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Comments and Addenda

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The ballast resistor

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This note supplements a recent analysis of the bistable ballast resistor, by Ross and Litster. Their discussion of static phase boundaries is extended to the case of motion with uniform velocity. The existence of an effective potential, entirely analogous to equilibrium, and dependent on detailed balance, is questioned.

Ross and Litster[1] have presented a theory of the ballast resistor along the lines of other recent treatments of systems far from equilibrium[2-4] and have thus presented a particularly simple example for such discussions. This note supplements the discussion by Ross and Litster in two directions. Ross and Litster, following the earlier work of Busch,[5] show that there is a particular value of current at which the resistor has a low-temperature phase (T_a) and a high-temperature phase (T_b) which can coexist statically. The current at this point of phase equilibrium satisfies a form of Maxwell construction. We show that this nonstochastic discussion can easily be extended to yield the velocity of the boundary between high- and low-temperature phases as a function of the degree of departure from the current required for static coexistence. This is then equivalent to a rate of melting, or a rate of magnetic switching, which depends on the deviation from equilibrium. We then go on to a discussion of the stochastic part of the Ross and Litster paper and explain our doubts about the existence of a potential function in this system. Bedeaux, Mazur, and Pasmanter[6] have provided another recent analysis of the ballast resistor, limited to deterministic (i.e., nonstochastic) considerations. We shall not comment explicitly on their work, though the contents of this note, as well as an earlier one,[7] are relevant to their work.

The basic equation of motion in this system [Eqs. (5) and (6) of Ref. 1, rewritten slightly so as to emphasize that we are concerned with the temperature T, and not just small deviations ΔT from a spatially uniform steady state] is

$$c_v \frac{\partial T}{\partial t} = -A(T) + i^2 R + \lambda \frac{\partial^2 T}{\partial x^2}. \tag{1}$$

For the sake of simplicity we have written the equation in a form which assumes a constant thermal conductivity λ, independent of T. Reference 1 shows that a static phase boundary between a low-temperature phase T_a and a high-temperature phase T_b can exist at a current i_0 at which

$$\int_{T_a}^{T_b} (A - i_0^2 R)\, dT = 0. \tag{2}$$

The treatment of Ref. 1 can be readily extended to the case of a uniformly moving phase boundary, if the external circuit is such as to maintain a constant flow in the ballast resistor, while the phase boundary moves along the wire.[8] We do this by following the spirit of other treatments of solitary waves[9-11] and assume a temperature profile

$$T(x,t) = T(x-ut), \tag{3}$$

and substitute this in Eq. (1). This yields

$$\lambda \frac{d^2 T}{dx^2} + c_v u \frac{dT}{dx} - A(T) + i^2 R(T) = 0. \tag{4}$$

This equation is entirely analogous to an equation of motion, in a damped potential

$$m\ddot{q} + \beta\dot{q} + \frac{\partial V}{\partial q} = 0. \tag{5}$$

Equation (4) deals with temperature as a function of position, whereas Eq. (5) deals with position as a function of time. Thus in the analogy T is replaced by q, x is replaced by t, λ by m, $c_v u$ by the damping constant β, and

$$V=\int(-A+i^2R)\,dT. \qquad (6)$$

The force $-\partial V/\partial q$ vanishes at the initial and final steady states, a and b. The values of the potential V at a and b will, however, be unequal unless Eq. (2) is satisfied. If Eq. (2) is satisfied, then we have a potential $V(T)=V(q)$ as shown by the solid line in Fig. 1. We then have a solution to the equations of motion, i.e., Eqs. (4) and (5), which departs from a and comes to rest at b, if and only if the damping vanishes, i.e., $u=0$. Denote the corresponding current by i_0 as in Fig. 1 of Ref. 1. Now consider what happens as i is changed away from i_0. For lower values of current, e.g., i_2 in Fig. 1 of Ref. 1, we have an effective potential as shown by the dashed line in Fig. 1. A system now makes the transition away from a, coming to rest at b, only if Eqs. (4) and (5) provide just enough damping, via $c_v u=\beta$, to account for the required energy loss. Thus u must be positive. If u is positive, the temperature profile moves to the right. Thus if the profile displays a transition from a to b, as x increases, then the resistor makes a transition from b to a in time. Thus for $i<i_0$, a is favored and the phase-boundary velocity increases with the difference $|i-i_0|$. Similarly, for $i>i_0$, b is favored. For small departures from i_0, and thus for small damping, the energy losses arising from Eq. (5) can be calculated from the unperturbed solution for q, resulting from $\beta=0$, and from the solid curve in Fig. 1. We shall not, however, take the space to display these equations explicitly.

We have, at this point, completed our discussion of the nonstochastic part of Ref. 1, and now move on to aspects where the fluctuations play an essential role. In equilibrium thermodynamics the equality of free energy of two phases assures their equilibrium along *all* paths which can convert one phase into another. In nonequilibrium situations we can have circulation, i.e., a departure from detailed balance, accompanying the steady-state distribution function. Such circulatory effects prevent the existence of a simple potential function[2] of the form put forth as a conjecture in the discussion following Eq. (13) of Ref. 1. Ross and Litster[1] have cited this remark, as made by this author; here we point out why the ballast resistor can exhibit circulatory effects. We shall do this by showing that Eq. (2), which corresponds to static phase boundaries, does not correspond to zero flux between phases a and b, along a different conversion path.

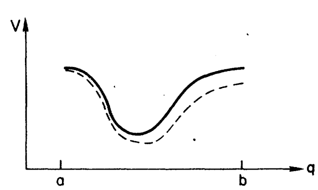

FIG. 1. Effective potential in Eq. (5). Solid curve corresponds to i_0 of Fig. 1 of Ref. 1. Dashed curve corresponds to the lower current i_2.

The actual ballast resistor, as considered by Busch,[5] is complicated by boundary conditions at the ends of the heated wire. These do not, however, appear in the discussion of Ref. 1. For simplicity, we will also discuss the analytically simpler system of Ref. 1 which, presumably, corresponds to periodic boundary conditions. (Our discussion is, however, readily extendable, in a qualitative way, to the complexities of the more realistic case.) Consider a spatially uniform transition from a to b, i.e., one in which T changes simultaneously all along the ballast resistor. If we are constrained to changes along such a one-dimensional set of intermediate states then the resulting distribution function $\rho(T)$ obeys a Fokker-Planck equation[12]

$$\frac{\partial\rho}{\partial t}=-\frac{\partial}{\partial T}\rho v+\frac{\partial}{\partial T}D\frac{\partial\rho}{\partial T}. \qquad (7)$$

v is the drift velocity of the distribution function ρ along the temperature axis. This is just $\dot{T}$, as given by Eq. (1), with $\partial^2T/\partial x^2=0$. D, in Eq. (7), represents the effects of noise, not shown in Eq. (1), and gives the diffusive behavior along the T axis. This term permits ensemble members, which are initially at the same value of T, to separate with time. In the steady state setting $\partial\rho/\partial t=0$, yields

$$\rho=A\exp\int\frac{v}{D}\,dT. \qquad (8)$$

If D were a constant in Eq. (8), then $\rho(a)=\rho(b)$ would correspond to

$$\int_a^b v\,dT=0. \qquad (9)$$

This, however, is identical to Eq. (2), which was derived from a deterministic, rather than a stochastic theory. D, however, is unlikely to be independent of T, as demonstrated by the equations in the appendix of Ref. 1. Thus we cannot expect $\int_a^b(v/D)dT=0$ to be satisfied at the same time as $\int_a^b v\,dT=0$. Hence the spatially uniform temperature transition does not vanish when the phase boundaries are stationary. We have, of course, in this discussion, considered only a very small subset of the totality of paths between the two terminal states, but that is all that is needed to demonstrate the existence of circulation.

We have oversimplified the situation slightly. Our Eq. (2) represents the condition for stationary phase boundaries. The total transition, however, from an initially uniform A state to a final uniform B state, via a moving boundary, also includes an initial nucleation of a phase boundary. This total process, including nucleation, cannot be completely independent of fluctuations, and cannot be described with complete accuracy by Eq. (2). Nevertheless, it is clear that noise sources are much more critical in the case of the spatially uniform transition. Hence we do not expect the qualitative distinction between the two alternative transition paths to disappear. It is true, of course, that in a long enough sample the rate of spatially uniform transitions will become very small compared to the rate of spatially inhomogeneous transitions. Thus in the long sample Eq. (2) (or its refinement which does justice to the nucleation event) may be a reasonable approximation.

As a subsidiary point we will also, here, discuss two of the points in the appendix of Ref. 1. The equations in the appendix of Ref. 1 imply that the voltage fluctuations generated in each section of the ballast wire affect only the heat generation in that section. Furthermore it is assumed that the noise generated in the remaining part of the circuit has no influence on the dissipation in the ballast resistor. If the ballast resistor current i is not allowed to fluctuate, these assumptions are valid, but in that case the heat fluctuations are given by ei, where e is the fluctuating voltage and i the fixed current. This is not the form of the equation for $\tilde{Q}_e(k,\omega)$ given in Ref. 1. A second and separate point: The appendix of Ref. 1. deals only with noise at a uniform steady state. But, as stressed elsewhere by the author,[13] the macroscopic kinetics, e.g., Eq. (1), and the noise sources at the steady states (a and b) do not necessarily determine the noise sources in between these steady states. This has already been illustrated in our discussion by the appearance, in our Eq. (8), of the diffusion constant D, for the intermediate states. An accurate determination of relative stability requires an explicit description of noise *along all the transition paths between two steady states.*

The author is indebted to B. Ross for stimulating correspondence on these questions. This exchange has narrowed the gap between our respective viewpoints, without necessarily leading to complete agreement.

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[1]B. Ross and J. D. Litster, Phys. Rev. A 15, 1246 (1977).
[2]H. Haken, Rev. Mod. Phys. 47, 67 (1975).
[3]*Fluctuations, Instabilities, and Phase Transitions*, edited by T. Riste (Plenum, New York, 1975).
[4]W. Ebeling, *Strukturbildung Bei Irreversiblen Prozessen* (Teubner, Leipzig, 1976).
[5]H. Busch, Ann. Phys. (Leipz.) 64, 401 (1921).
[6]D. Bedeaux, P. Mazur, and R. A. Pasmanter (unpublished).
[7]R. Landauer, J. Phys. Soc. Jpn. 41, 653 (1976).
[8]B. Ross has pointed out (private communication) that a similar treatment is also contained in his Ph. D. thesis (MIT, 1975) (unpublished).
[9]R. Landauer, Ferroelectrics 10, 237 (1976).
[10]H. Metiu, K. Kitahara, and J. Ross, J. Chem. Phys. 64, 292 (1976).
[11]R. Landauer ["Moebius Strip Coupling of Bistable Elements," IBM Tech. Rep., 1955 (unpublished)] discusses solitary waves in a one-dimensionally extended electronic system, far from equilibrium. This report is available upon request.
[12]R. Landauer, in *Synergetics*, edited by H. Haken (Teubner, Stuttgart, 1973). See Sec. 5 and the Appendix which discusses the appropriate form for the diffusion term, when the diffusion coefficient is not constant. N. G. van Kampen has criticized this material (unpublished). The criticism, however, even if valid, does not seem applicable to the case discussed in the present paper.
[13]R. Landauer, Ber. Bunsengesells. Phys. Chem. 80, 1048 (1976).