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Energy distribution of fast test particles slowing down in plasmas. I. Time-independent energy spectra in infinite medium*

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A model is developed to describe the scattering of fast test particles in an infinite background plasma in which conditions are such that the probability of elastic scattering by plasma ions much exceeds the probability of elastic scattering by plasma electrons. Encounters leading to large angles of scattering are excluded in calculating the scattering kernel and the slowing-down density. These functions are evaluated using the quantum-mechanical elastic scattering cross sections. The encounter density is obtained by solving the space- and time-independent Boltzmann equation in the presence and in the absence of fusion reactions between the test particles and the plasma ions. Asymptotic energy distributions in the slowing-down range below the source energy are obtained for both cases. The average density and mean energy of the test particles in the slowing-down range are also evaluated. The results are found for arbitrary sources of test particles, taking into account spread in the initial energy of the particles. Binary and multicomponent plasmas are considered.

I. INTRODUCTION

Evaluation of the energy distribution of fast test particles released in a background plasma is of interest in several areas; in particular, in neutral-beam injection modeling, in studying secondary fusion reactions, and in investigating the energetics of quasi-steady-state plasmas. A knowledge of the energy distribution of test particles is also necessary for calculating their average density, energy, kinetic pressure, and diffusion coefficients.

The effects on steady-state thermonuclear reactions of finite energy transfer between the charged primary fusion reaction products and the plasma particles have been considered by Jensen, Kofoed-Hansen, and Wandel.[1] In their analysis they have assumed that the reaction products are confined long enough to acquire Maxwellian distributions. The energy distribution of alphas produced by D-T fusion reactions in mirror machines has been evaluated numerically by Kuo-Petravic, Petravic, and Watson.[2] Adler and Dorning[3] have used Laplace transform methods and adopted scattering kernels obtained by Husseiny[4] and Husseiny and Forsen[5,6] to approximately evaluate the energy distribution of ions slowing down in a plasma. The spatial distribution of fast test ions resulting from tangential injection of a diffuse neutral beam into a tokamak has been discussed by Rome, Callen, and Clarke.[7] Other investigators have implicitly considered the energy distribution of test particles in the course of the numerical analysis of specific plasma problems[8,9]; however, no explicit data or expressions have been provided for such distribution.

In this work, the time-independent energy distribution of fast test particles in an infinite Maxwellian background plasma is derived. The average density and the average energy of the test particles are also evaluated. The results are valid for steady-state homogeneous plasmas at regions far from the boundaries when the leakage of the plasmas and the test particles can be neglected throughout the energy range of interest. The energy of the fast test particles and the kinetic plasma temperature are such that quantum-mechanical cross sections can be used.[5,6] The treatment is restricted to the slowing-down range; that is, the velocity of the test particle, V, is assumed to be larger than v_i, the velocity of plasma ions of species i. The electron kinetic temperature is such that the probability of scattering of the test particles by plasma ions is larger than the probability of scattering by electrons.[6,10]

In Sec. II a scattering model is developed, and the transport equation is formulated for a source of arbitrary energy dependence. The distribution of nonfussile[11] test particles is then evaluated in Sec. III, and asymptotic solutions far from the source are obtained for plasmas containing single ion species as well as multicomponent plasmas. The effects on the energy distribution of the initial energy spread of test particles are taken into account. The average density and energy of the test particles are also obtained. In Sec. IV, fussile[11] test particles are considered, and their energy distribution is found. The effect of secondary fusion reactions involving fussile test particles on the distribution is discussed. Concluding remarks and examination of the validity of the scattering model are then given in Sec. V.

II. THE SCATTERING MODEL

The number of test particles of type x that undergo elastic or nonelastic scattering with the particles of an l-component plasma background per unit volume per unit time per unit energy is given by the encounter density $\Psi_x(E)$, that is,

$$\Psi_x(E)=\sum_{\nu=1}^{l}\left[\Sigma_{Rx}(E,T_\nu)+\Sigma_{Sx}(E,T_\nu)\right]N(E)V(E)\,, \tag{1}$$

where $N(E)$, $V(E)$, and E are the density per unit energy, the velocity, and the energy of the test particles, respectively; T_ν is the kinetic temperature of plasma particles of species ν; and $\Sigma_{Rx}(E,T_\nu)$ and $\Sigma_{Sx}(E,T_\nu)$ are the total macroscopic nonelastic and elastic cross sections, respectively, for scattering of test particles by plasma particles of species ν. In the case of fast fussile test particles in fully ionized plasmas, the nonelastic cross section $\Sigma_{Rx}(E,T_\nu)\simeq\Sigma_{Fx}(E,T_i)\,n_i\sigma_{Fx}(E,T_i)$, where n_i and T_i are the density and the kinetic temperature of plasma ions of species i, respectively, and Σ_{Fx} and σ_{Fx} are the macroscopic and microscopic cross sections, respectively, of fusion reactions between the test particles and plasma ions of species i. Expressions for the elastic scattering cross sections have been derived for a wide energy range of test particles by Husseiny,[4] Husseiny and Forsen,[5,6] and Husseiny and Sabri.[10]

In the slowing-down range, the macroscopic elastic scattering cross section of test particles of type x and plasma particles of type ν is given in mks rationalized units by[4]

$$\Sigma_{Sx}(E,T_\nu)=\frac{q^2MkT_\nu}{8\pi\epsilon_0\hbar^2E}\,,\quad \mathrm{m}^{-1}\,, \tag{2}$$

where q and M are the charge and mass of the test particle, respectively, and T_ν is in °K.

Consider a fully ionized quasineutral single-ion-species plasma, with $T_e\simeq T_i\simeq T$, and assume that the velocity of the test particles is in the slowing-down range described above. The space- and time-independent transport equation of the test particles may be expressed in terms of the encounter density, that is, the rate of encounters per unit volume, per unit energy,

$$\Psi(E)=\int_E^{E/\alpha_1}dE'\,\frac{\Sigma(E'\to E)\Psi(E')}{\Sigma_F(E',T)+\Sigma_S(E',T)}+\theta(E) \tag{3}$$

where the subscripts x and i are deleted for simplicity in notation. Here, E' is the energy before encounter, α_1 is defined as

$$\alpha_1=\frac{E_{\min}}{E}=\left(\frac{M-m_i}{M+m_i}\right)^2\,, \tag{4}$$

$E_{\min}$ is the minimum energy with which the test particle can emerge after one encounter with an ion, and m_i is the mass of a plasma ion of species i. The function $\Sigma(E'\to E)$ is the scattering kernel for encounters leading to changes in the test particle energy from E' to E. The kernel is related to the energy-transfer probability function $\Gamma(E'\to E)$ by

$$\Sigma(E'\to E)=\Gamma(E'\to E)\Sigma_S(E',T)\,. \tag{5}$$

The function $\theta(E)$ of Eq. (3) is a source term which gives the number of test particles released into the plasma at energy E per unit volume per unit time.

The form of the source function depends on the origin of the test particles, whether they are produced by fusion reactions in the plasma or being injected by an external source. In both cases, the source can be represented by a δ function centered at an initial energy E_0; that is,

$$\theta(E)=\theta_0\delta(E-E_0)\,, \tag{6}$$

where θ_0 is the strength of the source giving the number of particles released in a unit plasma volume per unit time.

For a Maxwellian plasma background, the thermal motion of the plasma ions causes the initial distribution of the reaction particles to assume the shape of a Gaussian peaked at E_0, i.e., a dispersed δ function. Thus, the number of fast test particles released into the plasma per unit energy per unit time between an energy E and $E+dE$ may be explicitly expressed as

$$\theta(E)\,dE=\frac{\theta_0}{\sqrt{\pi}U}\exp\left[-\left(\frac{E-E_0}{U}\right)^2\right]\,, \tag{7}$$

where U is the width of the dispersed δ function and $U\ll E_0$. For reaction particles, $U=2kT_i$. In the case of neutral beam injection, the source is also expected to be a slightly dispersed monoenergetic beam. The degree of dispersion depends on the type of injector, and the source strength is determined from the beam parameters. In some cases the source may have more than one distinct component.

In the slowing-down range $\Gamma(E'\to E)$ has been obtained analytically[5] and is given by

$$\Gamma(E' \to E) = \begin{cases} \dfrac{\Delta}{(E' - E + \Delta)^2}, & \alpha_1 E' < E < E', \\ 0, & \text{otherwise}, \end{cases} \quad (8)$$

where $\Delta = \hbar^2/2m_i\lambda_i^2$, λ_i is the Debye radius; that is, $\lambda_i^2 = \epsilon_0 kT_i/n_i q_i^2$, and q_i is the charge of a plasma ion of species i. The energy-transfer probability function has a maximum corresponding to no energy transfer, and then it decreases slowly as the after-encounter energy decreases from E' to an energy $\alpha_2 E'$, where $\alpha_2 = 1 - \Delta/E'$. This energy range corresponds to distant encounters which are the most probable events. For after-encounter energies below $\alpha_2 E'$ the energy-transfer probability drops as $(E' - E)^{-2}$ until it reaches a minimum at an after-encounter energy of $\alpha_1 E'$ which corresponds to the unlikely event of head-on collision. Thus, the Lorentzian of Eq. (8) can be replaced by a scattering model of the form

$$\Gamma(E' \to E) \simeq \begin{cases} 1/\Delta, & \alpha_2 E' \le E \le E' \\ 0, & \text{otherwise}. \end{cases} \quad (9)$$

In this case, $\int_{\alpha_2 E'}^{E'} \Gamma(E' \to E)\, dE \simeq 1$, but the probability that the test particle will have an energy $E < \alpha_2 E'$ after an encounter is zero. The use of this model is equivalent to assuming that no energy is transferred from the fast test particles to the plasma ions through encounters at impact parameters $b < \lambda_i$ leading to deflection angles $\chi > \chi_0$; where $\chi_0 = \hbar/\mu V \lambda_i$ and μ is the reduced mass of the two interacting particles. On the other hand, encounters at impact parameters $b > \lambda_i$ which result in a deflection angle $\chi \le \chi_0$ approximately lead to an incremental energy loss Δ which is independent of E and E'. Examination of the dependence of the differential elastic scattering cross section on the angle of deflection shows that the elastic scattering cross section is large at $\chi = \chi_0$, and it changes rather slowly in the range $0 \le \chi \le \chi_0$, then it drops more rapidly for $\chi > \chi_0$.

The total elastic scattering cross section has been found to be $\sigma_{Si} = 4\pi\lambda_i^2$ in the range wherein the classical theory is applicable.[6] This cross section corresponds to a hard-sphere scattering. In the quantum-mechanical range this value is modified by a factor $(qq_i/4\pi\epsilon_0\hbar V)^2$, which is less than unity.[6] The nature of the encounters can be visualized as collisions between a charged test particle and neutral Debye spheres of radius λ_i.

Using the model of Eq. (9), we may rewrite Eq. (3) in the form

$$\Psi(E) = \frac{1}{\Delta}\int_E^{E+\Delta} dE' \frac{\Sigma_S(E',T)\Psi(E')}{\Sigma_F(E',T) + \Sigma_S(E',T)} + \theta(E). \quad (10)$$

We may also define a slowing-down density $\Upsilon(E)$, as the number of test particles slowing down past an energy E per unit volume per unit time; this is given in general by

$$\Upsilon(E) = \int_E^{E/\alpha_1} dE' \frac{\Sigma_S(E',T)\Psi(E')G(E',E)}{\Sigma_F(E',T) + \Sigma_S(E',T)}, \quad (11)$$

where $G(E',E)$ is the probability that a fast test particle of initial energy $E' > E$ emerges after an encounter with any energy in the range between $\alpha_1 E'$ and E; that is,

$$G(E',E) = \int_{\alpha_1 E'}^{E} dE''\, \Gamma(E' \to E''). \quad (12)$$

Substituting for the energy-transfer probability function from Eq. (8) into Eq. (12)

$$G(E',E) = \begin{cases} \dfrac{\Delta}{E' - E + \Delta} - \dfrac{\Delta}{E'(1-\alpha_1) + \Delta}, \\ \qquad E \le E' \le E/\alpha_1 \\ 0, \quad E/\alpha_1 < E' < E. \end{cases} \quad (13)$$

In the absence of absorption and if the test particles do not escape from the interaction volume, the slowing-down density is simply equal to the number of the test particles produced with initial energies, E_0, that is, $\Upsilon(E) = \theta_0$. Using the model described by Eq. (9), the relation given by Eq. (13) reduces to

$$G(E',E) \simeq \begin{cases} 1, & E' - \Delta \le E \le E' \\ 0, & \text{otherwise}, \end{cases} \quad (14)$$

which can be used into Eq. (11) to evaluate the slowing-down density.

III. ENERGY DISTRIBUTON OF NONFUSSILE PARTICLES

In absence of reactions between the fast test particles and the background plasma particles, Eq. (10) simplifies to the form

$$\Psi(E) \simeq \frac{1}{\Delta}\int_E^{E+\Delta} \Psi(E')\, dE' + \theta(E). \quad (15)$$

This relation applies to fast nonfussile charged particles released from fusion reactions, such as α particles and protons. It is also approximately valid for fast fussile particles at energy ranges for which the elastic scattering cross section is much greater than the fusion reaction cross section.

A. Distribution near the initial energy

To solve Eq. (15) for the encounter density, the first few encounters involving test particles having energies near their initial energy are considered, and an iterative method is used. If the test particles born with energy E_0 suffer an encounter at all, they will be slowed down to $E_0 - \Delta$ on the average after a single encounter. The probability that a test particle is scattered into an energy element dE at an energy E between $E_0 - \Delta$ and E_0 is $\Gamma(E_0 \to E)\,dE$. The probability of scattering outside this range after a single encounter is zero. Using the source function given by Eq. (6), the density of the first encounter is

$$\Psi_1(E) = \begin{cases} \dfrac{\theta_0}{\Delta}\displaystyle\int_{E-\Delta}^{E} dE'\,\delta(E' - E_0) = \dfrac{\theta_0}{\Delta}, \\ \quad E_0 - \Delta < E < E_0 \\ 0, \quad E < E_0 - \Delta . \end{cases} \tag{16}$$

Here, the subscript 1 indicates the encounter number. Following the sequence of encounters of θ_0 particles per unit volume per unit time as they slow down from an initial energy E_0 until they reach the cutoff energy E_c below which the slowing-down model is not valid, Eq. (15) can be solved in a stepwise fashion.[12] Thus, the energy range of interest is divided into intervals of equal width Δ with the upper energy taken as the initial energy.

To determine the density of the second encounter, $\Psi_2(E)$, two situations need to be considered, namely, when the energy before the second encounter, E, is such that $E_0 - \Delta \le E \le E_0$ and $E_0 - 2\Delta \le E \le E_0 - \Delta$; thus,

$$\begin{aligned} \Psi_2(E) &= \theta_0(E_0 - E)/\Delta^2, & E_0 - \Delta \le E \le E_0 \\ &= \theta_0(E - E_0 + 2\Delta)/\Delta^2, & E_0 - 2\Delta \le E \le E_0 - \Delta \\ &= 0, & E \le E_0 - 2\Delta . \end{aligned} \tag{17}$$

Similarly, the density of the third encounter, $\Psi_3(E)$, is

$$\begin{aligned} \Psi_3(E) &= \theta_0(E_0 - E)^2/2\Delta^3, & E_0 - \Delta \le E \le E_0, \\ &= \theta_0[6\Delta(E_0 - E) - 2(E_0 - E)^2 - 3\Delta^2]/2\Delta^3, & E_0 - 2\Delta \le E \le E_0 - \Delta, \\ &= \theta_0[(E_0 - E)^2 - 6\Delta(E_0 - E) + 9\Delta^2]/2\Delta^3, & E_0 - 3\Delta \le E \le E_0 - 2\Delta, \\ &= 0, & E \le E_0 - 3\Delta, \end{aligned} \tag{18}$$

and that of the fourth encounter, $\Psi_4(E)$ is

$$\begin{aligned} \Psi_4(E) &= \theta_0(E_0 - E)^3/6\Delta^4, & E_0 - \Delta \le E \le E_0 \\ &= \theta_0[4\Delta^3 - 3(E_0 - E)^3 + 12\Delta(E_0 - E)^2 - 12\Delta^2(E_0 - E)]/6\Delta^4, & E_0 - 2\Delta \le E \le E_0 - \Delta \\ &= \theta_0[3(E_0 - E)^3 - 24\Delta(E_0 - E)^2 + 60\Delta^2(E_0 - E) - 44\Delta^3]/6\Delta^4, & E_0 - 3\Delta \le E \le E_0 - 2\Delta \\ &= \theta_0[64\Delta^3 - (E_0 - E)^3 + 12\Delta(E_0 - E)^2 - 48\Delta^2(E_0 - E)]/6\Delta^4, & E_0 - 4\Delta \le E \le E_0 - 3\Delta \\ &= 0, & E \le E_0 - 4\Delta . \end{aligned} \tag{19}$$

Figure 1 shows the behavior of the encounter density for the first four encounters. This only describes those test particles which have already made an encounter. The complete solution for the total encounter density is clearly the sum over the source particles that have suffered no encounters plus the sum of all the encounter densities calculated separately for each encounter. Discontinuities in the individual encounter densities lead to discontinuities in the total encounter density. The complete solution thus requires going through the above calculation L times, where L is the number of encounters in the slowing-down range between E_c and E_0; that is, $L = (E_0 - E_c)/\Delta$. In the first step of energy loss, that is, in the interval between E_0 and $E_0 - \Delta$, the solution for the encounter density can be represented by the series

$$\Psi_I(E) = \theta_0 \sum_{\gamma=1}^{L} \frac{(E_0 - E)^{\gamma-1}}{(\gamma - 1)!\,\Delta^\gamma} . \tag{20}$$

where I designates the number of the interval. Similar series can be obtained for other intervals.

The number of intervals is usually very large; for example, for 3.03-MeV reaction protons released in a background deuteron plasma of n_D

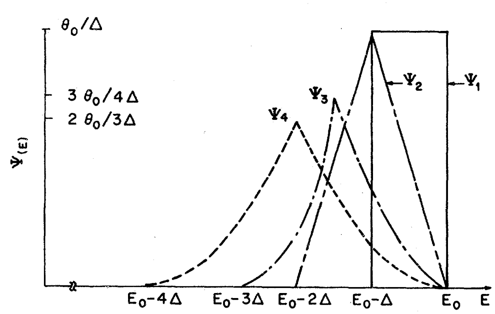

FIG. 1. Transient behavior of the encounter density of the test particles.

$=2\times10^{21}$ m^{-3} and $T_D=60$ keV, the number of encounters is $L=4.66\times10^{20}$. However, the first few encounters give the transient behavior which indicates the presence of oscillations in the encounter density similar to the well-known Placzek wiggles.[12] The four encounters described above take place in about 10^{-18} sec, while the encounter density tends to smooth out to a Gaussian when the test particles' energy approaches the thermal energy range, and hence, asymptotic solutions may be sought by considering scattering events below the source energy.

B. Asymptotic energy distribution in single-ion-species plasmas

For energies below the source energy E_0 a solution of the integral equation (15) is $\Psi(E)=C$, where C is a constant that can be obtained from Eqs. (11) and (14). The slowing-down density is independent of E, that is,

$$\Upsilon=\int_E^{E+\Delta} dE'\Psi(E')=C\Delta\,, \tag{21}$$

but the fact that the slowing-down density is also given by $\Upsilon=\theta_0$ in a nonabsorbing medium can be used to give the encounter density as

$$\Psi(E)=\theta_0/\Delta\,. \tag{22}$$

Thus, the density of test particles having energies between E and $E+dE$ is

$$N(E)\,dE\simeq\frac{\theta_0 m_i\sqrt{E}\,dE}{\pi n_i(qq_i/4\pi\epsilon_0)^2(2M)^{1/2}}\ \text{particles/m}^3\,. \tag{23}$$

The average number of test particles that have energies between E_c and some upper energy E_h below which the asymptotic solution is valid can be obtained from Eq. (23) by integrating over E, thus,

$$N=\frac{7.396\times10^{21}\theta_0 A_i}{n_i Z^2 Z_i^2\sqrt{A}}\,(E_h^{3/2}-E_c^{3/2})\ \text{particles/m}^3\,,\qquad E_c<E_h<E_0 \tag{24}$$

where A_i and A are the atomic masses of plasma ions and test particles, respectively, Z_i and Z are the charge numbers of plasma ions and test particles, respectively, and E_h and E_c are in MeV. The value of E_c is arbitrary; however, a value of $E_c=2T_i$, where T_i is in MeV, is found to be appropriate.[4] The upper limit can be selected very close to E_0, since the encounter density is expected to smooth after several encounters, while a slight reduction in the energy of the test particle is expected to occur after very many collisions. For deuteron plasma at 60 keV, we find that $\Delta\simeq6.24\times10^{-21}$ MeV and that 1.6×10^{17} encounters would take place to reduce the energy of a test particle by 1 keV.

The energy distribution of the test particles may be normalized by using Eq. (24), and hence Eq. (23) may be rewritten in the form

$$\frac{N(E)}{N}=\frac{3E^{1/2}}{2(E_h^{3/2}-E_c^{3/2})}\ \text{MeV}^{-1}\,. \tag{25}$$

The average energy $\overline{E}$ in the range between E_c and E_h can be directly found from Eq. (25); thus

$$\overline{E}=\frac{0.6(E_h^{5/2}-E_c^{5/2})}{E_h^{3/2}-E_c^{3/2}}\ \text{MeV}\,. \tag{26}$$

If the upper energy is such that $E_h\gg E_c$, then $\overline{E}\simeq0.6E_h$; hence, the average energy is close to the upper part of the energy distribution. However, this is only true in the slowing-down energy range. In this range $\overline{E}$ is rather insensitive to changes in the ion temperature.

As an example, let us consider reaction protons which are produced at an energy $E_0=3.03$ MeV by D-D fusion reactions. The number of reaction protons with energy between E_p and E_p+dE_p in an infinite homogeneous deuteron plasma of $n_{\rm D}=2\times10^{21}$ m^{-3} and $T_{\rm D}=60$ keV is obtained from Eq. (23); thus

$$N_p(E_p)\,dE_p\simeq7.8\times10^{20}\sqrt{E_p}\,dE_p\ \text{protons/m}^3\qquad 0.12<E_p<3.03\ \text{MeV} \tag{27}$$

where E_p is the proton energy in MeV. The average reaction proton density in the slowing-down range N_p can be found using Eq. (24) or Eq. (27). If we chose E_h as 3 MeV, then $N_p\simeq2.46\times10^{21}$ protons/m^3. The average number of reaction protons in the slowing-down range slightly exceeds the density of plasma deuterons. This is because the density of the deuterons is assumed to be constant while protons are continuously produced in an infinite nonabsorbing medium. On the other hand, the average proton energy $\overline{E}_p\simeq1.8$ MeV.

Since current interest in thermonuclear research is primarily devoted to plasmas composed of equal portions of deuterons and tritons, we may consider the case of reaction α particles released in a plasma of $T_{\rm D}=T_{\rm T}=T_i=20$ keV and $n_{\rm D}=n_{\rm T}=n_i/2=10^{21}$ m^{-3}, where subscripts D and T refer to the plasma ions D and T, respectively. The ion mass may be taken as $m_i=(m_{\rm D}+m_{\rm T})/2$. Since α particles are released at 3.52 MeV, the value of E_h may be taken as 3.5 MeV; thus $N_\alpha\simeq3.25\times10^{21}$ alphas/m^3 and $\overline{E}_\alpha=2.1$ MeV.

C. Asymptotic energy distribution in a multicomponent plasma

In a multicomponent plasma the encounter density at energies $E<E_0$ may be written as

$$\Psi(E) = \sum_\nu \Psi_\nu(E, T_\nu)$$
$$= \sum_\nu \frac{1}{\Delta_\nu} \int_E^{E+\Delta_\nu} dE' \frac{\Sigma_S(E', T_\nu)\Psi(E')}{\Sigma_S(E')}, \qquad (28)$$

where the summation is over all the $(l-1)$ ion species in an l-component plasma, and $\Sigma_S(E)$ is the sum of the macroscopic cross sections of elastic scattering of test particles with each of the plasma components. From Eq. (2),

$$\Sigma_S(E) = \sum_\nu \Sigma_S(E, T_\nu) = \frac{q^2 Mk}{8\pi\epsilon_0 \hbar^2 E} \sum_\nu T_\nu . \qquad (29)$$

The kinetic temperatures of the different species of ions are likely to be equal if the equilibration time is less than the plasma confinement time, and hence the sum in Eq. (29) may be replaced by $(l-1)T$ where $T \simeq T_\nu$. The total scattering cross section depends on neither the mass nor the density of the plasma constituents. The ratio $\Sigma_S(E, T_\nu)/\Sigma_S(E)$ is independent of energy since both of the corresponding microscopic cross sections $\sigma_S(E, T_\nu)$ and $\sigma_S(E)$ depend on E in the same manner. Similarly, the slowing-down density is

$$\Upsilon = \sum_\nu \frac{1}{\Delta_\nu} \int_E^{E+\Delta_\nu} dE' \frac{\Sigma_S(E', T_\nu)(E' - E + \Delta_\nu)\Psi(E')}{\Sigma_S(E')}, \qquad (30)$$

where $G(E', E)$ is evaluated using Eq. (12) for encounters involving plasma particles of species ν.

Using the same approach as in Sec. III B, the energy distribution of the test particles is

$$N(E) = \frac{16\pi\epsilon_0^2\theta_0 E^{1/2}}{Z^2 e^4 (2M)^{1/2} \sum_\nu (n_\nu Z_\nu^2/m_\nu)}, \quad E < E_0 \qquad (31)$$

where m_ν is the mass of a particle of the νth species. The average density in the energy interval between E_c and E_h is

$$N = \frac{7.396\times 10^{21}\,\theta_0 (E_h^{3/2} - E_c^{3/2})}{Z^2\sqrt{A}\, \sum_\nu (n_\nu Z_\nu^2/A_\nu)} \text{ particles/m}^2 , \qquad (32)$$

where A_ν and Z_ν are the atomic mass and the atomic number of the νth species. The average test particle energy is the same as that given by Eq. (26) for the single-ion-species plasmas. The choice E_c depends on the ion kinetic temperature for each plasmas species. If these are all equal to a single temperature T_i, then $E_c = 2kT_i$ can be chosen as before; otherwise, a number average temperature may be selected, namely

$$T_i = \sum_\nu n_\nu T_\nu / n_i . \qquad (33)$$

Here n_i is the total ion density. We may reconsider the example of reaction alphas in D-T plasmas which is discussed in Sec. III B. Using the mass of deuterons and tritons instead of the average mass and using Eq. (32) to determine n_α, the result given in Sec. III B is found to overestimate n_α by about 4.17%.

D. Dispersed source

Thus far a monoenergetic source of test particles has been considered; however, the solution of the encounter density is also the Green's function of any other source energy spectrum. Thus, for a test-particle source of an arbitrary energy dependence, the encounter density is

$$\Psi(E) = \int_E^{E_{\max}} dE'\, \theta(E')\Psi_U(E') + \theta(E) , \qquad (34)$$

where $\Psi_U(E')$ is the encounter density that arises from a source of unit strength at E', and the value of $E_{\max}$ extends to the upper bound of the energy range of the source spectrum. If $\Psi_U(E)$ is taken as its asymptotic value, that is, $1/\Delta$, then the energy distribution of the test particles is obtained using Eqs. (7) and (34); thus,

$$N(E)\,dE \simeq \frac{8\pi\epsilon_0\theta_0 E^{1/2}}{q^2(2M)^{1/2}} \times\left\{\frac{\hbar^2}{\sqrt{\pi}\,UkT_i}\exp\left[-\left(\frac{E-E_0}{U}\right)^2\right] + \frac{\epsilon_0 m_i}{n_i q_i^2}\left[1-\operatorname{erf}\left(\frac{E-E_0}{U}\right)\right]\right\} dE, \quad E \gtrless E_0 . \qquad (35)$$

For $E \sim E_0$, the second term between curly brackets is dominant while the first term is more significant at $E > E_0$. In the case of $E < E_0$ the energy distribution of Eq. (35) reduces to a form approximately given by Eq. (23).

IV. ENERGY DISTRIBUTION OF FUSSILE PARTICLES

In the case of neutral beam injection the test particles are fussile and may suffer fusion reactions in the course of slowing down; consequently, their energy distribution differs from that obtained in the absence of fusion reactions. The same situation takes place for fussile reaction particles, such as T and ^{3}He, produced by D-D reactions. The time-independent spatial-independent transport equation governing this process is

$$\Sigma_F(E, T_i)N(E)V(E)$$
$$= -\Sigma_S(E, T_i)V(E)n(E) + \frac{1}{\Delta}\int_E^{E+\Delta} \Sigma_S(E', T_i)N(E')V'(E')\,dE + \theta_0\delta(E - E_0) . \qquad (36)$$

Since $\sigma_F(E,T_i)<\sigma_S(E,T_i)$, the encounter density does not significantly deviate from its behavior in the absence of absorption, it can be expanded by a Taylor series about E. Retaining only the first two terms of the expansion, Eq. (36) becomes

$$\Sigma_F(E,T_i)N(E)V(E)\simeq\frac{\Delta}{2}\frac{d}{dE}[\Sigma_S(E,T_i)N(E)V(E)] + \theta_0\delta(E-E_0)\,, \tag{37}$$

which can be directly integrated to give

$$N(E)\simeq\frac{\theta_0\exp[-(2/\Delta)\int_E^{E_0}dE'\,\Sigma_F(E',T_i)/\Sigma_S(E',T_i)]}{\Delta\Sigma_S(E,T_i)V}\,. \tag{38}$$

This is essentially the distribution obtained for nonfussile test particles, Eq. (23), modified by the exponential term.

The energy distribution can be explicitly evaluated for a given plasma composition once the fusion cross section $\sigma_F(E',T_i)$ is specified. The dependence of the fusion cross section on T_i is important only when the energy of the fussile test particles approaches the plasma-ion temperature. For plasma ions of $T_i \ll E$, the cross section of fusion reactions involving test particles and plasma ions is independent of T_i and depends only on E. Hence, the cross section may be represented by segments of the form

$$\sigma_{Fi}(E)=a+uE^y,\quad E_1\le E\le E_2\,, \tag{39}$$

where E_1 and E_2 are the limits of the energy range for which the expression of Eq. (39) is valid and a, u, and y are constants to be specified by a fit to the cross section data. Using the approximate relation of Eq. (39) the energy distribution of fussile test particles may be obtained in the specific energy range between E_1 and E_2; that is,

$$N(E)\simeq\frac{(2M)^{1/2}\theta_0(E)^{1/2}}{2\Delta n_i\sigma_S(E_2,T_i)E_2}\exp\left[-\frac{1}{\Delta E_2\sigma_S(E_2,T_i)}\left(a(E_2^2-E^2)+\frac{2u(E_2^{y+2}-E^{y+2})}{y+2}\right)\right],\quad E_1\le E\le E_2 \tag{40}$$

where we used the fact that the product $\sigma_S(E_2,T_i)E_2=\sigma_S(E,T_i)E$ is constant. The overall energy distribution may be constructed by considering each energy segment separately.

Let us consider, for example, the case of deuteron plasma which is discussed above in Sec. III B and obtain the energy distribution of the tritons which are produced at 1.01 MeV from D-D reactions. We may take $E_c=120$ keV, $E_h=1$ MeV, and neglect the T-T fusion reactions between the reaction tritons. The fusion cross section $\sigma_{DT}(E_T)$ between reaction tritons and plasma deuterons may be approximated by the following relations[2]:

$$\sigma_{DT}(E_T)\simeq\begin{cases}8.3-27.8E_T, & 0.12<E_T<0.25\,,\\ 2.7-5E_T, & 0.25<E_T<0.5\,,\\ 0.2, & 0.5<E_T<1.0\,,\end{cases} \tag{41}$$

where E_T is the triton energy in MeV and $\sigma_{DT}(E_T)$ is in barns. Assuming that no significant reactions will take place above $E_T=1$ MeV, the number of reaction tritons having an energy between E_T and E_T+dE_T is

$$N_T(E_T)\simeq\begin{cases}1.714\times10^{17}\sqrt{E_T}\exp(92.96E_T^2-207.573E_T^3), & 0.12\le E_T\le0.25\\ 4.633\times10^{18}\sqrt{E_T}\exp(30.24E_T^2-37.3E_T^3), & 0.25\le E_T\le0.5\\ 4.775\times10^{19}\sqrt{E_T}\exp(2.24E_T^2), & 0.5\le E_T\le1.0\,.\end{cases} \tag{42}$$

In contrast, the number of tritons having an energy between E_T and E_T+dE_T, neglecting fusion reactions, can be obtained from Eq. (23), and the result is

$$N_T(E_T)\,dE_T=4.486\times10^{20}\sqrt{E_T}\,dE_T,\quad 0.12\le E_T\le1.0\text{ MeV}. \tag{43}$$

Comparison of Eqs. (42) and (43) shows a significant drop in the energy distribution as E_T approaches 0.12 MeV. The ratio between the energy distribution including fusion and the distribution in absence of absorption is 1 at 1 MeV, 0.186 at 0.5 MeV, 0.038 at 0.25 MeV, and 9.13×10^{-3} at 0.12 MeV.

For $u\simeq0$, the average number density in the energy range $E_1\le E\le E_2$ can be readily evaluated; that is,

$$N=\frac{1.1094\times10^{22}\theta_0A_i\beta(1,3)}{n_iZ^2Z_i^2\sqrt{A}}\exp\left(-\frac{aE_2}{\Delta\sigma_{Si}(E_2)}\right)\times[E_2^3\,{}_4F_4(\tfrac34,1,\tfrac54,\tfrac32;\,1,\tfrac54,\tfrac32,\tfrac74;\,aE_2^3/\Delta\sigma_{Si}(E_2))-E_1^3\,{}_4F_4(\tfrac34,1,\tfrac54,\tfrac32;\,1,\tfrac54,\tfrac32,\tfrac74;aE_1^4/\Delta E_2\sigma_{Si}(E_2))] \tag{44}$$

where $\beta(1,3)$ is the beta function and ${}_4F_4$ is the generalized hypergeometric series. Similarly, the mean energy in the range $E_1 \leq E \leq E_2$ is

$$\overline{E} = \frac{\beta(1,5)[E_2^5\, {}_4F_4(\frac{5}{4},\frac{3}{2},\frac{7}{4},2;\frac{3}{2},\frac{7}{4},2,\frac{9}{4};aE_2^3/\Delta\sigma_{Si}(E_2)) - E_1^5\, {}_4F_4(\frac{5}{4},\frac{3}{2},\frac{7}{4},2;\frac{3}{2},\frac{7}{4},2,\frac{9}{4};aE_1^4/\Delta E_2\sigma_{Si}(E_2))]}{\beta(1,3)[E_2^3\, {}_4F_4(\frac{3}{4},1,\frac{5}{4},\frac{3}{2};1,\frac{5}{4},\frac{3}{2},\frac{7}{4};aE_2^3/\Delta\sigma_{Si}(E_2)) - E_1^3\, {}_4F_4(\frac{3}{4},1,\frac{5}{4},\frac{3}{2},1,\frac{5}{4},\frac{3}{2},\frac{7}{4};aE_1^4/\Delta E_2\sigma_{Si}(E_2))]} . \tag{45}$$

Numerical integration may become necessary for cases when $u \neq 0$.

V. CONCLUSIONS

The encounter density used in the above calculations is based on the model given by Eq. (9) and is asymptotically given by Eq. (22). In this model, large-angle encounters are not taken into account. The encounter density derived by Adler and Dorning using the nonapproximate model of Eq. (8) is given in the present notations by[3]

$$\Psi(E) \simeq (\theta_0/\Delta) \exp[-(1/\eta)e^{(E_0-E)/\eta\Delta}] , \tag{46}$$

where $\eta = e^\gamma$, and $\gamma = 0.5772...$. Aside from some approximations used in the derivation of Eq. (46), the result given by this equation takes into account contributions from both large-angle scattering events and small-angle encounters. Comparison between the result obtained in Eq. (22) and the expression given by Eq. (46) shows that they differ by the exponential factor appearing in Eq. (46). The exponential factor drops very fast as $E_0 - E$ increases above Δ; hence $\Psi(E) \simeq 0$ for $E < E_0 - \Delta$, while the exponential drops from 0.57037 to 0.37367 as E changes from $E = E_0$ to $E = E_0 - \Delta$. Thus, the use of Eq. (22) to calculate the encounter density gives a value higher than that obtained from Eq. (46) in the range of small scattering angles. Nevertheless, contributions of the large-angle scattering events to the encounter density are negligible compared to contributions from events leading to small deflection angles. Consequently, the approximate model of Eq. (9) which is used here is fairly accurate for the above evaluation of the energy distribution and for the calculation of the average parameters of energetic test particles released in background plasmas in the slowing-down range.

The distributions derived above may be compared with results obtained from Fokker-Planck formalism. Thus far such results are not available in explicit form. However, an estimate of the space- and time-independent slowing-down energy distribution may be obtained from available expressions of the energy transfer rate evaluated by the Fokker-Planck approach. Comparison between the distribution given in Eq. (23) and the energy transfer rate derived using the Boltzmann collision operator[5] shows that

$$N(E)\,dE \simeq \frac{\theta_0 \ln\Lambda(E)\,dE}{(-dE/dt)} , \tag{47}$$

where

$$\Lambda(E) = \frac{hq_i}{\mu}\left(\frac{n_i M}{2\epsilon_0 kT E}\right)^{1/2} . \tag{48}$$

An expression has been obtained using the Rosenbluth, MacDonald, and Judd (RMJ) form of the Fokker-Planck equation.[13] Substituting that result into Eq. (47) we get for the range of validity of Eq. (23)

$$N(E)\,dE \simeq \frac{\theta_0 m_i \sqrt{E}\, \ln\Lambda(E)\,dE}{\pi n_i (q q_i/2\pi\epsilon_0)^2 (2M)^{1/2} \ln\Lambda_C} \tag{49}$$

where $\ln\Lambda_C$ is the classical Coulomb logarithm. Equation (49) is similar to Eq. (23) except from the factor $\ln\Lambda(E)/\ln\Lambda_C$. This factor approaches unity if Λ_C is replaced by the quantum-mechanical equivalent. In fact, using the formalism employed in deriving Eq. (23) for a Rutherford cross section model with a classical cutoff angle gives a result in full agreement with that estimated from the solution of the RMJ equation for the energy-transfer rate.[14] A detailed comparison between the energy transfer rates obtained from the Boltzmann formalism and those obtained by the Fokker-Planck approach has been given in Refs. 5, 10, and 14.

Monoenergetic nonfussile test particles released in an infinite plasma containing a single ion species are found to assume a discontinuous encounter density near their initial energy E_0, as shown in Fig. 1. After several encounters the distribution of these particles smoothens and asymptotically behaves as $E^{1/2}$ in the slowing-down range. This is given by Eq. (23). In the case of reaction particles produced by fusion events in a thermonuclear plasma, the average number of reaction particles in the energy range $2kT_i \leq E \leq E_0$ is sensitive to the plasma ion temperature, Eq. (24). Nevertheless, the mean energy does not vary significantly with T_i, and it has a value close to the initial energy as indicated by Eq. (26). Similar results are also obtained in Eqs. (31) and (32) for plasma containing several ion species, which is practically the case in all thermonuclear plasmas of interest.

Considering a spread in the initial energy of the test particles released in the background plasmas, the source is represented by an initial Gaussian distribution centered about the most probable energy of production, E_0, Eq. (7). After several encounters the energy distribution of the test particles, Eq. (35), assumes the same behavior obtained for monoenergetic reaction particles for energies lower than E_0. At energies higher than E_0 the distribution is essentially a Gaussian with high-energy tail. The average number of test particles in the energy range $2kT_i \leq E \leq \infty$ is of the same order as that obtained from monoenergetic test particles. Presence of fusion reactions between fussile test particles and plasma ions causes the asymptotic distribution, Eq. (38), to decay faster as the energy decreases.

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