PHYSICAL REVIEW A VOLUME 15, NUMBER 4 APRIL 1977

Energy distribution of fast test particles slowing down in plasmas. II. Time-dependent energy spectra in finite media*

A. A. Husseiny†
Nuclear Science and Engineering Division, Carnegie-Mellon University, Pittsburgh, Pennsylvania 15213

Z. A. Sabri
Chemical Engineering and Nuclear Engineering Department, Engineering Research Institute, Iowa State University, Ames, Iowa 50011

D. R. Harris‡
University of California, Los Alamos Scientific Laboratory, Los Alamos, New Mexico 87544
(Received 18 November 1974; revised manuscript received 15 October 1976)

The time-energy spectrum of fast particles released in background plasmas is evaluated for the cases of monoenergetic impulses and sources dispersed in time and energy, as well as for general source spectra. Distant encounters are assumed to contribute mainly to the energy degradation of the test particles in the slowing-down range in which ions are treated as fixed centers, and in which the probability of elastic scattering by electrons is insignificant compared to the probability of scattering by plasma ions. The results give a slowing-down time close to that evaluated by approaches that account for close encounters. Asymptotic solutions of the Boltzmann equation are also obtained, and the effects of the confinement time of the test particles on the time-energy spectrum are taken into account. Confinement times characteristic of open-ended devices, tokamaks, and Bohm-like plasmas are considered. Pitch-angle scattering is not considered here, and hence only a rough estimate of the parameters relevant to neoclassical processes is given. The average density and mean energy of the test particles are obtained in the slowing-down range for these three types of plasmas. Discrepancies resulting from the classical treatment of scattering as opposed to the quantum-mechanical treatment are pointed out. A multienergy approach is introduced for numerical calculations; such an approach is necessary in particular for model-dependent studies. The approach is rather flexible and can accommodate different types of spatial dependence and scattering processes.

I. INTRODUCTION

The space- and time-independent energy distribution of fast test particles slowing down in an infinite plasma background has been reported in a previous paper[1] which is here referred to as paper I. The purpose of the present work is to extend the earlier effort to evaluate the time-dependent spectrum and energy distribution of fast test particles slowing down in a finite plasma. The average values of the density and the energy of the test particles are also obtained. The results are of particular interest in studying the dynamics of neutral-beam heating experiments, the behavior of reaction particles released by fusion reactions, and the energetics of two-component thermonuclear plasmas.[2]

In the slowing-down range, the velocity of the test particles, V, is such that $(2kT_i/m_i)^{1/2} < V < (2kT_e/m_e)^{1/2}$, where T_i and T_e are the kinetic temperatures of plasma ions and electrons, respectively; m_i and m_e are the masses of the plasma ions and electrons, respectively; and k is the Boltzmann constant. The electron temperature is restricted to values for which the scattering probability of the test particles by electrons is negligible compared to the probability of scattering by plasma ions.[3,4] In this energy range, the time-dependent conservation equation of the test particles released in a single-ion-species plasma may be written as[5]

$$\frac{\partial N(E,t)}{\partial t} = -\Psi(E,t) + \int_E^{E/\alpha_1} dE' \frac{\Sigma(E'\to E)\Psi(E',t)}{[\Sigma_S(E')+\Sigma_F(E')]} - \frac{N(E,t)}{\tau_x(E)} + \Xi(E,t) \quad (1)$$

where $N(E,t)$ is the number density of test particles of species x per unit energy having an energy E at time t; $\Psi(E,t)$ is the encounter density per unit time; $\Sigma_S(E')$ and $\Sigma_F(E')$ are the macroscopic cross sections of elastic scattering and fusion reactions, respectively, which describe interactions involving the test particles and the plasma ions of species i; E' and E are the energies of the test particles before and after scattering; $\tau_x(E)$ is the confinement time of a test particle x at energy E; $\alpha_1 = [(M-m_i)/(M+m_i)]^2$; M is the mass of the test particle; $\Sigma(E'\to E)$ is the macroscopic removal cross section which describes the change in the energy of the test particle from E' to E due to elastic scattering events leading to transfer of energy to the plasma ions; and $\Xi(E,T)$ is the rate of release per unit volume per unit energy of test particles in the plasma background either by external means or by fusion reactions in thermonuclear plasmas. The functional dependence of the cross sections on T_i is not included in the notation for simplicity.

An approximate form of the energy transfer cross section $\Sigma(E'\to E)$ has been derived in paper I using a scattering model in which contributions of encounters at impact parameters $b<\lambda_i$ are neglected (λ_i is the Debye radius). This form is used here and is given by

$$\Sigma(E'\to E)\simeq\begin{cases}\Sigma_S(E')/\Delta, & \alpha_2E'\le E\le E'\\ 0, & \text{otherwise},\end{cases} \tag{2}$$

where $\Delta=\hbar^2/2m_i\lambda_i^2$ and $\alpha_2=1-\Delta/E'$. The macroscopic elastic cross section is explicitly given by[2]

$$\Sigma_S(E')=\xi\sqrt{M}/\Delta E'. \tag{3}$$

Here ξ is a parameter defined as

$$\xi=\pi\left(\frac{Zz_ie^2}{4\pi\epsilon_0}\right)^2\frac{n_i\sqrt{M}}{m_i}, \tag{4}$$

where Z and z_i are the charge numbers of the test particle and the plasma ion, respectively; n_i is the number density of plasma ions; and the rest of the notation is used in the standard manner. The dimensions are given in the rationalized mks units unless stated otherwise.

Equation (1) can be solved to give the time-dependent energy spectrum of fast test particles in different situations and for different types of sources. In Sec. II, the time-dependent energy spectrum is evaluated in the limit of $\tau_x(E)>\tau_s(E)$ for an impulse of monoenergetic test particles released in a finite plasma where $\tau_s(E)$ is the slowing-down time. Dispersion in the energy spectrum of the source of test particles is considered together with a spread in the impulse duration. The effect of the confinement time of the plasma is taken into account. The solutions are then generalized to include any form of time dependence of the source term $\Xi(E,t)$. To probe the detailed structure of the energy spectrum during the slowing-down process, asymptotic solutions of the spectrum are obtained in Sec. III for both of the cases of an impulse and of a steady-state source of test particles. The effects of diffusion of test particles are then taken into consideration in Sec. IV. Three types of confinement times are considered, namely, those of the open-ended, tokamak, and Bohm-like plasmas. The finiteness of the devices is incorporated in the calculations through the adoption of the empiric Bohm-diffusion coefficient and through the use of coefficients estimated from detailed spatial Fokker-Planck treatments. In Sec. V the average density and the mean energy of the test particles are determined using the results obtained in Sec. IV. Comparison between the results of the analysis of Secs. IV and V and the results available in the literature is made in Sec. VI. The use of energy multigroup analysis to evaluate the energy distribution is discussed in Sec. VII. Finally, summation of the results is given in the concluding section VIII.

II. DIFFERENT SOURCE SPECTRA [$\tau_x(E)\to\infty$]

If the rate of production of test particles exceeds the rate of their leakage from the plasma and if their mean time of slowing-down is less than $\tau_x(E)$, the medium can be treated as being infinite. In the limit of an infinite $\tau_x(E)$, Eq. (1) reduces to

$$\frac{\partial N(E,t)}{\partial t}=-\Psi(E,t)+\frac{1}{\Delta}\int_E^{E+\Delta}dE'\,\Psi(E',t)+\Xi(E,t), \tag{5}$$

where the fusion cross section is neglected and Eq. (2) is used to eliminate the energy-transfer cross section.

To reduce the integro-differential equation given by Eq. (5) to a partial differential equation, we may assume that the introduction of the time variable causes $\Psi(E,t)$ to deviate slightly from the steady-state encounter density $\Psi(E)$ and that the time-dependent encounter density $\Psi(E,t)$ does not appreciably change over an energy interval Δ; thus, $\Psi(E',t)$ may be expanded in a Taylor series about E, that is,

$$\Psi(E',t)=\Psi(E,t)+(E'-E)\left.\frac{\partial\Psi(E',t)}{\partial E'}\right|_{E'=E}+\cdots. \tag{6}$$

Retaining the first two terms only, substituting Eq. (6) into Eq. (5), and integrating, we get

$$\frac{\partial N(E,t)}{\partial t}\simeq\frac{\Delta}{2}\frac{\partial\Psi(E,t)}{\partial E}+\Xi(E,t). \tag{7}$$

This equation can be solved directly by Laplace transform methods for different time and energy dependences of the source term.

A. Impulse

A spatially homogeneous monoenergetic impulse of test particles may be represented by

$$\Xi(E,t)=\Xi_0\,\delta(E_0-E)\,\delta(t-t_0), \tag{8}$$

where Ξ_0 is the strength of the source, the energy distribution $\delta(E_0-E)$ is a δ function centered at the most probable initial energy E_0, and $\delta(t-t_0)$ is the shape of the impulse which is a δ function centered at some initial time t_0. Equation (8) can be used to explicitly give the source term of Eq. (7). The right side of this equation may be rewritten in terms of the encounter density, since $\Psi(E,t)=\Sigma_S(E)\,V(E)N(E,t)$, and since both $V(E)$ and $\Sigma_S(E)$ are time independent; therefore,

$$\frac{2}{\Delta\Sigma_S(E)\,V(E)}\frac{\partial\Psi(E,t)}{\partial t}-\frac{\partial\Psi(E,t)}{\partial E}\simeq\frac{2}{\Delta}\Xi_0\,\delta(E_0-E)\,\delta(t)\,, \quad (9)$$

where the initial time t_0 is taken as zero. By means of the Laplace transform in time of Eq. (9) the time-dependent energy distribution of the test particles can be readily obtained, that is,

$$\Psi(E,t)\simeq-\frac{2\Xi_0}{\Delta}\int_E^{E_0}dE'\,\delta(E_0-E')\times\delta\left(t-\frac{(2M)^{1/2}}{\Delta}\int_E^{E'}\frac{dE''}{\Sigma_S(E'')E''^{1/2}}\right),\quad t<\tau_I$$

$$\simeq 0,\quad t>\tau_I \quad (10)$$

where τ_I is the integral term in the time δ function. Thus, the number density per unit energy per unit time of test particles at energy E at time t is

$$N(E,t)=(\Xi_0/\xi\sqrt{2})\,E^{1/2}\,\delta[t-\tau_s(E)]\,, \quad (11)$$

where τ_s is the slowing-down time given by

$$\tau_s(E)=(\sqrt{2}/3\xi)\,(E_0^{3/2}-E^{3/2})\,, \quad (12)$$

which is the time required for the energy of a test particle to decrease from an initial value E_0 to an energy E. The value of τ_s obtained in Eq. (12) is an approximate value of more accurate expression obtained earlier by Husseiny and Forsen.[6] The above derivation, which is similar to Fermi's age theory of spatial neutron slowing-down,[5] represents a simple and straightforward tool to calculate directly the mean slowing-down time without a prior knowledge of the time-dependent spectrum of the test particles. Comparison of the results of Eqs. (11) and (12) with the average test-particle density N, which has been obtained in paper I for the asymptotic time-independent case, that is,

$$N=\frac{7.396\times10^{21}\Xi_0 A_i}{n_i Z^2 z_i^2\sqrt{A}}(E_h^{3/2}-E_c^{3/2}),\ \ \mathrm{m}^{-3}, \quad (13)$$

shows that $\tau_s(E_c)\simeq N/\Xi_0$, which gives the mean time for a test particle to slow down from an energy E_h to an energy E_c. Here, E_h is the energy upper limit below which the asymptotic solution of Eq. (13) is valid; E_c is the lower cut-off energy below which the slowing-down model is invalid; and A and A_i are the mass numbers of the test particles and plasma ions, respectively. The value of E_h is very close to E_0, and E_c is usually taken as $2kT_i$.[1]

The time-dependent energy distribution given by Eq. (11) indicates that an initial impulse of test particles remains as a line spectrum about the slowing-down time, that is, the spectrum maintains its initial shape and merely moves down the energy scale and each value of energy is uniquely correlated with a particular time. However, Eq. (11) does not give much information about the structure of the spectrum.

B. Dispersed energy source

The initial energy of the source may have a spread in energy as that given by the Gaussian source considered in paper I. In addition, the time dependence may be in the form of a dispersed δ function centered at some initial time t_0, that is,

$$\Xi(E,t)=\frac{\Xi_0}{\pi t_1 U}\exp\left[-\left(\frac{t-t_0}{t_1}\right)^2-\left(\frac{E-E_0}{U}\right)^2\right] \quad (14)$$

where the widths U and t_1 of the energy and time spectra, respectively, are to be determined from the source characteristics. In the case of reaction particles produced by fusion in thermonuclear plasmas, $U=2kT_i$ and $t_1=2\tau$ where τ is the plasma confinement time. Taking $t_0=0$ and replacing the source term in Eq. (7) by the expression given in Eq. (14), the time-dependent conservation equation of reaction particles takes the form

$$\frac{2}{\Delta\Sigma_S(E)\,V(E)}\frac{\partial\Psi(E,t)}{\partial t}-\frac{\partial\Psi(E,t)}{\partial E}\simeq\frac{\Xi_0}{2\Delta\pi kT_i\tau}\exp\left[-\left(\frac{t}{2\tau}\right)^2-\left(\frac{E-E_0}{2kT_i}\right)^2\right]. \quad (15)$$

This differential equation can be solved by Laplace transform in time to give

$$N(E,t)\simeq-\frac{\Xi_0E^{1/2}}{4\sqrt{2\pi}\,\tau KT_i\xi}\int_E^\infty dE'\exp\left[-\left(\frac{E_0-E'}{2kT_i}\right)^2\right]\exp\left(-\frac{[t-(\sqrt{2}/3\xi)(E'^{3/2}-E^{3/2})]^2}{4\tau^2}\right),\quad t>(\sqrt{2}/3\xi)(E'^{3/2}-E^{3/2})$$

$$\simeq 0,\quad t<(\sqrt{2}/3\xi)(E'^{3/2}-E^{3/2})\,. \quad (16)$$

The first exponential function in Eq. (16) peaks at $E'=E_0$ and then drops very fast for $E'\gtrless E_0$. The argument of the second function assumes a value of $(t/2\tau)^2$ at the lower limit of the integral, and it changes toward the upper limit, passing by the value $[(t-\tau_s)/2\tau]^2$ at $E'=E_0$. Since the main practical interest is in a time scale of the order of τ_s and since on this time scale the main contribution to the integral comes from the first exponential function while the second exponential function can be taken outside the integral evaluated at $E'=E_0$, the number of test particles having an energy between E and $E+dE$ at a time between t and $t+dt$ is given for $t\sim\tau_s$ as

$$N(E,t)\simeq\frac{\Xi_0E^{1/2}}{4\sqrt{2}\pi\tau\xi}\left[1-\operatorname{erf}\left(\frac{E-E_0}{2kT_i}\right)\right]\times\exp\left[-\left(\frac{t-\tau_s}{2\tau}\right)^2\right]. \tag{17}$$

This result shows that the spectrum remains essentially a Gaussian centered about $t=\tau_s$ and merely moves down the energy scale as in the case of the impulse.

After a long period of time, that is, for $t-\tau_s\gg2\tau$, Eq. (16) can be evaluated at the limit $\tau\to0$; thus,

$$N(E,t)=\frac{\Xi_0E^{1/2}}{2\xi\sqrt{2\pi}kT_i}\times\exp\left[-\left(\frac{E_0-[E^{3/2}+(3/\sqrt{2})\xi t]^{2/3}}{2kT_i}\right)^2\right]. \tag{18}$$

The time dependence preserves the shape of the energy distribution but causes a shift in the peak from $E=E_0$ to $E=E_0(1-3\xi t/\sqrt{2}E_0^{3/2})^{2/3}$; that is, the location of the peak is shifted to lower energy. The shift is approximately equal to $(1-t/\tau_s)^{2/3}$ for $E_c\ll E_0$; hence, at a time $t=\tau_s$, the peak reaches $E\simeq E_0$.

C. General source spectrum

The solution of Eq. (10) is the Green's function of Eq. (7) for arbitrary source spectrum. Thus, Eq. (10) may be rewritten in the general form,

$$N(E,t)=\frac{\sqrt{2}\,\Xi_0}{\xi}E^{1/2}\int_E^{E_0}dE'f(E')F(t-\tau_I),\qquad t>\tau_I, \tag{19}$$

where the shape functions $f(E')$ and $F(t-\tau_I)$ give the energy and the time dependence, respectively, of the source spectrum. If $f(E')$ is represented by $\delta(E_0-E')$, then Eq. (19) reduces to

$$N(E,t)=(\Xi_0/\sqrt{2}\xi)E^{1/2}F(t-\tau_s), \tag{20}$$

in which the shape function $F(t-\tau_s)$ can assume any form of time dependence and τ_s is given by Eq. (12).

For a steady-state rate of production of test particles, that is

$$\Xi(E,t)=\Xi_0\delta(E_0-E), \tag{21}$$

and $F(t-t_0)=1$, the distribution given by Eq. (20) reduces to the solution obtained asymptotically in paper I for the time-independent Boltzmann equation. This supports the argument presented in paper I that the asymptotic solutions may be extended to $E=E_0$ without a significant error.

Using the technique described above in solving the time-dependent Boltzmann equation, the details of the shape of the spectrum cannot be determined unless several terms in the Taylor expansion of Eq. (6) are used. However, even if the third term is included in the analysis, the problem becomes too complicated for analytical evaluation. However, more information about the shape of the time-dependent spectrum of test particles can be obtained by evaluating the asymptotic shape of such spectrum.

III. ASYMPTOTIC TIME-ENERGY SPECTRUM

If we consider a source term of the form given by either Eq. (8) or Eq. (21) and assume that the test particles are perfectly confined for a time, $\tau_x=\tau_c$, and that $E<E_0$ and $t_0\leqslant t_c$, Eq. (7) reduces to

$$\frac{2M}{\Delta\Sigma_S(V)}\frac{\partial VN(V,t)}{\partial t}\simeq\frac{\partial VN(V,t)}{\partial V}-3N(V,t), \tag{22}$$

in which the variable E is replaced by the variable V. This partial differential equation may be transformed to a first-order ordinary differential equation by using the method of combination of variables. Thus, by introducing a dimensionless variable θ, defined as

$$\theta=\frac{\Delta\Sigma_S(V)\,t}{2MV}=\xi_1\frac{t}{V^3}, \tag{23}$$

where $\xi_1=\xi/M^{3/2}$, Eq. (22) becomes

$$(1+3\theta)\frac{d\Phi(\theta)}{d\theta}+3\Phi(\theta)=0, \tag{24}$$

where $\Phi(\theta)$ is a dimensionless flux given by

$$\Phi(\theta)=\frac{VN(V,t)}{V_cN(V_c,\tau_c)}. \tag{25}$$

Here, the subscript c designates parameters evaluated at $t=\tau_c$. Equation (24) can be easily solved using the boundary condition $\Phi(\theta_c)=1$ to give

$$\Phi(\theta)=(1+3\theta_c)/(1+3\theta). \tag{26}$$

Defining the νth moment of the dimensionless flux as

$$\Pi_\nu = \int_0^{\theta_c} d\theta\, \theta^\nu \Phi(\theta)\,, \tag{27}$$

the zeroth moment may be evaluated. Thus,

$$\Pi_0 = \tfrac{1}{3}(1+3\theta_c)\ln(1+3\theta_c)\,. \tag{28}$$

However, for a specific value of V, Eq. (28) may be written explicitly as

$$\Pi_0 = \frac{\Delta\Sigma_S(V)}{2MV_cN(V_c,\tau_c)}\int_0^{\tau_c} dt N(V,t)\,, \tag{29}$$

where the value of the integral is equal to the time-independent asymptotic velocity distribution obtained in paper I and, consequently,

$$\Pi_0 = \frac{1}{V_cN(V_c,\tau_c)}\,. \tag{30}$$

Using the definition of the dimensionless flux given by Eq. (25) and the values of the zeroth moment of Eqs. (28) and (30),

$$N(V,t) \simeq 3[V(1+3\xi_1 t/V^3)\ln(1+3\xi_1\tau_c/V^3)]^{-1}\,. \tag{31}$$

where τ_c is equal to the slowing-down time $\tau_s(E_c)$ required for the test particles energy to change from E_h to E_c. The details of the dependence of the spectrum on both t and V are well displayed in Eq. (31).

As an example let us consider the case of alphas

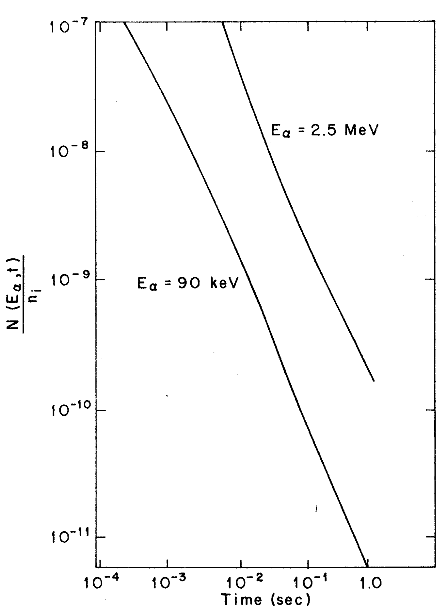

FIG. 1. α-particle spectrum at different energies in a nonlossy finite system.

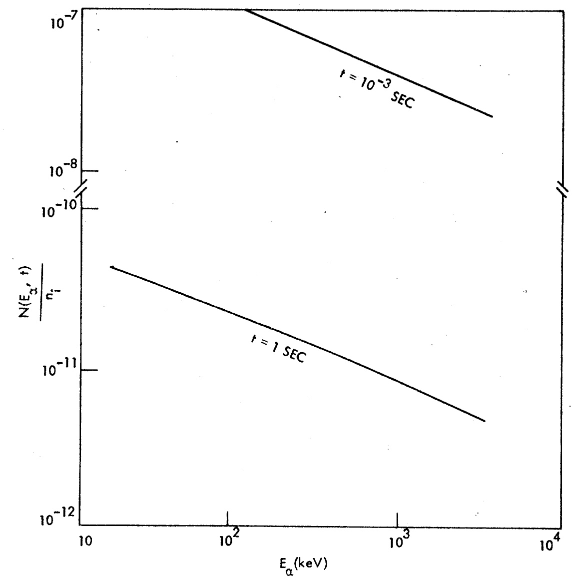

FIG. 2. Energy distribution at different times.

produced at $E_0 = 3.52$ MeV from fusion reactions in a plasma composed of equal portions of tritons and deuterons at $T_i = 10$ keV and plasma ion density of 10^{21} m^{-3}. The α spectrum is shown in Fig. 1 as function of time at two different energies. The alpha distribution is also plotted versus E_α at two different times in Fig. 2.

IV. EFFECT OF DIFFUSION OF TEST PARTICLES

Fast test particles released in a background plasma are likely to continuously diffuse out of the confinement. Since the confinement time of the test particles $\tau_x(E)$ generally depends on their energy E and on their type, as well as on the parameters of the background plasma, the value of $\tau_x(E)$ may be comparable to or less than the slowing-down time. In this case, the results of Secs. II and III are not valid, and the possibility of the leakage of the test particles throughout the slowing-down process needs to be taken into account. To do that, the leakage term given in Eq. (1) must be included. For a source of nonfussile test particles of the form given by Eq. (8) or Eq. (21), at energies $E \leq E_h < E_0$ and at time $t > t_0$, Eq. (1) reduces to

$$\frac{2M}{\Delta\Sigma_S(V)}\frac{\partial VN(V,t)}{\partial t} = \frac{\partial VN(V,t)}{\partial V} - 3[1+\kappa(V)]N(V,t) \tag{32}$$

where the parameter $\kappa(V)$ is defined as

$$\kappa(V) = V^3/[3\xi_1\tau_x(V)]\,. \tag{33}$$

A comparison between Eqs. (33) and (12) shows that $\kappa(V) \simeq \tau_s(V)/\tau_x(V)$. Equation (32) is similar to Eq. (22) save the presence of the additional term, including the parameter $\kappa(V)$. This term can be neglected in situations wherein $\tau_x(V) > \tau_s(V)$ or when $\tau_x(V) \to \infty$. For $\tau_x(V) \lesssim \tau_s(V)$, the solution of Eq. (32) is determined by the functional dependence of $\tau_x(V)$ on V. Thus, we may consider confinement times typical of open-ended and tokamak devices as well as plasma confinement systems in which the test particles follow the Bohm diffusion regime.

A. Classical diffusion

Plasma diffusion in open-ended devices is characterized by a classical confinement time which is proportional to the mean deflection time $\tau_D(V)$. This is defined as the mean time required for a particle to be deflected by 90° due to multiple Coulomb collisions. The constant of proportionality, $\psi = \tau_D(V)/\tau_x(V)$, depends on the particular geometry of the device and on the intensity of the confining magnetic field.

In the range of energy of interest here,[8]

$$\tau_D(V) = \frac{n_i M}{8\xi_1 m_i n_e} \frac{V^3}{\ln\Lambda(V)}, \tag{34}$$

where n_e is the plasma electron number density, the function $\Lambda(V)$ is given in this particular case by

$$\Lambda(V) = \mu\lambda_i V/\hbar, \tag{35}$$

and μ is the reduced mass of a test particle and a plasma ion. Hence, Eq. (33) may be explicitly written as

$$\kappa(V) = \frac{8 n_e m_i \psi}{3 M n_i} \ln\Lambda(V). \tag{36}$$

Since the logarithmic function is not sensitve to moderate changes in V, it is often taken as the Coulomb logarithm for which the argument

$$\Lambda = \Lambda(T_e) = 12\pi \left(\frac{\epsilon_0 k T_e}{e^2 n_e}\right)^{3/2} n_e \tag{37}$$

is used. In this case, $\ln\Lambda$ and hence κ are independent of V.

In the case of tokamak devices for which the neoclassical theory is applicable, the dependence of $\tau_x(V)$ on V and $\Lambda(V)$ is the same as in the case of open-ended devices; however, the constant of proportionality is different due to the presence of pitch-angle scattering processes. Thus, the confinement time of the test particle may be expressed in a general form which is applicable for devices characterized by the loss-cone or classical diffusion, and which would provide a rough estimate only for tokamak devices; that is,

$$\tau_x(V) = \upsilon V^3/\ln\Lambda(V), \tag{38}$$

where υ is a constant that depends on the type of plasma geometry. This confinement time may be used also as an estimate of neoclassical effects. Thus, Eq. (33) may take the form

$$\kappa(V) = (1/3\upsilon\xi_1)\ln\Lambda(V), \tag{39}$$

which reduces to the constant

$$\kappa(T_e) = \ln\Lambda(T_e)/(3\upsilon\xi_1) = \kappa_0 \tag{40}$$

when the argument of the logarithm is approximated by the Coulomb value given in Eq. (37).

Let us first consider the case when $\kappa(T_e) = \kappa_0$ is used and apply the same method used in solving Eq. (22). Thus, the dimensionless flux takes the form

$$\Phi(\theta) = \left(\frac{1 + 3\theta_c}{1 + 3\theta}\right)^{\kappa_0 + 1}, \tag{41}$$

where θ and $\Phi(\theta)$ are defined by Eqs. (23) and (25), respectively. The time velocity distribution is similarly obtained and is given by

$$N(V, t) = \frac{3\kappa_0}{V(1+3\theta)^{\kappa_0+1}[1 - 1/(1+3\theta_c)^{\kappa_0}]} \tag{42}$$

which reduces in the limit of $\kappa_0 \to 0$ to Eq. (31) for $\tau_x(V) > \tau_s(V)$.

In order to take into consideration the dependence of the logarithmic function on E, let us rewrite Eq. (1) for nonfussile test particles and use the expansion of Eq. (6) and the source term given by Eq. (8) with $t_0 = 0$; thus,

$$\frac{\sqrt{2E}}{\xi} \frac{\partial\psi(E,t)}{\partial E} = \frac{\partial\psi(E,t)}{\partial E} - \frac{\sqrt{2E}}{\xi\tau_x(E)}\psi(E,t) + \frac{2}{\Delta}\Xi_0\,\delta(E_0 - E)\,\delta(t) \tag{43}$$

which can be solved by Laplace transform. The result is

$$\psi(E,t) = \frac{\Xi_0}{\Delta}\delta(t - \tau_s)\exp\left(-\frac{\sqrt{2}}{\xi}\int_E^{E_0} \frac{dE'\sqrt{E'}}{\tau_x(E')}\right), \tag{44}$$

where τ_s is given by Eq. (12). Thus, the leakage of the test particles does not affect the slowing-down time. Substituting for $\tau_x(E)$ from Eq. (38), carrying out the integration in the argument of the exponential function, and then integrating the result over time,

$$N(E) = \frac{1}{\sqrt{2\xi}}\Xi_0 E^{1/2} \exp\left(-\frac{1}{\upsilon\xi_1}[\ln^2\Lambda(E_0) - \ln^2\Lambda(E)]\right). \tag{45}$$

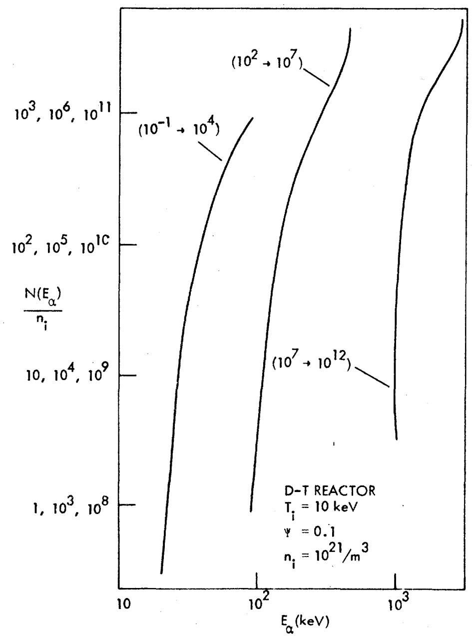

FIG. 3. Energy distribution of alphas in a D-T plasma in an open-ended device $\psi = 0.1$.

The exponent is not sensitive to slight changes in energy; however, over the energy range $E_c \leq E \leq E_0$ the energy distribution drops with a decrease in energy which is faster than that for the case when the confinement of test particles is long. For the D-T plasma example given in Sec. III, the α particle's energy distribution is plotted versus E_α in Fig. 3. The plasma is assumed to be confined in an open-ended device, and a value of $\psi = 0.1$ is selected.

B. Bohm plasmas

For plasma devices characterized by Bohm diffusion, the confinement time of the test particles may be expressed as $\tau_x = l\tau_B$, where τ_B is the Bohm confinement time and l is a numerical factor that depends on the type of device. Since both τ_B and τ_x are independent of the energy of the test particle, the exponent in Eq. (44) can be easily integrated to give $N(E,t)$. Integrating the result over time we get

$$N(E) = \frac{1}{\sqrt{2}\xi}\Xi_0 E^{1/2} \exp\left(-\frac{2\sqrt{2}}{3\tau_B \xi l}(E_0^{3/2} - E^{3/2})\right). \tag{46}$$

This energy distribution is more sensitive to changes in E than that obtained for classical and neoclassical plasma diffusion, Eq. (45). For the example given above for reaction α particles slowing down in a D-T plasma, Fig. 4 shows the

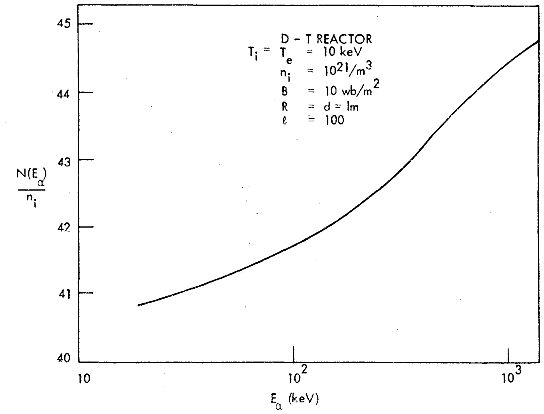

FIG. 4. Alpha distribution in a Bohm-like D-T plasma.

energy spectrum of the alphas for $l = 100$ and $\tau_B = 0.749$ sec.

The solution obtained in Eq. (46) is also valid for other plasma devices for which the test-particle's confinement time does not depend on the velocity of the test particles. In this case, the ratio between the actual confinement time and τ_B gives the value of l.

V. AVERAGE DENSITY AND MEAN ENERGY

The average number density of test particles in the slowing-down range can be directly obtained by integrating the energy distribution over energy. For plasmas in which the test particles diffuse in the plasma according to the classical or neoclassical theory, the average number density of test particles is obtained from Eq. (45); thus, in the range $2kT_i \leq E \leq E_0$,

$$N = \frac{(\xi_1 \upsilon)^{1/2}}{\xi}\Xi_0 \times \left[E_0^{3/2}\mathfrak{D}(Y_{01}) - (2kT_i)^{3/2}\mathfrak{D}(Y_{i1}) \times \exp\left(-\frac{1}{2\upsilon\xi_1}[\ln^2\Lambda(E_0) - \ln^2\Lambda(2kT_i)]\right)\right] \tag{47}$$

where

$$Y_{01} = \frac{1}{(2\upsilon\xi_1)^{1/2}}[\ln\Lambda(E_0) + 3\xi_1\upsilon], \tag{48}$$

$$Y_{i1} = \frac{1}{(2\upsilon\xi_1)^{1/2}}[\ln\Lambda(2kT_i) + 3\xi_1\upsilon], \tag{49}$$

and $\mathfrak{D}(Y)$ is the Dawson integral. Tables of $\mathfrak{D}(Y)$ for various values of Y are available in Jahnke and Emde.[10] For $E_0 \gg 2kT_i$ and in cases of practical interest, the second term can be neglected since the exponential function is very small and the Dawson integral in this term is less than 1. For the D-T plasma example, the exponential function in the second term is of the order of 10^{-200}, and the Dawson integral in the same term is 0.025; hence, neglecting this term is justifiable. If the plasma is contained in an open-ended device of $\psi \simeq 0.1$, the average number of α particles is $n_\alpha = 7.5\times10^{18}$ m^{-3}.

In the energy range of thermonuclear interest, the asymptotic value of the Dawson integral may be used; that is, $\mathfrak{D}(Y) \simeq 0.5/Y$, and consequently Eq. (47) reduces to

$$N \simeq \frac{\sqrt{2}\,\Xi_0 E_0^{3/2}}{6\xi\,[1+\ln\Lambda(E_0)/3\xi_1\upsilon]}\,. \qquad (50)$$

The mean energy of test particles in the range $2kT_i \leq E \leq E_0$ is

$$\bar{E} = \frac{m_i m\Delta}{\mu^2}\exp\left(-\frac{1}{4\xi_1\upsilon}\right) \times\left(\frac{\mathfrak{D}(Y_{02})\,e^{Y_{02}^2} - \mathfrak{D}(Y_{i2})\,e^{Y_{i2}^2}}{\mathfrak{D}(Y_{01})\,e^{Y_{01}^2} - \mathfrak{D}(Y_{i1})\,e^{Y_{i1}^2}}\right), \qquad (51)$$

where

$$Y_{02} = \frac{1}{(2\upsilon\xi_1)^{1/2}}[\ln\Lambda(2kT_i) + \tfrac{5}{6}\xi_1\upsilon]\,, \qquad (52)$$

$$Y_{i2} = \frac{1}{(2\upsilon\xi_1)^{1/2}}[\ln\Lambda(2kT_i) + \tfrac{5}{6}\xi_i\upsilon]\,. \qquad (53)$$

Applying the same approximation used to obtain Eq. (50) from Eq. (47), the average energy of the test particles given by Eq. (51) may be reduced to

$$\bar{E} \simeq E_0 Y_{01}/Y_{02} \qquad (54)$$

and for the example given above for reaction alphas, $\bar{E}_\alpha \simeq 0.9E_0$. This may be compared to $\bar{E}_0 \simeq 0.6E_0$ which has been obtained in paper I for $\tau_s^{(\alpha)}(E) < \tau_\alpha(E)$ or for $\tau_\alpha(E) \to \infty$; where $\tau_s^{(\alpha)}(E)$ and $\tau_\alpha(E)$ are the slowing-down time and the confinement time of the reaction α particles, respectively. Thus, the diffusion of the test particles

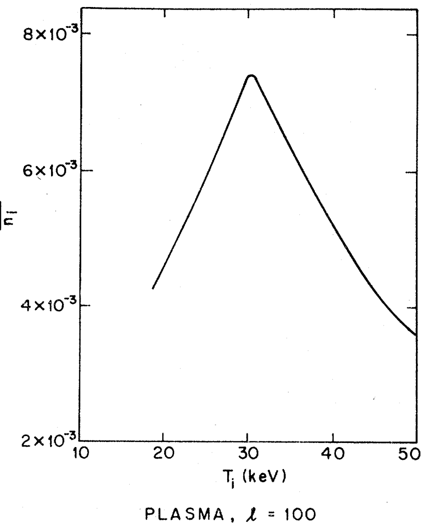

FIG. 5. Ratio of the mean density of α particles to plasma ions in a Bohm-like plasma, $l = 100$.

from the confinement results in shifting the mean energy to higher energies.

In the case of Bohm-like plasmas, the average number density of test particles can be obtained from Eq. (46); thus,

$$N = l\tau_B\,\Xi_0\left\{1 - \exp\left(-\frac{2\sqrt{2}}{3l\tau_B\xi}[E_0^{3/2} - (2kT_i)^{3/2}]\right)\right\}. \qquad (55)$$

For values of $l \geq 1$ the exponent is a large number, and the term between braces is close to unity. For the D-T example discussed above, $n_\alpha \simeq 1.9\times10^{18}\,l$ m^{-3}, which is about $0.25l$ times the value obtained for the open-ended device. Figure 5 shows the behavior of the average α-particle density as function of T_i for $l = 100$.

The average energy in the slowing-down range of test particles released in Bohm-like plasmas is obtained from Eq. (46) and is given by

$$\bar{E} \simeq \frac{\sqrt{2}E_0^{3/2}\exp(-2\sqrt{2}\,E_0^{3/2}/3l\tau_B\xi)}{l\tau_B\xi\,(1-\exp\{-2\sqrt{2}\,[E_0^{3/2}-(2kT_i)^{3/2}]/3l\tau_B\xi\})}\sum_{j=0}^{\infty}\frac{(2\sqrt{2}E_0^{3/2}/3l\tau_B\xi)^j - (2kT_i/E_0)^{5/2}[(4kT_i)^{3/2}/3l\tau_B\xi]^j}{j!(\frac{5}{2}+\frac{3}{2}j)}\,. \qquad (56)$$

The value of $\bar{E}$ is very close to E_0 since the Bohm diffusion causes a significant hardening in the spectrum compared to the cases of classical and neoclassical diffusion.

VI. COMPARISON WITH PREVIOUS WORK

Considering the case of test particles with a confinement time $\tau_x(E) \lesssim \tau_s(E)$, the results of Secs. IV and V may be compared with the analytical results which have been reported thus far. Using a deductive approach, Rose and Clark[7] have obtained an energy distribution similar in form to the time integral of Eq. (44). Since they used the Coulomb logarithm instead of $\ln\Lambda(E)$ in their analysis, the same is done in this section using the results of our work in order to provide a common ground for comparison. Doing so, Eq. (44) can be used to provide the solution for $N(E,t)$ near the source energy; that is,

$$N(E,t)=\frac{\sqrt{2}\,\Xi_0}{\xi E_0^{3\kappa_0/2}}\,\delta(t-\tau_s)\,E^{3\kappa_0/2+1/2}. \qquad (57)$$

For reaction α particles released from D-T fusion reactions, Eq. (57) is integrated over time; thus,

$$N(E)=\frac{\sqrt{2}\,\pi\epsilon_0^2 m_i\, n_i\,\langle\sigma v\rangle_{\rm DT}}{e^4\sqrt{M_\alpha}\,E_{\alpha 0}^{3\kappa_0/2}}\,E_\alpha^{3\kappa_0/2+1/2}, \qquad (58)$$

where $\langle\sigma v\rangle_{\rm DT}$ is the product of the D-T fusion cross section times the relative velocity averaged over the velocity distributions of the interacting ion species. Equation (58) is of a form similar to the reaction α-particle distribution which has been estimated for a steady-state mirror device by Rose and Clark,[7] where they take into account the effects of α thermalization via scattering by plasma electrons. Their results differ from that of Eq. (58) due to the difference in the values of the elastic scattering cross section which are used here. While they have treated the scattering classically, the results given here are based on quantum-mechanical considerations. The necessity to treat the scattering probabilities quantum mechanically has been discussed by Husseiny and Forsen[3] and Husseiny and Sabri.[4] The difference between the two approaches has led to different values of κ_0 and of the constants appearing in Eq. (58).

Actually, the use of the Coulomb logarithm as opposed to the use of $\ln\Lambda(E)$, as with the argument given by Eq. (35), leads to a disagreement between the result of Ref. 7 and the result of Eq. (45) with respect to the dependence of $N(E)$ and E. In addition, the use of the Coulomb logarithm leads to underestimation of the average population of the test particles in the slowing-down range as demonstrated below. Integrating Eq. (58) over the energy range $2kT_i \le E \le E_0$ gives

$$N_\alpha \simeq \frac{2\sqrt{2}\,\pi\epsilon_0^2 m_i\, n_i\,\langle\sigma v\rangle_{\rm DT}}{3e^4\sqrt{M_\alpha}\,(1+\kappa_0)}\,E_0^{3/2} \qquad (59)$$

where the approximation

$$E_0^{3(\kappa_0+1)/2}-(2kT_i)^{3(\kappa_0+1)/2}\simeq E_0^{3(\kappa_0+1)/2}$$

is used. For devices of practical interest κ_0 is a small number since $\psi \lesssim 0.1$. Using the approximation $\kappa=0$ is equivalent to assuming that the α particles degrade to very low energy before escaping; that is, $\tau_\alpha(E) \gg \tau_s^{(\alpha)}(E)$. In this case, Eq. (59) must give the value of N_α for $\tau_\alpha(E)\to\infty$; however, using the parameters given in the example discussed above for D-T plasmas, $N_\alpha \simeq 2.65\times10^{12}$ $\rm m^{-3}$, which is orders of magnitude lower than the values obtained for $\tau_\alpha < \tau_s^{(\alpha)}$ as well as for $\tau_\alpha \to \infty$.

Although no analytical results have been obtained in Ref. 7 for Bohm-type plasmas, we may apply the technique which has been used in Ref. 7 for mirror machine to derive an expression for $N_\alpha(E)$ in the case of Bohm-like plasmas. Thus, the number of reaction alphas in a D-T Bohm-like plasma with energies between E and $E+dE$ is

$$N_\alpha(E)\,dE$$

$$=\frac{1.25\times10^{17} n_i\,\langle\sigma v\rangle_{\rm DT} T_e^{3/2} E_\alpha^{1/2}\,(E_\alpha^{3/2}+296T_e^{3/2})^{K-1}}{(E_0^{3/2}+296T_e^{3/2})^K}\,dE \qquad (60)$$

where

$$K=10^{18}\,T_e^{3/2}/(3n_i l\tau_B)\,. \qquad (61)$$

and T_e and E_α are in keV. The average density is obtained from Eq. (60) and is given by

$$N_\alpha = 4\times10^{-6} n_i^2 l\tau_B\,\langle\sigma v\rangle_{\rm DT} \times\left(1-\frac{1}{1+(E_0^{3/2}/296T_e^{3/2})^K}\right). \qquad (62)$$

For the plasma parameters discussed above, $N_\alpha = 3\times10^{13} l$ $\rm m^{-3}$, which also underestimates the α concentration as was the case for the mirror device.

Although the results reported in Ref. 7 have been roughly estimated, they cover a wider range of energy than the slowing-down range considered here. In addition, encounters of impact parameters smaller than λ_i are included while encounters at larger impact parameters are neglected, which is exactly opposite to the model considered here in Eq. (2). Contributions of scattering events at small impact parameters to the distribution of test particles are less than contributions of events at large impact parameters as we have shown in paper I. Consequently, excluding scattering events at large impact parameters would lead to the underestimation of the average number of the test particles which is found here.

VII. MULTIGROUP CONSIDERATIONS

In the preceding analysis the energy is treated as a continuous variable and the energy scale is divided into three groups, namely, a group near the source energy, an asymptotic slowing-down range, and the thermalization range. So far, the last energy range is not considered. Division of the energy scale into more groups is expected to yield more accurate results. In the multigroup approach test particles are treated as if they diffuse monoenergetically in each energy group and then get transported from one group to another by elastic and nonelastic scattering. This method enables us to consider inhomogeneities in the plasma core and nonelastic scattering, which plays an important role, especially in the presence of neutrals and impurities. The number of energy groups can be selected according to the desired degree of accuracy.

Let us divide the energy scale into L groups with widths ζ, which is generally a variable, and define the flux of the test particles in the jth group, ϕ_j, as

$$\phi_j = \int_{E_j}^{E_{j-1}} dE'\, \phi(E') \tag{63}$$

where $\phi(E) = V(E)N(E)$ and E_{j-1} and E_j are the lower energy boundaries of the $(j-1)$th and jth groups, respectively. Integrating Eq. (1) over the jth group, assuming a steady-state homogeneous plasma and rearranging the result,

$$\int_{E_j}^{E_{j-1}} \frac{dE\, \phi(E)}{V(E)\tau_x(E)} + \int_{E_j}^{E_{j-1}} dE\, \Sigma(E)\, \phi(E) = \int_{E_j}^{E_{j-1}} dE\, \Xi(E) + \int_{E_j}^{E_{j-1}} dE \int_0^\infty dE'\, \Sigma(E' \to E)\, \phi(E') . \tag{64}$$

However, the conservation of the test particles in the jth energy group gives

$$\frac{\phi_j}{[V(E)\tau_x(E)]_j} + \Sigma_j \phi_j = \Xi_j + \sum_{\substack{i=1\\ i\neq j}}^{L} \Sigma(i \to j)\, \phi_j . \tag{65}$$

Comparison of these two equations yields the group parameters,

$$\Sigma_j = \frac{1}{\phi_j} \int_{E_j}^{E_{j-1}} dE\, \Sigma(E)\, \phi(E) , \tag{66}$$

$$\frac{1}{V(E)\tau_x(E)} = \frac{1}{\phi_j} \int_{E_j}^{E_{j-1}} dE\, \frac{\phi(E)}{V(E)\tau_x(E)} , \tag{67}$$

$$\Xi_j = \int_{E_j}^{E_{j-1}} dE\, \Xi(E) , \tag{68}$$

$$\Sigma(i \to j) = \frac{1}{\phi_i} \int_{E_j}^{E_{j-1}} dE \int_{E_i}^{E_{i-1}} dE'\, \Sigma(E' \to E)\, \phi(E') . \tag{69}$$

The total cross section $\Sigma(E)$ is the sum of the fusion cross section $\Sigma_F(E)$ and the scattering cross section $\Sigma_S(E)$, and Σ_j includes a fusion cross section Σ_{Fj} and a removal cross section Σ_{rj} which represents the probability to transfer a test particle from group j to any other group i and is given by

$$\Sigma_{rj} = \sum_{i\neq j} \Sigma(j \to i) . \tag{70}$$

The source term Ξ_j includes a contribution from external injection of test particles in group j, that is, $\Xi_{\text{ext},j}$, and a contribution from fusion reactions resulting in production of test particles in the same group, $\Xi_{F,j}$. However, the fusion source term is usually confined to one energy group, such as group 1; thus $\Xi_j = \Xi_{\text{ext},j}$ for $j \neq 1$, while for $j = 1$

$$\Xi_1 = \Xi_{\text{ext},1} + \frac{1}{2} \sum_{\upsilon,\upsilon'} \frac{n_\upsilon n_{\upsilon'} \langle \sigma v \rangle_{\upsilon\upsilon',1}}{(1+\delta_{\upsilon\upsilon'})} \tag{71}$$

where $\delta_{\upsilon\upsilon'}$ is the Kronecker delta and the second term represents the number of test particles of type x produced by fusion reactions between the plasma ions of species υ and υ'. If the reaction particles are produced in different energy groups due to different fusion events, a parameter $\chi^{j\to i}$ may be used to represent the fraction of reaction particles produced with energies in group i due to fusion in group j. In this case the source term for $j \neq 1$ includes a term similar to the second term in Eq. (71) multiplied by $\chi^{i\to j}$ and summed over all energy groups i.

Parameters similar to those defined by Eqs. (66) through (70) can be assigned to each of the energy groups. According to the model introduced in paper I, the probability of removal of a test particle from group j to the next group $j+1$ is much more than the probability that the particle will be removed to any further group; that is, the scattering will not lead to group skipping. This is particularly true if $\zeta \gg \Delta$; thus, it can be assumed that $\Sigma(j \to i) \simeq \Sigma(j \to j+1)$. In the slowing-down range, scattering can be assumed to lead only to energy loss. For the special case in which no external injection takes place and the test particles are monoenergetically produced from fusion reactions in group 1, the conservation equation for the jth group is simplified to the form

$$\frac{\phi_j}{[V(E)\tau_x(E)]_j} + \Sigma_{Fj}\phi_j + \Sigma(j \to j+1)\, \phi_j = \Sigma(j-1 \to j)\, \phi_{j-1} + \Xi\, \delta_{j1}, \quad j = 1, 2, \ldots, L . \tag{72}$$

Equation (72) can be solved for $j = 1$ to give

$$\phi_1 = \Xi_1 \{\Sigma_{F1} + \Sigma(1 \to 2) + 1/[V(E)\tau_x(E)]_1\}^{-1} , \tag{73}$$

and for $j = 2$

$$\phi_2 = \Sigma(1 \to 2)\, \phi_1 \{\Sigma_{F2} + \Sigma(2 \to 3) + 1/[V(E)\tau_x(E)]_2\}^{-1} . \tag{74}$$

Expressions similar to Eqs. (73) and (74) can be found for the fluxes of the other groups. Numerical calculations become necessary if L is taken to be a large number. However, these detailed numerical analyses are only necessary if the model of the plasma device is considered and if the spatial dependence of the plasma parameters is known.

VIII. CONCLUSIONS

In the scattering model adopted here only distant encounters are considered. The incremental loss in the energy of a test particle is assumed to be so small that the change in the distribution due to such energy loss can be treated as a small perturbation. This assumption is used to evaluate the time-energy spectrum for various shapes of the source spectra of the test particles. Assuming that the confinement time of the test particles exceeds their slowing-down time, the spectrum is obtained for a monoenergetic impulse of test particles and is given by Eq. (11). Comparison between this result and the time-independent energy distribution of the test particles in a Maxwellian infinite plasma which has been derived in paper I yielded an expression for the slowing-down time of the test particles. This is given by Eq. (12). The slowing-down time obtained here is found to be approximately equal to an expression which has been obtained earlier by Husseiny and Sabri[4] using a scattering model wherein both close and distant encounters have been included. Considering the spread in the initial energy and the duration of the source impulse, the Boltzmann equation is solved to get the time-energy spectrum of the test particles which is given by Eq. (16). Then, a generalized form of the spectrum is obtained in Eq. (20) for a source of test particles having arbitrary energy and time functional dependence.

The time dependence of the spectrum for different sources indicated that the spectrum maintains its initial shape, which becomes centered about the energy-dependent slowing-down time; that is, the initial source spectra move down the energy scale as the particles slow down. The detailed structure of the time dependence appeared in the asymptotic solution which is obtained below the source energy and is given by Eq. (31) for either a monoenergetic impulse or a steady-state production of monoenergetic test particles. The results are plotted in Figs. 1 and 2 for α particles slowing down in D-T plasmas.

Taking into account the possibility of test particles having confinement times smaller than their slowing-down times, the time-energy spectra are obtained for classical diffusion of test particles. The result is given by Eq. (42) for a confinement time that varies as the product of the cube of the velocity of the test particle, V^3, and the Coulomb logarithm. Equation (44) or (45) gives the time-energy spectrum for the case when the confinement time varies as $V^3 \ln\Lambda(V)$, where $\Lambda(V)$ is given by Eq. (35). The diffusion of the test particles causes the distribution to drop faster as the energy decreases. For test particles obeying the Bohm-diffusion law the energy distribution is given by Eq. (46). The slowing-down time is found to be the same regardless of the value of the test particles confinement time.

Values of the average density and mean energy of the test particles in the slowing-down range are given by Eq. (50) for the classical diffusion regime and by Eqs. (55) and (56) for both of the Bohm diffusion and the constant confinement time regimes. Equation (51) gives a rough estimate of the corresponding values in the case of toroidal-type confinement. The specific scattering phenomena characteristic of neoclassical processes are implicitly incorporated in a undetermined proportionality constant. To evaluate explicitly this constant pitch angle scattering has to be included as well as spatial dependence in the Boltzmann collision operator.

It is customary, to *a priori* evaluate diffusion parameters and then use the results to obtain the energy distribution including the slowing-down tail.[11] This is done here for both open-ended and toroidal systems wherein estimates of diffusion coefficients obtained by spatially dependent Fokker-Planck calculations are employed to calculate the proper confinement time, and then the Boltzmann equation is used to find the energy distribution. The results obtained in the limit of time and space independence can be estimated also from energy-transfer rates derived using Fokker-Planck formalism.[1,12,13] Both approaches has been employed in calculating the thermonuclear reaction rates taking into account slowing-down effects in the STEEP 4 code.[14] The quantum-mechanical effects can be incorporated in a Fokker-Planck collision operator yielding a quantum-mechanical form of the kinetic equation.[15] Nevertheless, the Boltzmann formulation provide a powerful tool for numerical calculations of interaction rates as well as analytic evaluation.

Taking $\ln\Lambda(E)$ as the Coulomb logarithm is found to lead to underestimation of both the mean energy and the average density of the test particles. This underestimation involved several orders of magnitude in a mirror confinement as demonstrated by comparing the results of this work with those estimated by Rose and Clark.[7] Such comparison also indicated that the classical treatment of scattering wherein distant encounters are excluded by the use of a cutoff, leads to values of the density and mean energy which are lower than those obtained by the quantum-mechanical treatment used in this work. This is in support of earlier findings.[1,3,4]

To obtain results more accurate than those given here, one needs to resort to the laborious multigroup approach for which numerical calculations are indispensible. However, the present results are satisfactory for the exploration of the dynamic behavior of fast test particles in a background plasma and for the evaluation of their average density and mean energy in the slowing-down range. Further work is necessary to account for plasma inhomogeneities and to evaluate the energy distribution, average density, and mean energy of the test particles in the transition range between the slowing-down and the thermalization ranges.

---

*Work supported in part by the Engineering Research Institute, Iowa State University.

†Present address: Chemical Engineering nnd Nuclear Engineering Department, Iowa State University, Ames, Ia. 50011.

‡Present address: Nuclear Engineering Dept., Rensselaer Polytechnic Institute, Troy, N. Y. 12181.

[1]A. A. Husseiny, Z. A. Sabri, and D. R. Harris, preceding paper, Phys. Rev. A 15, 1659 (1976).

[2]A. A. Husseiny and Z. A. Sabri, Trans. Am. Nucl. Soc. 17, 43 (1973).

[3]A. A. Husseiny and H. K. Forsen, Phys. Rev. A 2, 2019 (1970).

[4]A. A. Husseiny and Z. A. Sabri, Phys. Rev. A 10, 644 (1974).

[5]A. A. Husseiny, Ph.D. thesis (University of Wisconsin, Madison, 1970) (unpublished).

[6]A. A. Husseiny and H. K. Forsen, Phys. Rev. A 1, 1483 (1970).

[7]D. J. Rose and M. Clark, Jr., *Plasmas and Controlled Fusion* (MIT, Cambridge, 1961).

[8]L. Spitzer, Jr., *Physics of Fully Ionized Gases*, 2nd ed. (Wiley, New York, 1967).

[9]L. A. Artsimovich, Nucl. Fusion 12, 215 (1972).

[10]E. Jahnke and F. Emde, *Tables of Functions with Formulas and Curves* (Dover, New York, 1945).

[11]M. M. R. Williams, *Mathematical Methods in Particle Transport Theory* (Wiley, New York, 1971).

[12]A. A. Husseiny, Z. A. Sabri, and D. R. Harris, Trans. Am. Nucl. Soc. 20, 42 (1975).

[13]A. A. Husseiny, Z. A. Sabri, and D. R. Harris, Los Alamos Scientific Laboratory Report No. LA-5988, 1975 (unpublished).

[14]D. R. Harris, D. E. Dei, A. A. Husseiny, Z. A. Sabri, and G. M. Hale, Los Alamos Scientific Laboratory Report No. LA-6344-MS, 1976 (unpublished).

[15]H. DeWitt, in *Lectures in Theoretical Physics*, edited by W. E. Brittin (Gordon and Breach, New York, 1967), Vol. 9C, p. 627.