Generalized dressed-atom approach to atom–strong-field interactions—application to the theory of lasers and Bloch-Siegert shifts*

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A generalized dressed-atom approach (GDAA) is developed to study the interaction of two or more strong fields with atomic and molecular systems. In the GDAA, part of the atom-field interaction is incorporated into the free-atom, free-field Hamiltonian and this expanded Hamiltonian is said to characterize a "dressed atom." The remaining atom-field interaction may then be thought to constitute a probe field interaction for this dressed atom. Provided that the dressed atom constitutes a reasonably good approximation to the true interaction, the GDAA should offer a relatively simple means for semiquantitatively predicting the positions of absorption resonances in quite complex atom-field systems. The GDAA is applied to the theory of a high-intensity, single-mode, standing-wave gas laser, the theory of Bloch-Siegert shifts, and the analysis of Autler-Townes or saturation spectroscopy experiments. The agreement between the GDAA and computer-determined values for laser resonance positions and Bloch-Siegert shifts is found to be surprisingly good over the entire range of applied field strengths. Some methods for using the GDAA to study complex spectroscopic systems are discussed.

I. INTRODUCTION

When an atom is placed in a strong optical field, its energy levels are effectively split by the field. This phenomenon, referred to as an ac Stark effect[1] or "light shift",[2] has also been studied by considering the strong field to act as a "dressing" field for the atom.[3] One can then apply additional (usually weak) fields to probe the structure of the dressed atoms. Experiments of this type can yield values for dipole matrix elements, in addition to providing information on the level spacing in atoms or molecules.

Saturation spectroscopy,[4] "two-photon Doppler-free spectroscopy,"[5] Autler-Townes[1] type experiments,[6] and recent resonance fluorescence experiments[7] can be explained by such an approach under limited conditions. In all cases, the dressing field is strong and in near resonance with a given transition so that the antiresonance component of the field may be neglected. The probe field is a weaker field applied to the same or a coupled transition (in resonance fluorescence, the probe is the vacuum field). As the probe field is tuned, the splittings induced by the dressing field can be obtained.

In many saturation absorption and two-photon experiments, the probe field strength is large enough to invalidate this approach, and the separation into "dressing" plus "probe" fields becomes somewhat artificial. Given the high precision of these experiments, it is important to determine any changes in the line shapes induced by a "strong" probe field. It is the purpose of this paper to develop a theory which will permit the use of a "generalized dressed-atom approach" (GDAA) when two or more intense fields are incident on an atomic system. This GDAA allows one to gain insight into complex physical systems with a minimum of effort; it can also aid in constructing computational schemes for quantitative solutions of a given problem, but itself is not a substitute for the analytic or numerical solutions of those problems. The GDAA is discussed below and applied to the theory of a high-intensity, single-mode, standing-wave gas laser. Since a standing wave may be thought of as two oppositely directed traveling waves of *equal intensity*, designation of one as the dressing field and one as the probe field becomes all but meaningless in this case. A new type of dressed-atom approach is needed.

In conventional dressed-atom approaches, the system of two-level atom plus strong field is solved exactly (neglecting the antiresonance component of the field) using either a quantized or classical field. The "dressed atom" may be viewed as the original atom with each level of the transition effectively split into two levels (Fig. 1). This interpretation of level splittings follows from a solution of the time-dependent Schrödinger equation for the atom and strong field.[8] As is well known, the probability amplitudes $b_\alpha(t)$ ($\alpha = a, b$; see Fig. 1) in this case have a time dependence given by

$$b_\alpha(t) = A_{\alpha 1} e^{-iE_{\alpha 1} t/\hbar} + A_{\alpha 2} e^{-iE_{\alpha 2} t/\hbar}$$

with the A's and E's constants. Consequently,

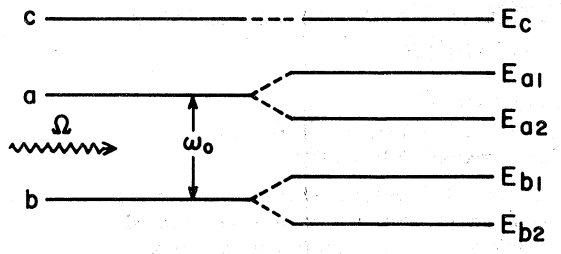

FIG. 1. A strong traveling-wave optical field with frequency Ω in near resonance with the b-a transition effectively splits each of the levels a and b into two separate levels.

each state α is effectively split into two states with energies $E_{\alpha 1}$ and $E_{\alpha 2}$. A calculation of the off-diagonal density matrix element $\rho_{ab} = b_a b_b^*$ reveals that there are four possible oscillation frequencies for this dressed atom ($E_{ai} - E_{bj}$; $i,j = 1,2$), two of which turn out to be equal.

Having accounted for the dressed atom, the probe field is treated in perturbation theory; resonances with the dressed atom occur at the new positions of the level spacings. Consequently, a probe field tuned near the a-c transition in Fig. 1 exhibits two resonances; and one tuned near the a-b transition, four resonances (two of which are equal). Such structure, extremely well known in the microwave domain,[1] has been observed recently in the optical regime.[6,7]

The conventional dressed-atom technique represents a method for carrying out perturbation theory. In reality, one can never separate the strong and weak fields since they are applied to the system at the same time. However, an expanded dressed-atom Hamiltonian consisting of the free-atom–free-field Hamiltonian plus the strong-field interaction accounts for the major part of the interaction; it makes sense and offers physical insight to think of the probe field interacting with the atom + strong-field system. This philosophy can be carried over to the case when there are two or more strong fields interacting with the atoms. In that case, it may still be possible to break up the total interaction of the fields into a part (dressing field interaction) giving a fairly good approximation to the true interaction and a remainder (probe field interaction). If such a separation is possible, then despite the fact that the dressing and probe "fields" may no longer coincide with any particular physical field, the separation into dressing and probe fields can still be made on physical grounds and, in this respect, the approach parallels the conventional dressed atom theory.

In this spirit, a general dressed-atom approach (GDAA) is applied to the specific case of two strong oppositely directed traveling waves interacting with a two-level system. The limiting case of a standing-wave field (equal intensities and frequencies for the applied fields) is discussed in detail since it connects with previous theories of a high-intensity single-mode laser.[9-12] The dressing-field interaction is taken to consist of the two waves interacting independently with the atoms but sharing the same available population of atoms (sometimes called[13] the rate equation approximation—REA) since the REA provides a fairly good approximation to the true interaction. The probe field interaction consists of the difference between the total and REA interactions.

Let us emphasize that the theory we present is not a conventional one, even though the general philosophy is similar to that of traditional dressed-atom theories. The "probe field" of our theory does not coincide with any real physical field and is, in effect, a mathematical construct. Moreover, the "probe" field may not be weak; normal perturbation theory carried out in powers of the probe field diverges in certain cases. It turns out, however, that the GDAA is very useful in providing a simple physical picture to explain the appearance and position of various resonances (see discussion below) that exist in atom–strong-field systems. We have not been able to obtain a formal proof as to why the GDAA works so well for the problem mentioned above, but believe it is related to the fact that the REA does provide a good approximation to the total atom-field interaction, validating its use as a "dressing field."

Using this technique, we are able to predict the positions of resonances occurring in an atomic medium subject to a high-intensity standing-wave laser field. Previous computational theories[9-12] have established the existence of structure in graphs of the medium's spatially averaged population inversion vs atomic axial velocity, and we are able to semiquantitatively explain this structure as a direct consequence of the GDAA. As such, our theory offers an alternative explanation for the appearance of this structure to that given by Feldman and Feld,[10] who relate it to sidebands generated by atoms subject to a strong amplitude modulated field. On the other hand, it is the natural extension of the work of other authors[14] who explain similar structure in saturated absorption curves on the basis of the ac Stark effect.

Another interesting feature of the problem of an atomic medium subjected to a nearly resonant standing-wave optical field is its relation to calculations of Bloch-Siegert[15] shifts. The precise relationship between the two problems and the manner in which the GDAA can be used to provide predictions for Bloch-Siegert shifts are discussed

in the Appendix.

Of course, exact calculations[16] of Bloch-Siegert shifts as well as exact solutions to the single-mode standing-wave laser[9-12] problem exist. The GDAA is not intended to and cannot provide a substitute for those calculations. On the other hand, the GDAA *is* intended to provide an explanation or interpretation of those results which some people might find useful. The GDAA may also be extended to more complex physical problems where exact solutions do not yet exist. In this work, the GDAA is used to provide new predictions of possible resonances in saturation spectroscopy experiments.

Section II contains the Schrödinger equations of motion for density matrix elements written in a form enabling one to apply the GDAA. Resonance predictions are developed in Sec. III, and the comparison of these predictions with exact calculations is given in Sec. IV. Also to be found in Sec. IV is a discussion of methods for testing some GDAA predictions that may be provided by laser and saturation spectroscopy experiments.

II. EQUATIONS OF MOTION

In a typical laser calculation,[9-13] one obtains the contributions to the population inversion and polarization of an atomic sample as a function of both the position in the laser cavity and the velocity of the atoms giving rise to the contribution. If the inversion and polarization are spatially averaged, one obtains these quantities solely as a function of velocity. Graphs of the spatially averaged population inversion or polarization vs velocity for high laser intensities exhibit many resonances.[9-12] In this section and the next, we demonstrate how to anticipate the position of these resonances. A direct experimental verification of these resonances is not possible since the laser output depends on the velocity averaged polarization of the sample. However, there are ways to monitor the resonances and these methods are mentioned in Sec. IV.

In order to use the GDAA to predict the resonances as a function of velocity, we consider two fields:

$$\vec{E}_+(z,t) = \hat{i}E_+ \cos(\Omega_1 t - k_1 z), \tag{1a}$$

$$\vec{E}_-(z,t) = \hat{i}E_- \cos(\Omega_2 t + k_2 z), \tag{1b}$$

with propagation vectors $k_1\hat{z}$ and $-k_2\hat{z}$ and frequencies $\Omega_1 = k_1 c$ and $\Omega_2 = k_2 c$, respectively, interacting with a two-level nondegenerate atom moving with axial velocity v. The free atom has an electric dipole resonance with frequency ω_0 between an upper state a and lower state b. The equations of motion for density matrix elements follow from Schrödinger's equation, $i\hbar\dot{\rho} = [H,\rho]$. Employing the rotating-wave approximation and working in the atomic rest frame ($z = vt$), one obtains

$$\frac{\partial \rho_{ab}}{\partial t} = -i\omega_0\rho_{ab} + \tfrac{1}{2}i[\chi_+ e^{-i(\Omega_1 - k_1 v)t} + \chi_- e^{-i(\Omega_2 + k_2 v)t}](\rho_{bb} - \rho_{aa}), \tag{2a}$$

$$\frac{\partial \rho_{aa}}{\partial t} = \tfrac{1}{2}i[\chi_+ e^{-i(\Omega_1 - k_1 v)t} + \chi_- e^{-i(\Omega_2 + k_2 v)t}]\rho_{ba} + \text{c.c.}, \tag{2b}$$

where

$$\chi_+ = \mu_{ab}E_+/\hbar, \quad \chi_- = \mu_{ab}E_-/\hbar, \tag{3}$$

μ_{ab} is the x component of the electric dipole matrix element, and equations for ρ_{ba} and ρ_{bb} are obtained by interchanging $a \rightleftharpoons b$, $\omega_0 \rightleftharpoons -\omega_0$, and $(\Omega_i \pm k_i v) \rightleftharpoons -(\Omega_i \pm k_i v)$ in Eqs. (2). All decay processes have been neglected since they will not significantly affect the position of the resonances in the strong-field limit $\chi_\pm \gg$ (decay rates).[17]

It is impossible to extract the rate equation approximation (REA) directly from Eqs. (2). Since we wish to use the REA as our dressing-field interaction, we expand our basis by writing

$$\rho_{ab}(t) = \rho^+_{ab}(t)e^{-i(\Omega_1 - k_1 v)t} + \rho^-_{ab}(t)e^{-i(\Omega_2 + k_2 v)t}. \tag{4}$$

The quantities ρ^+_{ab} and ρ^-_{ab} are chosen to satisfy the corresponding equations of motion had fields $\vec{E}_1$ or $\vec{E}_2$ acted separately [i.e., ρ^+_{ab} is a solution of Eqs. (2) with $\chi_- = 0$ and ρ^-_{ab} is a solution of Eqs. (2) with $\chi_+ = 0$]. In terms of the real variables

$$C^\pm = \rho^\pm_{ab} + (\rho^\pm_{ab})^*, \tag{5a}$$

$$S^\pm = i[\rho^\pm_{ab} - (\rho^\pm_{ab})^*], \tag{5b}$$

$$N = \rho_{aa} - \rho_{bb}, \tag{5c}$$

Eqs. (2) may be written

$$\dot{C}^\pm = -\delta_\pm S^\pm, \tag{6a}$$

$$\dot{S}^\pm = \delta_\pm C^\pm + \chi_\pm N, \tag{6b}$$

$$\dot{N} = -\chi_+ S^+ - \chi_- S^- + [-\chi_- C^+ \sin\bar{\delta}t - \chi_- S^+ \cos\bar{\delta}t + \chi_+ C^- \sin\bar{\delta}t - \chi_+ S^- \cos\bar{\delta}t], \tag{6c}$$

where

$$\delta_+ = \omega_0 - \Omega_1 + k_1 v, \quad \delta_- = \omega_0 - \Omega_2 - k_2 v \tag{7}$$

and

$$\bar{\delta} = \delta_- - \delta_+. \tag{8}$$

Equations (6), excluding the terms in brackets in Eq. (6c) describe two traveling waves sharing the same population of the atom but in all other ways acting independently with the atom and constitutes the REA for the system. We take the solution of the REA to represent our "dressed" atom

and the terms in brackets in Eq. (6c) to represent a probe-field interaction, in the sense described in the Introduction.

III. PREDICTION OF RESONANCES

It was noted in the Introduction that the new resonant frequencies of the dressed atom are found by determining those frequencies appearing in the time-dependent polarization arising from the atom and dressing-field interaction. Equations (6) in the REA [neglect of bracketed term in Eq. (6c)] represent five coupled linear first-order differential equations. When trial solutions of the form

$$[S^{\pm}(t), C^{\pm}(t), N(t)] = \sum_{n=1}^{5} [S_n^{\pm}, C_n^{\pm}, N_n] e^{i\sigma_n t}$$

are substituted into these equations, one finds eigenvalues

$$\sigma_1 = 0 \ , \tag{9a}$$

$$\sigma_{2,3} = 2^{-1/2}\{(\delta_+^2 + \delta_-^2 + \chi_+^2 + \chi_-^2) \pm [(\delta_+^2 - \delta_-^2)^2 + (\chi_+^2 + \chi_-^2)^2 + 2(\chi_+^2 - \chi_-^2)(\delta_+^2 - \delta_-^2)]^{1/2}\}^{1/2} \ . \tag{9b}$$

$$\sigma_4 = -\sigma_2, \quad \sigma_5 = -\sigma_3 \ . \tag{9c}$$

Since $S^{\pm}(t)$ and $C^{\pm}(t)$ vary as $\exp(\pm i\sigma_j t)$ $(j = 1,2,3)$, it follows from Eqs. (5a) and (5b) that $\rho_{ab}^{\pm}(t)$ also vary as $\exp(\pm i\sigma_j t)$. Using this result in Eq. (4), one finds that $\rho_{ab}(t)$ and, consequently, the polarization oscillates at ten frequencies in the atom's rest frame given by

$$\omega' = \Omega_i + (-1)^i k_i v \pm \sigma_j \ ,$$
$$i = 1,2 \ , \quad j = 1,2,3 \ . \tag{10}$$

Thus, one can consider interactions of the probe or correction fields with this atom that has had ω_0 replaced by ten frequencies as a result of the dressing field's action. To compare our predictions with some numerical results of laser[9–12] and Bloch-Siegert theory,[16] we consider the limiting case of a standing wave

$$\chi_+ = \chi_- = \chi \ , \quad \Omega_1 = \Omega_2 = \Omega \ , \quad k_1 = k_2 = k \ . \tag{11}$$

In this limit, Eqs. (9) and (10) reduce to

$$\omega' = \Omega \pm kv \pm \sigma_j, \quad j = 1,2,3 \tag{12}$$

with

$$\sigma_1 = 0 \ , \tag{13a}$$

$$\sigma_{2,3} = [\Delta^2 + k^2 v^2 + \chi^2 \pm (4\Delta^2 k^2 v^2 + \chi^4)^{1/2}]^{1/2} \ , \tag{13b}$$

where the detuning Δ is defined by

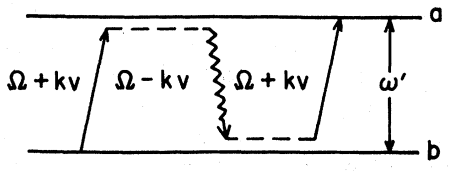

FIG. 2. Diagram showing a "three-photon absorption." There is absorption of two photons of one of the traveling waves and stimulated emission involving one photon of the other traveling wave leading to an overall net absorption. The frequency ω' is one of the "dressed-atom" frequencies given by Eq. (12).

$$\Delta = \Omega - \omega_0 \ . \tag{14}$$

Resonances of the "probe" field with the dressed atom occur whenever the probe-field frequency seen in the atomic rest frame equals one of the ω' given by Eqs. (12) and (13). Since the probe field contains parts of both traveling waves, the appropriate frequencies in the atomic rest frame are $\Omega \pm kv$. Thus, the resonance condition for the absorption of one probe photon by the dressed atom is

$$\Omega \pm kv = \omega' \tag{15}$$

with ω' given by (12) and (13). Multiphoton absorption resonances are generated when the dressed atom absorbs n photons at frequency $(\Omega \pm kv)$ and undergoes stimulated emission with $(n-1)$ photons at frequency $\Omega \mp kv$ as shown in Fig. 2. The resonance condition is

$$n(\Omega \pm kv) - (n-1)(\Omega \mp kv) = \omega' \ . \tag{16}$$

(It should be clear that any number of absorption-emission pairs of one traveling-wave component can be inserted in Fig. 2 without changing the resonance condition. Such pairs are invariably present when strong fields are involved and our nomenclature of "one-photon" or "n-photon" resonance is meant to apply only to the unpaired absorptions and emissions.)

Equations (15) and (16) can be satisfied only by atoms having the proper axial velocity v. Using Eqs. (16), (12), and (13), one finds that atoms with axial velocities given by

$$kv = 0 \ , \tag{17a}$$

$$k^2 v^2 = (4n^2 - 1)^{-2}\{(4n^2 + 1)\Delta^2 + (4n^2 - 1)\chi^2 \pm [16n^2\Delta^4 + 4(4n^2 - 1)\Delta^2\chi^2 + (4n^2 - 1)^2\chi^4]^{1/2}\} \tag{17b}$$

experience resonant interactions with the fields.

Interpretation

Large detunings—$|\Delta| \gg \chi$. For large detunings, each of the traveling waves essentially interacts

with a *separate* velocity group of atoms (e.g., $kv = \pm\Delta$ for single photon processes). Consequently, to lowest order, the results should reduce to a conventional dressed-atom picture where an off-resonant traveling-wave field dresses the atom and leads to a shift in the frequency position for maximum absorption of the resonant traveling wave. For example, an atom moving with axial velocity $kv \approx \Delta$ sees two fields at frequencies $\Omega \pm kv \approx \Omega \pm \Delta = \omega_0$, $\omega_0 + 2\Delta$ in its rest frame. The off-resonant field at $\omega_0 + 2\Delta$ "dresses" the atom and slightly shifts the absorption resonance at ω_0. Similar pictures are possible for the multiphoton processes, but some care must be taken in their interpretation as is discussed below.

For large detunings, Eq. (17a) represents nonresonant Rayleigh scattering and is relatively unimportant, while Eq. (17b) takes on the asymptotic limit

$$(kv)_n^+ \sim \pm\frac{\Delta}{2n-1}\left[1+\frac{(2n-1)}{4n}\left(\frac{\chi^2}{\Delta^2}\right)\right], \tag{18a}$$

$$(kv)_n^- \sim \pm\frac{\Delta}{2n+1}\left[1+\frac{(2n+1)}{4n}\left(\frac{\chi^2}{\Delta^2}\right)\right], \tag{18b}$$

where + or − refers to the two roots of Eq. (17b). Resonances occur near $\pm\Delta/(2n-1)$, which for $n=1$ represents single-photon absorption of one of the traveling waves and for $n>1$ represents absorption of n photons of one traveling wave and emission of $n-1$ photons of the other. All resonance positions are slightly shifted by the field.

One notices that for $n>1$, $(kv)_n^+$ and $(kv)_{n-1}^-$ exhibit resonances near $\pm\Delta/(2n-1)$. These roots actually correspond to the same resonance and, to get the total shift in the position of the resonance, one must combine the shifts of $(kv)_n^+$ and $(kv)_{n-1}^-$ in some manner. One could simply add the shifts, but adding the shifts in quadrature produces a somewhat more satisfying result in the asymptotic high-field limit.[18] Consequently, resonances are predicted to occur at

$$(kv)_1 = (kv)_1^+ \sim \pm\Delta[1+\tfrac{1}{4}(\chi^2/\Delta^2)], \tag{19a}$$

$$(kv)_n = \{[(kv)_n^+]^2 + [(kv)_{n-1}^-]^2 - [\Delta/(2n-1)]^2\}^{1/2}$$

$$\sim \pm\frac{\Delta}{2n-1}\left[1+\frac{(2n-1)^2}{4n(n-1)}\left(\frac{\chi^2}{\Delta^2}\right)\right], \quad n>1 . \tag{19b}$$

The shifts appearing in Eqs. (19) are equal to those found in conventional dressed-atom theories. However, if we were to expand Eq. (17) of our GDAA to include the χ^4/Δ^3 term, the results would be in agreement with the exact results while those of conventional dressed atom theories, where only one field acts as the dressing field, would differ.

Small detunings—$|\Delta| \ll \chi$. If $|\Delta| \ll \chi$, the dressing field is absorbed by atoms with axial velocities in the range $-\chi \lesssim kv \lesssim \chi$. This power broaded absorption "hole" is well known in hole-burning[19] theories of laser operation or saturation spectroscopy. The probe field creates *additional* structure on this broad absorption band at velocities obtained from Eq. (17) in the limit $|\Delta| \ll \chi$:

$$(kv)_n^+ \sim \pm\left(\frac{2\chi^2}{4n^2-1}\right)^{1/2}\left[1+\frac{(4n^2+3)}{4(4n^2-1)}\left(\frac{\Delta^2}{\chi^2}\right)\right], \tag{20a}$$

$$(kv)_n^- \sim \pm\Delta/(4n^2-1) . \tag{20b}$$

This structure is seen as narrow absorptions superimposed on the broad absorption background produced by the dressing field (to be discussed below).

It is not obvious if $(kv)_n^+$ and $(kv)_{n-1}^-$ still correspond to the same resonance as was evident for the case $|\Delta| \gg \chi$ above. However, we use the same prescription employed in arriving at Eqs. (19) to obtain

$$(kv)_1 \sim \pm(2/3)^{1/2}\chi[1+(7/6)(\Delta^2/\chi^2)], \tag{21a}$$

$$(kv)_n \sim \pm\left(\frac{2\chi^2}{4n^2-1}\right)^{1/2}\left[1+\frac{(8n^3-4n^2+14n-7)}{4(2n+1)(4n^2-8n+3)}\left(\frac{\Delta^2}{\chi^2}\right)\right], \quad \text{for } n>1 . \tag{21b}$$

In the limit of zero detuning the resonance positions are at

$$(kv)_n = \pm[2\chi^2/(4n^2-1)]^{1/2}. \tag{22}$$

It is interesting to note that, had one used a conventional dressed-atom approach, Eq. (22) would be replaced by $(kv)_n = \pm[\chi^2/(4n^2-1)]^{1/2}$. The GDAA takes into account the fact that both traveling fields are strong and gives a much better approximation to the resonances than the conventional theory.

Intermediate detunings—$|\Delta| \approx \chi$. For intermediate detunings, neither of the above pictures is strictly valid. The dressing-field–probe-field concept is still applicable in the more general sense that the dressing-field interaction provides a fairly good approximation to the true interaction and the probe-field interaction monitors the new frequencies of the dressed atom. Using the same prescription for combining $(kv)_n$ and $(kv)_{n-1}$ for $n>1$ that was used in Eqs. (19) and (21), we determine the GDAA resonance predictions for arbitrary detunings and field strengths to be

$$(kv)_1 = \pm\tfrac{1}{3}[5\Delta^2+3\chi^2+(16\Delta^4+12\Delta^2\chi^2+9\chi^2)^{1/2}], \tag{23a}$$

$$(kv)_n = \pm\left(\sum_{m=n-1}^{n}(4m^2-1)^{-2}\{(4m^2+1)\Delta^2+(4m^2-1)\chi^2 + (-1)^{m-n}[16m^2\Delta^2+4(4m^2-1)\Delta^2\chi^2+(4m^2-1)^2\chi^4]^{1/2}\} - \frac{\Delta^2}{(2n-1)^2}\right)^{1/2}, \quad n>1 . \tag{23b}$$

IV. COMPARISON WITH EXACT RESULTS—EXPERIMENTAL CONSIDERATIONS

Equations (23) give the GDAA resonance predictions. The GDAA, as described above, offers no estimate as to the amplitude or width of the resonances. Consequently, Eqs. (23) should be viewed as providing *possible* resonance positions. If the detuning is large compared with the field strength χ or χ is less than or comparable to some appropriate decay parameter in the system, it is difficult to observe more than one or two resonances. While experimental verification of the resonance positions may be difficult, it is a relatively simple matter to compare the GDAA predictions with computer solutions of Eq. (6). Taking $N(t_0)=-1$, $S^{\pm}(t_0)=C^{\pm}(t_0)=0$ in Eqs. (6) (i.e., atoms in lower state at $t=t_0$), we solved the equations numerically to obtain $N(t)$. The solutions were then averaged over t_0 and t to determine those values of kv giving rise to absorption maxima. The exact resonance positions obtained from the computer solutions are compared with the predictions of the GDAA [Eqs. (23)] in Table I. Several values of n ($1\leq n\leq 5$) corresponding to $(2n-1)$-photon resonances [see Eq. (16)] have been tabulated along with a range of field strength parameters.

The agreement between the GDAA predictions and the exact results is truly remarkable. For the main resonance ($n=1$) the GDAA and exact results differ by no more than 2% over the entire range of field strengths. For the higher-order resonances ($2\leq n\leq 5$) the difference between the exact results and GDAA predictions is no more than 6%. For these higher-order resonances, the GDAA prediction is admittedly dependent on the manner of combining shifts given in Eq. (19b); however, any reasonable algorithm for combining the shifts does not lead to errors greater than 10% to 20% over the entire range of field strengths. We can offer no rigorous explanation as to why the GDAA produces such excellent approximations for the resonance positions. One might have expected an overall qualitative agreement, but the quantitative accuracy is surprising.

For arbitrary values of n, some general observations can be made. For large detunings $|\Delta/\chi| \gg 1$, Eqs. (23) may be expanded [the lead terms are given in Eqs. (19)] and one finds agreement with the exact results to order χ^4/Δ^4. Thus, the GDAA is very accurate for large detunings.

For large field strengths $|\chi/\Delta| \gg 1$, the GDAA

TABLE I. Positions of absorption maxima kv/Δ as a function of field strength χ/Δ when a standing-wave field is applied to a two-level system. The quantities $(kv/\Delta)_E$ are the exact (to 1%) resonance positions determined from computer solutions while the $(kv/\Delta)_G$ represent the generalized dressed-atom approach prediction for the resonance positions. The quantity n through the factor $(2n-1)$ determines the number of unpaired photons taking part in the absorption (i.e., $n=2$ corresponds to a three-photon absorption process).

n	$\frac{\chi}{\Delta}$	$\left(\frac{kv}{\Delta}\right)_G$	$\left(\frac{kv}{\Delta}\right)_E$	% difference
1	0	1.00	1.00	0.0
1	0.5	1.06	1.06	0.0
1	1.0	1.25	1.25	0.0
1	2.5	2.23	2.26	1.3
1	5.0	4.18	4.24	1.4
1	10.0	8.21	8.36	1.8
1	∞	0.816[a]	0.832[a]	1.9
2	0	0.333	0.333	0.0
2	0.5	0.414	0.413	0.2
2	1.0	0.572	0.560	2.1
2	2.5	1.05	1.01	4.0
2	5.0	1.91	1.86	2.7
2	10.0	3.64	3.65	0.3
2	∞	0.365[a]	0.362[a]	0.8
3	0	0.2	0.2	0.0
3	0.5	0.244	0.245	0.4
3	1.0	0.331	0.340	2.6
3	2.5	0.644	0.650	0.9
3	5.0	1.22	1.19	2.5
3	10.0	2.40	2.33	3.0
3	∞	0.239[a]	0.231[a]	3.5
4	0	0.143	0.143	0.0
4	0.5	0.174	0.174	0.0
4	1.0	0.236	0.242	2.5
4	2.5	0.472	0.479	1.5
4	5.0	0.905	0.878	3.1
4	10.0	1.79	1.71	4.7
4	∞	0.177[a]	0.170[a]	4.1
5	0	0.111	0.111	0.0
5	0.5	0.135	0.136	0.7
5	1.0	0.184	0.188	2.1
5	2.5	0.375	0.376	0.3
5	5.0	0.720	0.693	3.9
5	10.0	1.43	1.35	5.9
5	∞	0.142[a]	0.134[a]	4.5

[a] Corresponds to kv/χ rather than kv/Δ.

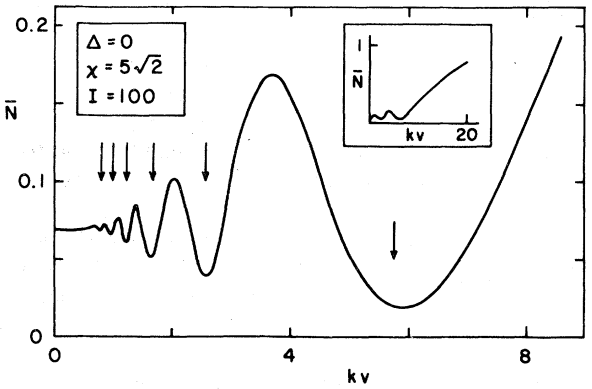

FIG. 3. Computer solution for the steady-state population inversion $\bar{N}$ of two-level atoms in a standing-wave field as a function of axial velocity. (The graph is symmetric about $kv=0$.) A normalization is chosen such that $\bar{N}=1.0$ in the absence of the laser field, and the field strength χ, detuning Δ, and kv are given in units of some atomic decay parameter γ, while the dimensionless intensity $I=2\chi^2/\gamma$. For this graph, $\chi=5\sqrt{2}$, $I=100$, and $\Delta=0$. The insert shows a wider range of kv revealing that the detailed structure near $kv=0$ is actually superimposed on a very broad hole centered at $kv=0$. The arrows on the diagram indicate the positions of the resonances predicted from the generalized dressed-atom approach.

resonance prediction is $(kv)_n=[2\chi^2/(4n^2-1)]^{1/2}$ while the exact resonance positions occur at the zeros of the Bessel function $J_0(4\chi/kv)$.[16] Thus the exact resonance positions occur at $(kv)_n/\chi=0.8317$, 0.3623, 0.2311, 0.1696, 0.1339 for $n=1,2,3,4,5$, respectively, while the GDAA prediction is $(kv)_n/\chi=0.8165$, 0.3651, 0.2390, 0.1768, 0.1421. For large n, the exact resonance position occurs at $(kv)_n/\chi=0.636/n$ and the GDAA prediction at $(kv)_n/\chi=0.707/n$ so that the two results differ by about 10%. Thus, even for large field strengths and high-order resonances, the GDAA and exact results are in good agreement.

To conclude this section, we examine the implications of our results for laser theory, and saturation spectroscopy or Autler-Townes-type experiments.

A. Laser theory

The equations which have been considered are essentially the same as those encountered in the theory of a high-intensity single-mode standing-wave laser.[9-12] In laser theory, one would add some decay terms to Eqs. (2), solve Eqs. (2) assuming an atom located at $z=z_0$ is excited to a given state at $t=t_0$, and then integrate over the rate of excitation to obtain the steady-state population inversion density $N(z,v)=\rho_{aa}(z,v)-\rho_{bb}(z,v)$ and polarization of the system. The spatially averaged population inversion $N(v)=\langle N(z,v)\rangle_z$ is not directly measurable in the laser output, but can be monitored by other means (see below). A graph of $\bar{N}=N(v)/N_0(v)$ vs kv, where $N_0(v)$ is the population inversion in the absence of the laser field, will reveal all "holes" burned into the population inversion by the standing-wave field.

The GDAA has provided predictions for the position of these holes or resonances. The inclusion of decay rates for the levels does not significantly alter the resonance positions for $\chi \gg$ decay rates,[17] but may determine whether the amplitude of certain resonances are large enough to distinguish. In Figs. 3 and 4, computer solution graphs of $\bar{N}$ vs kv for detunings $\chi/\Delta=\infty$ and $\chi/\Delta=\sqrt{2}$, respectively, are shown. The corresponding GDAA predictions for the resonance positions are shown by the arrows in the figures, and the agreement between the GDAA and exact results is seen to be excellent. For $\chi/\Delta=\infty$, at least six resonances are visible and, for $\chi/\Delta=\sqrt{2}$, four resonances may be seen. [In obtaining Figs. 3 and 4, some decay rate γ was assumed for levels a and b. All variables are expressed in units of γ (i.e., $\chi=5\sqrt{2}\gamma$, $\Delta=0.0\gamma$ in Fig. 3; $\Delta=5\gamma$ in Fig. 4; and kv/γ ranges from 0.0 to 20). [A dimensionless intensity $I=2\chi^2/\gamma^2$, which is generally used in laser theory,[9-12] has the value 100 in Figs. 3 and 4, corresponding to a typical power density $\approx 10^5$ W/m^2.]

B. Saturation or Autler-Townes experiments

As was mentioned above, one cannot experimentally directly measure the structure shown in Figs. 3 and 4. In order to probe this structure, one must saturate a given transition with a standing-wave field and monitor the absorption of a weak traveling-wave field on the same or a coupled

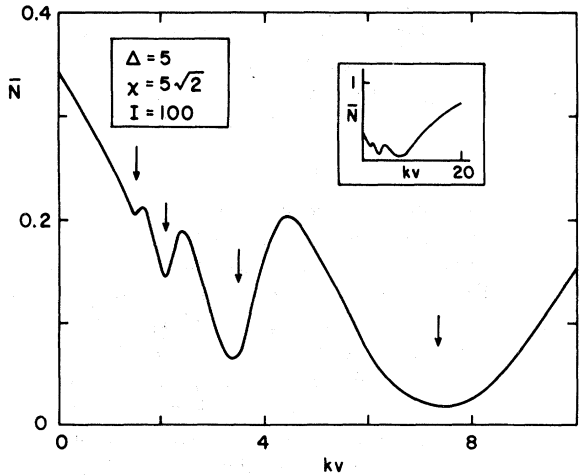

FIG. 4. A graph similar to that in Fig. 3, but with $\Delta=5$.

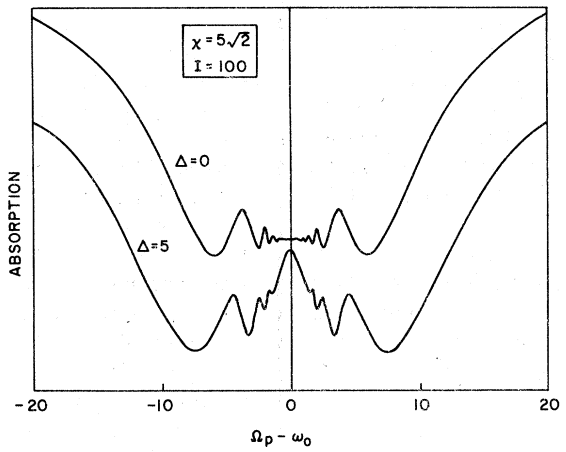

FIG. 5. Two graphs giving a *schematic* picture of the reduction in probe-beam absorption when a traveling-wave probe beam is applied to a transition that is simultaneously driven by a strong standing wave field. The transition frequency is ω_0, the probe frequency Ω_p, the standing wave field strength $\chi = 5\sqrt{2}$ and graphs are drawn for two detunings, $\Delta = 0$ and $\Delta = 5$ of the standing-wave field. The ordinate is in arbitrary units and χ, Δ, and $\Omega_p - \omega_0$ are in units of the decay parameter γ mentioned in the caption of Fig. 3.

transition. A theoretical analysis of this problem using an exact computer solution for the standing-wave field was given by Feldman and Feld.[20] Experimental work of this nature is just being reported.[6]

An exact theoretical analysis of the problem is quite complex. The GDAA enables one to at least get a semiquantitative prediction of resonances which may appear in the probe beam absorption profile.

Two approaches can be used. From a simple hole-burning picture, there is a resonance in the probe absorption whenever the probe interacts with atoms having axial velocity given by Eqs. (23), corresponding to the resonance dips in Figs. 3 and 4. That is, resonances occur whenever

$$\Omega_p - \omega_p = (kv)_n, \quad n \geq 1, \tag{24}$$

where Ω_p is the probe field frequency, ω_p the transition being probed and $(kv)_n$ given by Eqs. (23).

The predictions of Eq. (24) are shown schematically in Fig. 5 corresponding to the standing-wave saturation conditions of Figs. 3 and 4, respectively. The standing-wave field saturates the $b \rightarrow a$ transition and the probe absorption on the same transition ($\omega_p = \omega_0$) is monitored. Whenever condition (24) is satisfied, there are fewer atoms in state b leading to less absorption of the probe. Thus each dip shown in Figs. 3 (zero detuning) and 4 ($\Delta = 5$) is mirrored in the probe absorption profiles shown in Fig. 5. The resonances move out and broaden with increasing pump-field strength.[21] One notes that power broadening of the higher-order resonances is substantially reduced, providing another means for obtaining narrow Doppler-free resonances. The distance between dips can be used to determine dipole matrix elements or calibrate laser field strengths.

Our predictions are in good agreement with the exact calculations of Feldman and Feld.[20] However, their work, as well as recent experimental studies,[6] have revealed only the main resonances shown in Fig. 5 and none of the higher-order resonances. Admittedly, the hole-burning picture is an oversimplification, and the actual line shape and widths of the resonances are determined in detail by complex interference effects (e.g., the exact result[20] corresponding to $\Delta = 0$ of Fig. 5 exhibits a sharp dip at $\Omega_p - \omega_0 = 0$). However, we feel that both theoretical and experimental studies carried out at higher intensities than have been used previously will reveal the additional structure shown in Fig. 5.

The alternative approach to predicting the probe-field resonances is the use of an ac Stark effect or dressed-atom picture. In that case, the dressing field produces modified resonant frequencies of the atoms given through Eqs. (9) and (10) or (12) and (13). By considering interactions of the correction fields plus the applied traveling-wave probe field with the dressed atom, one can determine probe absorption or amplification resonances as a function of velocity. This type of analysis leads to the result that some splitting of the velocity averaged absorption resonances shown in the $\Delta = 5$ curve of Fig. 5 is possible. Dressed-atom or ac-Stark-effect approaches can help determine the conditions necessary to produce splitting,[22] in addition to providing a semiquantitative prediction for the magnitude of the splitting. Experimental verification of the splittings has been achieved.[6]

It might be noted that coherent transient experiments using standing-wave excitation could also be used to observe the above phenomena.

V. CONCLUSIONS

A generalized dressed-atom approach (GDAA) has been proposed to study systems in which two or more intense fields interact with an atomic system. The major purpose of the GDAA is to provide a simple interpretation of the interactions which enable one to give a semiquantitative prediction of the system's output characteristics. While the GDAA gives no information on the amplitudes or widths of resonances, it does yield a reasonable approximation and explanation for position of resonances.

For the case of a standing-wave field interacting with a two-level system considered above, the GDAA provided an unusually good approximation for the resonance positions over the entire range of field strengths. Moreover, the resonance structure shown in Figs. 3 and 4 arises naturally in the GDAA and aids one's understanding of these curves. In the past, this structure had not been anticipated and its connection with the multiquantum Bloch-Siegert resonances had not been appreciated. Finally, we presented some predictions of the GDAA (Fig. 5) which can be tested both theoretically and experimentally. We should note that Eqs. (10), (15), and (16) can be used for predicting resonance positions for the case of two oppositely directed traveling waves of arbitrary amplitude and frequency interacting with a two-level system.

The GDAA should be applicable to a wide range of problems involving atom–strong-field interactions. By choosing a "dressing-field" interaction which is exactly soluble and provides a good first approximation to the true interaction, one can obtain resonance predictions by considering interactions of a "probe field" with this dressed atom. In particular, this method may prove useful in the theory of multimode lasers and multiphoton processes. Since the computational methods for treating those problems are by no means straightforward, the GDAA may provide some insight into the path the calculations should follow.

APPENDIX: BLOCH-SIEGERT SHIFTS

In typical[16] calculations of Bloch-Siegert shifts, one considers an isolated stationary spin-$\frac{1}{2}$ system subjected to a magnetic field of the form

$$\vec{B} = B_0\hat{k} + B_1 \cos\Omega' t\, \hat{i} \ . \tag{A1}$$

The static component of the field provides an energy separation

$$\omega_0' = -\gamma B_0 \tag{A2}$$

(γ is the gyromagnetic ratio) for the $\pm\frac{1}{2}$ spin states and the time-varying component is used to probe this separation. In the rotating-wave approximation, single and multiphoton resonances with the time-varying field occur when

$$\Omega' = \omega_0'/(2n+1) \tag{A3}$$

for integer n. The shift in these resonance positions due to the presence of the "antiresonance" (or counter-propagating) component of the oscillating field are termed Bloch-Siegert shifts. The relationship between the calculation of Bloch-Siegert shifts and the calculation of velocity resonances presented in the text is shown below.

The Hamiltonian for the spin system is

$$H = -\gamma\hbar\vec{S}\cdot\vec{B} \ , \tag{A4}$$

where $\vec{S}$ is the Pauli spin operator. For a spin-$\frac{1}{2}$ system with the $+\frac{1}{2}$ and $-\frac{1}{2}$ states labeled by a and b, respectively, the equations of motion for density matrix elements given by $ih\dot{\rho} = [H, \rho]$ are

$$\dot{\rho}_{aa} = i\chi' \cos(\Omega' t)(\rho_{ba} - \rho_{ab}) \ , \tag{A5a}$$

$$\dot{\rho}_{ab} = -i\omega_0'\rho_{ab} + i\chi' \cos(\Omega' t)(\rho_{bb} - \rho_{aa}) \ , \tag{A5b}$$

$$\dot{\rho}_{bb} = -i\chi' \cos(\Omega' t)(\rho_{ba} - \rho_{ab}) \ , \tag{A5c}$$

$$\rho_{ba} = \rho_{ab}^* \ , \tag{A5d}$$

where

$$\chi' = 2\gamma B_1 \ . \tag{A6}$$

Writing

$$\rho_{ab}(t) = \rho_{ab}^+(t)e^{-i\Omega' t} + \rho_{ab}^-(t)e^{i\Omega' t} \ , \tag{A7}$$

where $\rho_{ab}^+(t)\exp(-i\Omega' t)$ and $\rho_{ab}^-(t)\exp(i\Omega' t)$ are chosen as solutions of Eqs. (A5) in the rotating-wave and "antirotating" wave approximations, respectively; and defining

$$C'^{\pm} = \rho_{ab}^{\pm} + (\rho_{ab}^{\pm})^*, \tag{A8a}$$

$$S'^{\pm} = i[\rho_{ab}^{\pm} - (\rho_{ab}^{\pm})^*], \tag{A8b}$$

$$N' = \rho_{aa} - \rho_{bb}, \tag{A8c}$$

one uses Eqs. (A5)–(A8) to obtain

$$\dot{C}'^{\pm} = -\delta_{\pm}'S'^{\pm}, \tag{A9a}$$

$$\dot{S}'^{\pm} = \delta_{\pm}'C'^{\pm} + \chi'N', \tag{A9b}$$

$$\dot{N}' = -\chi'S'^{+} - \chi'S'^{-} + [-\chi'C'^{+}\sin\bar{\delta}'t - |\chi'S'^{+}\cos\bar{\delta}'t + \chi'C'^{-}\sin\bar{\delta}'t - \chi'S'^{-}\cos\bar{\delta}'t], \tag{A9c}$$

where

$$\delta_{\pm}' = \omega_0' \mp \Omega', \tag{A10a}$$

$$\bar{\delta}' = 2\Omega'. \tag{A10b}$$

Equations (A9) are equivalent to Eqs. (6) taken in the standing-wave limit $\Omega_1 = \Omega_2 = \Omega$, $k_1 = k_2 = k$, $\chi_+ = \chi_- = \chi$, provided the correspondence $\chi \rightarrow \chi'$,

$$\Delta = \Omega - \omega_0 \rightarrow -\omega_0' , \tag{A11a}$$

$$kv \rightarrow -\Omega' \tag{A11b}$$

is made. Thus, solutions of the Bloch-Siegert problem and the problem of the interaction of an intense standing-wave field with an atomic system are mathematically equivalent. (If the two levels are not isolated (as might occur in electric dipole atomic transitions), additional terms arising from interactions involving states outside the two-state subspace can contribute to the shift. Such terms

(e.g., light shifts) can contribute an amount that is comparable with the Bloch-Siegert shifts calculated above.)

Thus the GDAA also provides predictions for the positions of Bloch-Siegert shifts through Eqs. (23), (21), or (19) when the substitutions (A11) are used. The GDAA agrees with exact calculations[16] to order $(\chi/\omega_0')^4$ for small field strengths and provides a very good approximation for the resonance positions over the entire range of Ω'/ω_0' as given by Table I with the substitutions $kv/\Delta \rightarrow \Omega'/\omega_0'$, $\chi/\Delta \rightarrow \chi/\omega_0'$.

In some cases, it is more convenient to vary ω_0' (i.e., by varying a magnetic field) than to vary the field frequency Ω', and one would like the resonance condition stated for ω_0'/Ω' as a function of χ/Ω'. One could get such a condition by inverting Eqs. (23). One finds that resonances occur at $\omega_0'/\Omega' = 2n-1$ for low field strengths and decrease monotonically to zero at a field strength $\chi/\Omega' = [(4n^2-1)/2]^{1/2}$. For higher field strengths the resonance disappears; from the GDAA picture, one says that the resonance condition cannot be met while from more conventional interpretations, one says that power broadening has washed out the resonance. We should point out that, for values $\chi/\Omega' > n/2$, the GDAA prediction for ω_0'/Ω' resonances obtained by inverting Eqs. (23) is not as accurate as the Ω'/ω_0' predictions in Table I, owing to the fact that small errors in determining Ω' from Eqs. (23) can lead to larger errors in determining the corresponding ω_0'/Ω'-vs-χ/Ω' curves. Still, the GDAA provides a relatively simple means for obtaining the Bloch-Siegert shift for ω_0'/Ω' accurate to fourth order in χ/Ω' for weak field strengths and reasonably accurate (10 to 20%) over the entire range of χ/Ω'.

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