to Eq. (6), independent of m and equal to Γ. Moreover the $3j$ symbols of Eq. (6) can be associated with the operators of the type P_{ij} which appear in Eq. (1′); precisely, we define

$$R_q = (2j+1)^{1/2} \sum_{m,m'} (-1)^{j'-m'} \begin{pmatrix} j' & 1 & j \\ -m' & q & m \end{pmatrix} P_{j'm',jm} ,$$

$$R_q^\dagger = \text{H.c.} , \tag{8}$$

where the P operators are as in Eq. (4):

$$P_{j'm',jm} = \sum_{\alpha=1,N} |j'm'\rangle_{\alpha\alpha}\langle jm| . \tag{9}$$

With these definitions, the master equation of the collection of N atoms can be simply written

$$\dot{\rho}(t) = \frac{\Gamma}{2} \sum_q \{[R_q, \rho(t)R_q^\dagger] + \text{H.c.}\} , \tag{10}$$

or, in a more detailed way where the populating and depopulating terms are separated,

$$\dot{\rho}(t) = \Gamma \sum_q \{R_q \rho(t) R_q^\dagger - \tfrac{1}{2}[\rho(t)R_q^\dagger R_q + R_q^\dagger R_q \rho(t)]\} . \tag{10$'$}$$

The operators R_q and $R_q^\dagger$ are closely related to respectively the energy-decreasing and -increasing part of the components of the collective dipole operator $\vec{\mathrm{D}}$ of the N atoms. Precisely one notices that these operators are components of two tensor operators of rank 1 with respect to the total angular momentum $\vec{\mathrm{J}}$ of the N atoms, which is expressed in terms of P operators by

$$J_z = \sum_m m P_{jm,jm} + \sum_{m'} m' P_{j'm',j'm'} ,$$

$$J_\pm = \sum_m [j(j+1) - m(m\pm 1)]^{1/2} P_{jm\pm 1, jm} + \sum_{m'} [j'(j'+1) - m'(m'\pm 1)]^{1/2} P_{j'm'\pm 1, j'm'} ; \tag{11}$$

the operators R_q and $R_q^\dagger$ can then be written

$$R_q = R_q^{(1)}\,,$$
$$R_q^\dagger = (-1)^q \tilde{R}_{-q}^{(1)}\,, \qquad (12)$$

where $\tilde{R}^{(1)}$ is the Hermitian-conjugate tensor operator of $R^{(1)}$.[17] The components of the total dipole operator $D^{(1)}$ are related to those of $R^{(1)}$ and $\tilde{R}^{(1)}$ by

$$D_q^{(1)} = \frac{1}{(2j+1)^{1/2}} (j' \| d^{(1)} \| j) [R_q^{(1)} + \tilde{R}_q^{(1)}]\,; \qquad (13)$$

the three values of q correspond to the three independent polarizations π, σ_+, and σ_- (the choice of the quantization axis is arbitrary, unless the polarization properties of the initial excitation determine a privileged axis), and $(j' \| d^{(1)} \| j)$ is the reduced matrix element of the monatomic dipole operator.

The isotropic character of spontaneous emission appears in the fact that the master equation is invariant under the rotations and then under the SU(2) group related to the total angular momentum $\vec{J}$; the two operators $R^{(1)}$ and $\tilde{R}^{(1)}$ appear in Eq. (10) or (10′) only as coupled in a scalar product and in particular the depopulating terms involve the following scalar operator:

$$X = \sum_q R_q^\dagger R_q = (\tilde{R}^{(1)} \cdot R^{(1)})\,. \qquad (14)$$

It is interesting to compare the master equation (1) with the equation one would obtain in the corresponding nondegenerate case, that is by removing the degeneracy of the two levels j and j' in such a way that each atom has no longer degenerate transitions. In this case the only nonzero $\Gamma_{ik \to jl}$ coefficients are $\Gamma_{mm \to m'm'}$ and the master equation for a collection of N such atoms can be written

$$\dot{\rho}(t) = \frac{\Gamma}{2} \sum_{m,m'} \{ [R_{mm'}, \rho(t) R_{mm'}^\dagger] + \text{H.c.} \}\,, \qquad (15)$$

with

$$R_{mm'} = (2j+1)^{1/2} (-1)^{j'-m'} \begin{pmatrix} j' & 1 & j \\ -m' & q & m \end{pmatrix} P_{j'm'jm}\,,$$
$$R_{mm'}^\dagger = \text{H.c.}\,. \qquad (16)$$

The comparison of Eqs. (10) and (15) makes clear that the only difference between the degenerate and nondegenerate cases is due to the interference effects on the light emitted or absorbed with a given polarization in different transitions between Zeeman states $|jm\rangle$ and $|j'm'\rangle$.

C. Expectation values for the radiated field

The problem of the evaluation of the statistical properties of the field radiated by a collection of atoms with two degenerate levels j and j' is quite analogous to the same problem in the nondegenerate case. As in this latter case, the validity conditions of the master equation imply that the field follows the atoms adiabatically[18]: all expectation values characterizing the field can then be obtained from the reduced density operator $\rho(t)$ which is a solution of the master equation (10). The only new point is that we have to take into account the polarization of the emitted light. If the observation is made with a polarization $\vec{\epsilon}$, the expectation values of normally ordered creation and annihilation field operators are proportional to terms of the type:

$$\langle [(\vec{R} \cdot \vec{\epsilon})^\dagger]^m (\vec{R} \cdot \vec{\epsilon})^l \rangle (t)\,,$$

where $\langle X \rangle (t) = \text{tr}[X\rho(t)]$ and $\vec{R}$ is the vector operator associated to the operator $R^{(1)}$ [Eqs. (8) and (12)], related to the energy-decreasing part of the collective dipole operator [Eq. (13)]. In particular the intensity radiated in all directions with a polarization $\vec{\epsilon}$ is

$$I_{\vec{\epsilon}}(t) = I_0 \langle (\vec{R} \cdot \vec{\epsilon})^\dagger (\vec{R} \cdot \vec{\epsilon}) \rangle (t)\,, \qquad (17)$$

I_0 being the intensity radiated in all directions with all polarizations by an isolated excited atom; for a particular direction of observation α, one has to insert the factor $(3/8\pi)(1 - |(\vec{\epsilon} \cdot \vec{\alpha})|^2)$, which describes the spatial distribution of the fluorescence light of polarization $\vec{\epsilon}$. In the same way, the expression for the squared relative dispersion of the quantum fluctuations[19] of the intensity emitted with a polarization $\vec{\epsilon}$ can be generalized

$$\sigma_{\vec{\epsilon}}^2(t) = \frac{Q_{\vec{\epsilon}}(t) - [I_{\vec{\epsilon}}(t)]^2}{[I_{\vec{\epsilon}}(t)]^2}\,, \qquad (18)$$

where

$$Q_{\vec{\epsilon}}(t) = I_0^2 \langle [(\vec{R} \cdot \vec{\epsilon})^\dagger]^2 (\vec{R} \cdot \vec{\epsilon})^2 \rangle (t)\,. \qquad (19)$$

In expressions (17) and (19), the polarization $\vec{\epsilon}$, which depends on the observation conditions only, appears mixed in scalar products with the operator $\vec{R}$. It is convenient to better separate the role of these quantities; this is done by using the tensorial properties of $\vec{R}$ and $\vec{\epsilon}$ and by recoupling the products of scalar products which appear in these expressions.[20] One has finally

$$(\vec{R} \cdot \vec{\epsilon})^\dagger (\vec{R} \cdot \vec{\epsilon}) = \sum_k (-1)^k (\{\tilde{\epsilon}^{(1)} \cdot \epsilon^{(1)}\}^{(k)} \cdot \{\tilde{R}^{(1)} \cdot R^{(1)}\}^{(k)})\,, \qquad (20)$$

$$[(\vec{R} \cdot \vec{\epsilon})^\dagger]^2 (\vec{R} \cdot \vec{\epsilon})^2 = \sum_{k_1, k_2, k} (-1)^k (\{\{\tilde{\epsilon}^{(1)} \cdot \tilde{\epsilon}^{(1)}\}^{(k_1)} \cdot \{\epsilon^{(1)} \cdot \epsilon^{(1)}\}^{(k_2)}\}^{(k)} \cdot \{\{\tilde{R}^{(1)} \cdot \tilde{R}^{(1)}\}^{(k_1)} \cdot \{R^{(1)} \cdot R^{(1)}\}^{(k_2)}\}^{(k)})\,. \qquad (21)$$

The intensity and the quantum fluctuations corresponding to any value of $\vec{\epsilon}$ can thus be expressed as linear combinations of the following mean values:

$$\mathcal{G}_q^{(k)}(t)=I_0\langle\{\tilde{R}^{(1)}\cdot R^{(1)}\}_q^{(k)}\rangle(t)\,, \tag{22}$$

$$Q^{k_1k_2\,(k)}_{\;\;\;\;\;\;q}(t)=I_0^2\langle\{\{\tilde{R}^{(1)}\cdot\tilde{R}^{(1)}\}^{(k_1)}\cdot\{R^{(1)}\cdot R^{(1)}\}^{(k_2)}\}_q^{(k)}\rangle(t). \tag{23}$$

We do not make explicit here the coefficients of these linear combinations, which are given by Eqs. (20) and (21), for all possible polarizations. We only give here the relationship between the intensities corresponding to the π, σ_+, and σ_- polarizations and the mean values $\mathcal{G}_q^{(k)}(t)$:

$$I_\pi(t)=-\frac{1}{\sqrt{3}}\,\mathcal{G}_0^{(0)}(t)+\frac{\sqrt{2}}{\sqrt{3}}\,\mathcal{G}_0^{(2)}(t)\,,$$

$$I_{\sigma\pm}(t)=-\frac{1}{\sqrt{3}}\,\mathcal{G}_0^{(0)}(t)\pm\frac{1}{\sqrt{2}}\,\mathcal{G}_0^{(1)}(t)-\frac{1}{\sqrt{6}}\,\mathcal{G}_0^{(2)}(t)\,; \tag{24}$$

it can be seen from these equations that the polarization of the emitted light at the time t depends on the relative magnitudes of the various $\mathcal{G}_q^{(k)}(t)$. One can show, more generally, that the term $\mathcal{G}_0^{(0)}(t)$ corresponds to an isotropic and nonpolarized radiation; if any other term is different from zero, the emitted light becomes polarized and nonisotropic.

III. SOLUTION OF THE MASTER EQUATION: FORMAL DESCRIPTION USING TENSORIAL FORMALISM

A. Collective states

In order to solve the master equation, one has first to define a basis of states for the collection of atoms. The problem of labeling the collective states of N two-level atoms confined in a small volume is much more difficult for degenerate levels than for the nondegenerate case. The Lie group which plays the same role as the SU(2) group spanned by the R_3, R_+, and R_- operators defined by Dicke is spanned here by the operators R_q and $R_q^\dagger$ [Eqs. (8)]. It is easy to show that this group is a $\mathrm{SU}(2j+2j'+2)$ group,[21] whose infinitesimal operators are the $(2j+2j'+2)^2$ $P_{j_1m_1,\,j_2m_2}$ (where j_1 and j_2 run over the values j and j'). This group can then be called the "cooperation group" of the collection of atoms and its role is quite analogous to that of the SU(2) group in the Dicke model: the collective states can be chosen as basis functions of irreducible representations (irreps) of this group and the operators R_q and $R_q^\dagger$ being infinitesimal operators do not mix states of different irreps.[22] Therefore the irreps $\{\lambda\}$ of $\mathrm{SU}(2j+2j'+2)$ play the same role as the cooperation numbers r of Dicke's model. However, the labeling of the states of these irreps is difficult in the general case (that is for any j and j'), in particular because of the internal degeneracies which may appear in the reduction of $\{\lambda\}$ according to a chain of subgroups of the cooperation group. In this paper we do not try to solve this general problem and we label the collective states of the irreps $\{\lambda\}$ with only a few quantum numbers, whose physical meaning is obvious: the energy and the total angular momentum of the ensemble of atoms. We shall see that this approach allows to take advantage of the SU(2) invariance properties of the master equation; precisely the use of tensorial formalism simplifies the evolution equations and allows to show some qualitative results.

The Hamiltonian of the collection of atoms (without field) can be written

$$H=\frac{E_0}{2}\left(\sum_m P_{jm,\,jm}-\sum_{m'} P_{j'm',\,j'm'}\right)\,, \tag{25}$$

where E_0 is the energy difference between the two levels. The total angular momentum $\vec{J}$ commutes with H; so does the operator X defined in Eq. (14), which, being scalar, commutes also with $\vec{J}$. Moreover, all these operators, H, $\vec{J}$, and X, can be written as linear combinations of operators $P_{j_1m_1,\,j_2m_2}$ and are consequently diagonal with respect to the irreps $\{\lambda\}$ of the cooperation group. Therefore the operators H, J^2, J_z, and X can be simultaneously used for the labeling of the basis functions of an irrep of the cooperation group. They are not in general sufficient (except for particular small values of j and j') for a complete characterization of these states. However, in this first approach, we define the collective states belonging to an irrep $\{\lambda\}$ only as eigenfunctions of H, X, J^2, and J_z and we call them $|\{\lambda\}\mu\gamma JM)$; μ is related to the energy E of the collection of atoms by $E=E_0\mu$ (μ takes values varying from $\frac{1}{2}N$ to $-\frac{1}{2}N$ by integer steps; the case of complete population inversion corresponds to $\mu=\frac{1}{2}N$) and γ represents a set of quantum numbers which are not stated explicitly. If we call $X(\{\lambda\}\mu\gamma J)$ the eigenvalues of the operator X, the master equation (10′) becomes, in matrix form,

$$\begin{aligned}\{d/dt+\tfrac{1}{2}\Gamma[X(\mu\gamma J)+X(\mu'\gamma'J')]\}(\mu'\gamma'J'M'\,|\rho(t)|\,\mu\gamma JM)\\ =\Gamma\sum_{q,\gamma_0,J_0,\gamma_0',J_0'}(\mu'\gamma'J'M'\,|R_q^{(1)}|\,\mu'+1\,\gamma_0'J_0'M'-q)\\ \times(\mu\gamma JM\,|R_q^{(1)}|\,\mu+1\,\gamma_0J_0M-q)(\mu'+1\,\gamma_0'J_0'M'-q\,|\rho(t)|\,\mu+1\,\gamma_0J_0M-q)\,,\end{aligned} \tag{26}$$

where we have omitted the symbol $\{\lambda\}$ since only one representation should appear and where we have taken into account the obvious action of $R_q^{(1)}$ on the quantum numbers μ and M.

One notices here that the different matrix elements between states of given energy values μ and μ' are not coupled together but only with density matrix elements of energy values $\mu+1$ and $\mu'+1$: Eq. (26) can therefore be solved, in principle, step by step starting from the highest values of μ (the difficulty comes from the large number of density matrix elements). This characteristic of Eq. (26) is due to the fact that the operator X is diagonal in the chosen basis, which is thus very convenient for the resolution of these equations: therefore the complete characterization of the collective states should be done with a chain of subgroups of the cooperation group containing the SU(2) group of the total angular momentum and such that X is diagonal with respect to the irreps of these subgroups.

B. Interference effects

Mathematically, the aforementioned interference effects appear in the evolution equations of the density matrix elements, (26), by the presence, in the right-hand sides, of two different matrix elements of the collective dipole operator. Such products represent interference terms between the probability amplitude of two different transitions of the collective system, these transitions having the same frequency (the resonance frequency of the atoms) and the same polarization (characterized by q).

Such products of different matrix elements of $R^{(1)}$ appear obviously when the density matrix element of the left side of Eq. (26) is nondiagonal, that is a coherence. The corresponding interference effect is called "coherence transfer" by Cohen-Tannoudji[13] and it appears also for a collection of atoms with two (nondegenerate) levels. However the density matrix elements useful for the evaluation of mean values of the radiated field such as intensity or quantum fluctuations of the intensity are diagonal in energy and in irreps of the cooperation group. In the case of two nondegenerate levels this property implies that the interesting matrix elements are, at least in the basis of Dicke's states, only populations; the coherence transfer can thus be ignored. The situation is quite different if the two levels are degenerate; it is in general impossible to simultaneously diagonalize the various operators of expressions (17) or (19) which correspond to different polarizations: the corresponding mean values involve coherences and the coherence transfer therefore plays an explicit role.

Furthermore, interference terms may occur in the right-hand sides of Eq. (26) even if the density matrix element of the left-hand side is a population. This is due to the fact that it is generally impossible to diagonalize simultaneously the three products of operators $R_q^\dagger R_q$ corresponding to different polarizations: it is then impossible to find a basis such that the action of each operator R_q on a given basis state gives always a single basis state. As a consequence, the evolution of the populations is coupled, via the interference effects, to that of some coherences, whatever the choice of basis states. This result appears as another difference between the case of degenerate levels and the two-level model, which is clearly due to the presence in spontaneous emission of three independent polarizations (instead of one in the two-level model).

These interference effects are of course quite analogous to the level-crossing interference effects.[23] The great difference between the case of superradiance and the case of a single atom (or noninteracting atoms) is that, because of the collective character of superradiance, the interfering paths have to be found in the collective energy-level diagram. Therefore the number of elementary interference processes is much larger than in the case of a single atom. Let us take as an example the symmetric representation $\{N\}$ of $\mathrm{SU}(2j+2j'+2)$, whose states are symmetric under the permutations of the N atoms. The states of highest energy of this representation correspond to a complete population inversion ($\mu=\frac{1}{2}N$) and there are $N+1$ equidistant energy levels; since the dimension of this representation is given by

$$\mathfrak{N}=\frac{(N+2j+2j'+1)!}{N!\,(2j+2j'+1)!}\ ,$$

it turns out that for large values of N there is in general much more than one state for a given value of μ and of M. As shown before, the interference effects we are interested in concern the different transitions of a same polarization between two consecutive energy levels: the number of such transitions and thus of elementary interference processes becomes very large when j and j' and especially N are important.

C. Decomposition of the density operator on a set of irreducible tensor operators

In order to take advantage of the isotropic character of the collective spontaneous emission described by the master equation (10), that is of the invariance under the total angular momentum SU(2) group of this equation, it is convenient to use tensorial formalism and to decompose the density operator on a basis of irreducible tensor operators $T_q^{(k)}$; we write

$$\rho(t)=\sum_{\substack{k=0,1,\ldots,+\infty\\ q=-k,-k+1,\ldots,k}} \rho_{kq}(t)T_q^{(k)}. \tag{27}$$

The SU(2) invariance of the master equation implies that the evolution equations of the different $\rho_{kq}(t)$ are not coupled together and that these equations are identical for all values of q corresponding to a same value of k. This allows one to write the decomposition of $\rho(t)$ in a different way, which does not involve a particular basis of irreducible tensor operators:

$$\rho(t)=\sum_{\substack{k=0,1,\ldots,+\infty\\ q=-k,-k+1,\ldots,k}} a_{kq}\rho_q^{(k)}(t); \tag{28}$$

the coefficients a_{kq} are some constants (depending on the initial conditions only) and the $\rho_q^{(k)}(t)$ are irreducible tensor operators of rank k; the values of a_{kq} and of the reduced matrix elements of $\rho^{(k)}(t)$ can be deduced from the matrix elements of $\rho(t)$ by

$$\begin{aligned} a_{kq}(\{\lambda\}\mu'\gamma'J'\|\rho^{(k)}(t)\|\{\lambda\}\mu\gamma J) &= \sum_{M,M'}(2k+1)(-1)^{J'-M'}\begin{pmatrix} J' & k & J\\ -M' & q & M\end{pmatrix}\\ &\quad\times(\{\lambda\}\mu'\gamma'J'M'|\rho(t)|\{\lambda\}\mu\gamma JM), \end{aligned} \tag{29}$$

the dependence on k of a_{kq} being arbitrary. One notices that since the trace of an irreducible tensor operator of rank $k\neq 0$ is zero, the condition on the trace of the density operator, $\operatorname{tr}[\rho(t)]=1$, implies that

$$a_{00}\operatorname{tr}[\rho_0^{(0)}(t)]=1, \tag{30}$$

or equivalently

$$\sum_{\{\lambda\},\mu,\gamma,J} a_{00}(2J+1)^{1/2}(\{\lambda\}\mu\gamma J\|\rho^{(0)}(t)\|\{\lambda\}\mu\gamma J)=1, \tag{31}$$

but there is no analogous condition for the components of rank $k\neq 0$.

The SU(2) invariance of the master equation implies that the role of the quantum numbers M or q in its matrix form is inessential. This property is made clear if one uses the development (28) of $\rho(t)$; precisely a simple calculation[20] using the properties of the 3-j symbols contained in the different matrix elements of Eq. (26) shows that the reduced matrix elements of the operators $\rho^{(k)}(t)$ satisfy the following equation:

$$\begin{aligned} &\{d/dt+\tfrac{1}{2}\Gamma[X(\mu\gamma J)+X(\mu'\gamma'J')]\}(\mu'\gamma'J'\|\rho^{(k)}(t)\|\mu\gamma J)\\ &\quad=\Gamma\sum_{\gamma_0',J_0',\gamma_0,J_0}(-1)^{J+J_0'+k+1}\begin{Bmatrix} J_0' & k & J_0\\ J & 1 & J'\end{Bmatrix}\\ &\qquad\times(\mu'\gamma'J'\|R^{(1)}\|\mu'+1\,\gamma_0'J_0')(\mu\gamma J\|R^{(1)}\|\mu+1\gamma_0J_0)(\mu'+1\,\gamma_0'J_0'\|\rho^{(k)}(t)\|\mu+1\,\gamma_0J_0). \end{aligned} \tag{32}$$

Concerning the mean value of a tensor operator $V_q^{(k)}$ it is clear, since the trace of a tensor operator vanishes unless the rank is zero, that

$$\langle V_q^{(k)}\rangle(t)=\operatorname{tr}[a_{k-q}\rho_{-q}^{(k)}(t)V_q^{(k)}(t)], \tag{33}$$

or, in a more detailed expression (and for $V^{(k)}$ diagonal in μ):

$$\langle V_q^{(k)}\rangle(t)=\sum_{\{\lambda\},\mu,\gamma',J',\gamma,J}\frac{1}{2k+1}(-1)^{J-J'+q}a_{k-q}(\{\lambda\}\mu\gamma'J'\|\rho^{(k)}(t)\|\{\lambda\}\mu\gamma J)(\{\lambda\}\mu\gamma J\|V^{(k)}\|\{\lambda\}\mu\gamma'J'). \tag{34}$$

The interest of the present decomposition of $\rho(t)$ in irreducible tensor operators is then twofold: first, it eliminates from Eq. (26) the useless quantum numbers M and q; secondly, the mean values of tensor operators with rank k involve only the component $\rho^{(k)}(t)$ and the evaluation of such mean values requires the solution of Eq. (22) only for the considered value of k: for example the evaluation of the radiated intensity [for any polarization observation, cf. Eqs. (17) and (20)] requires only the values $k=0$, 1, and 2 and for the quantum fluctuations of the intensity we have to consider in general k varying from 0 to 4 [cf. Eqs. (18), (19), and (21)].

IV. REMAINING PROBLEMS AND QUALITATIVE CONCLUSIONS

In the formalism presented here, the derivation of the properties of the radiated field (intensity, polarization, quantum fluctuations...) requires the solution of Eq. (32) for particular values of k. In order to actually perform this resolution, we are faced with two main difficulties.

The first one is a consequence of the choice of a quantum model and appears in the determination of a suitable basis of collective states, that is, as shown before, a basis of states of an irrep of the cooperation group with given energy and angular momentum and for which the depopulating operator X is diagonal. This implies that one has first to find a chain of subgroups of $\mathrm{SU}(2j+2j'+2)$ such that X is diagonal in the irreps of these subgroups and which contains the SU(2) group related to the total angular momentum of the atoms. Moreover one has to deal with the internal degeneracies that generally occur in the reduction of $\mathrm{SU}(2j+2j'+2)$ in this chain; this latter problem can in principle be solved, at least for the symmetric representations, by assigning conditions to the fractional parentage coefficients which relate the symmetric states of N atoms to these of $N-1$ atoms; however, one has to consider large values of N and the problem becomes very complicated. Finally, except for particular simple values of $j\rightarrow j'$ (namely, $\frac{1}{2}\rightarrow\frac{1}{2}$, $1\rightarrow 0$, and $0\rightarrow 1$, for which the corresponding group theoretical studies and some numerical results will be published later), we have not yet obtained such a convenient basis of collective states and we are thus unable to compute in general the reduced density matrix elements of $R^{(1)}$ which appear in the evolution equations (32) and in the expressions of the expectation values of the field [cf. Eqs. (22), (23), and (34)].

The second difficulty lies in the complexity of Eq. (32), which is much greater than for the nondegenerate case[24]: first, one has to solve several independent systems of equations corresponding to the different values of k which appear in the expectation values of the field ($k=0,1,2$ for the intensity and $k=0,1,2,3,4$ for the quantum fluctuations); second, the states of the irreps of the cooperation group are characterized by several quantum numbers instead of only one in the nondegenerate case; the number of different reduced matrix density elements for a given k may become very large, all the more since one has to deal not only with populations but also with some coherences, because of the interference effects; finally, the presence of a sum in the right-hand side of Eq. (32) complicates also the solution of these equations. Consequently, even in the most simple cases mentioned before, the direct methods of solution such as for example the Laplace transform method can be used only for small N. For large values of N, it might be however possible to consider the quantum numbers as continuous parameters and Eq. (32) as partial differential equations.

The problems will be considered in more detail, for particular cases, in another paper. We conclude here by a few qualitative remarks concerning the expected results.

Since the first-order linear differential coupled equations (32) corresponding to different values of k are not coupled together, it is clear that if all reduced matrix elements of $\rho^{(k)}(t)$ are initially zero for a given k, the same remains true at any time. This is particularly interesting if it occurs for all k values except $k=0$ [it cannot occur for $k=0$ because of the trace condition on $\rho(t)$, cf. Eq. (31)]: in this case the emitted radiation remains isotropic and nonpolarized at any time; furthermore it is simple to show that this situation appears if, at $t=0$, the only nonzero density matrix elements in the basis $|\{\lambda\}\mu\gamma JM\rangle$ are populations and do not depend on M (which implies that the initial excitation had created neither orientation nor alignment). Except for such particular initial conditions, the emitted radiation is polarized and non isotropic (cf. Sec. II C); moreover the polarization state will vary with t. In other words, if one observes the radiated light in different polarization directions, the form of the corresponding signals will in principle be different.

However, as far as delay times are concerned, these differences are expected to be small, the superradiance appearing in a single burst except for very particular initial conditions. In effect, since any atomic transition between Zeeman sublevels has either an upper and/or lower level in common with other transitions, the competition effects mentioned in the Introduction play an important role; the superradiant emission on each transition will be influenced (either inhibited, either initiated) by the others. A more precise discussion of these competition effects can be done by a simplified solution of the master equation without interferences (15); one can show that the maximal intensities of the superradiant pulses corresponding to different observation polarizations can be different but correspond to a single delay time (small differences may remain, but they can be masked by the quantum fluctuations of the intensity, which are expected to be large if the initial population inversion is important[19]). Furthermore one can show in the same way that this delay time is always greater than the one which would be evaluated by assuming that the two levels were nondegenerate; the enhancement factor depends of course on the initial conditions and can reach the order of magnitude of the degeneracy of the lower level. Finally we stress that these qualitative results take into account the competition effects only and ignore the interference effects which are the specific consequence of the level degeneracy; however, the role of these effects cannot be studied alone and their influence cannot be predicted, even qualitatively, without a more complete treatment.

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[1] R. H. Dicke, Phys. Rev. 93, 99 (1954).

[2] N. Skribanowitz, I. P. Herman, J. C. MacGillivray, and M. S. Feld, Phys. Rev. Lett. 30, 309 (1973).

[3] M. Gross, C. Fabre, P. Pillet, and S. Haroche, Phys. Rev. Lett. 36, 1035 (1976).

[4] D. Dialetis, Phys. Rev. A 2, 599 (1970); G. S. Agarwal, Phys. Rev. A 2, 2038 (1970); R. Bonifacio, P. Schwendimann, and F. Haake, Phys. Rev. A 4, 302 (1971); 4, 854 (1971); R. Bonifacio and L. A. Lugiato, Phys. Rev. A 11, 1507 (1975); 12, 587 (1975).

[5] V. Ernst and P. Stehle, Phys. Rev. 176, 1456 (1968); R. H. Lehmberg, Phys. Rev. A 2, 883 (1970); N. E. Rehler and J. H. Eberly, Phys. Rev. A 3, 1735 (1971) (see also references therein); R. H. Picard and C. R. Willis, Phys. Rev. A 8, 1536 (1973); R. J. Glauber and F. Haake, *Cooperative Effects: Progress in Synergetics* (North-Holland, Amsterdam, 1974); G. Banfi and R. Bonifacio, Phys. Rev. A 12, 2068 (1975); C. T. Lee, Phys. Rev. A 12, 575 (1975); 13, 1657 (1976).

[6] R. Friedberg and S. R. Hartmann, Phys. Lett. 38A, 227 (1972); C. R. Stroud, J. H. Eberly, W. L. Lama, and L. Mandel, Phys. Rev. A 5, 1094 (1972); N. Skribanowitz, I. P. Herman, J. C. MacGillivray, and M. S. Feld, in *Proceedings of the Vail Conference on Laser Spectroscopy* (Plenum, New York, 1974); J. C. MacGillivray and M. S. Feld, Phys. Rev. A 14, 1169 (1976); see also R. J. Glauber and F. Haake, in Ref. 5.

[7] Y. C. Cho, N. A. Kurnit, and R. Gilmore, in *Coherence and Quantum Optics,* edited by L. Mandel and E. Wolf (Plenum, New York, 1973); T. M. Makhviladze and L. A. Shelepin, Phys. Rev. A 9, 538 (1974); I. R. Senitzky, Phys. Rev. A 10, 1868 (1974); R. Gilmore, C. M. Bowden, and L. M. Narducci, Phys. Rev. A 12, 1019 (1975).

[8] R. Friedberg, S. R. Hartmann, and J. T. Manassah, Phys. Lett. 40A, 365 (1972); R. Friedberg and S. R. Hartmann, Phys. Rev. A 10, 1728 (1974).

[9] C. K. Rhodes, A. Szöke, and A. Javan, Phys. Rev. Lett. 21, 1151 (1968); S. L. McCall and E. L. Hahn, Phys. Rev. 183, 457 (1969); F. A. Hopf, C. K. Rhodes, and A. Szöke, Phys. Rev. B 1, 2833 (1970); G. J. Salamo, H. M. Gibbs, and G. G. Churchill, Phys. Rev. Lett. 33, 273 (1974); H. M. Gibbs, S. L. McCall, and G. J. Salamo, Phys. Rev. A 12, 1032 (1975).

[10] G. S. Agarwal, Phys. Rev. A 4, 1791 (1971); J. H. Eberly, Nuovo Cimento Lett. 1, 182 (1971); E. Ressayre and A. Tallet, Phys. Rev. Lett. 30, 1239 (1973); R. Jodoin and L. Mandel, Phys. Rev. A 9, 873 (1974); A. Flusberg, R. Friedberg, and S. R. Hartmann, Phys. Rev. A 10, 1904 (1974).

[11] B. R. Mollow and M. M. Miller, Ann. Phys. (N.Y.) 52, 464 (1969).

[12] G. S. Agarwal, in *Quantum Optics, Springer Tracts in Modern Physics 70* (Springer-Verlag, Berlin, 1974).

[13] C. Cohen-Tannoudji, in Proceedings of the Summer School in Theoretical Physics of Les Houches, 1975 (to be published).

[14] This general case is studied in Ref. 13 and the particular case of one degenerate and one nondegenerate level in Ref. 12.

[15] This j and the one used for characterizing the upper level in the following are not to be confused.

[16] We have omitted here the dependence of $\Gamma_{ik\to jl}$ on $\omega=\omega_{ij}=\omega_{kl}$, which does not play any role in the following since we consider only two energy levels.

[17] A. R. Edmonds, *Angular Momentum in Quantum Mechanics* (Princeton U.P., Princeton, N.J., 1957).

[18] See, for example, R. J. Glauber and F. Haake, Phys. Rev. A 13, 357 (1976).

[19] J. H. Eberly and N. E. Rehler, Phys. Rev. A 2, 1607 (1970); V. Degiorgio, Opt. Commun. 2, 362 (1971); A. M. Ponte-Gonçalves and A. Tallet, Phys. Rev. A 4, 1319 (1971); F. Haake and R. J. Glauber, Phys. Rev. A 5, 1457 (1972); G. Oliver and A. Tallet, Phys. Rev. A 7, 1061 (1973); M. E. Smithers and E. Y. C. Lu, Phys. Rev. A 9, 790 (1974); R. J. Glauber and R. Haake, Ref. 18.

[20] See, for example, B. R. Judd, *Operator Techniques in Atomic Spectroscopy* (McGraw-Hill, New York, 1963).

[21] In a same way, Cho, Kurnit and Gilmore, and also Makhviladze and Shelepin in Ref. 7 utilize a SU(3) group for a collection of three (nondegenerate) level atoms.

[22] This statement is the exact analog of the conservation of the Dicke's number of cooperation r, and it is no longer exact for extended systems; see C. T. Lee in Ref. 5.

[23] See for example, F. K. Lamb and D. Ter Haar, Phys. Rep. 4, 253 (1971).

[24] For the resolution of the master equation in the two-(nondegenerate) level case, see for example, V. Degiorgio and F. Ghielmetti, Phys. Rev. A 4, 2415 (1971).