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Comparison between the classical and quantum theories of optical coherence

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The comparison between the classical and quantum theories of optical coherence is presented by using an idea used in the problems of modulation of light beams of classical fields. In such problems it is assumed that it is possible to vary the mean light intensity of a beam without changing its statistical properties defined by a set of coherence functions. We generalize this idea and introduce precisely the class of optical fields which are consistent for modulation. It is shown that all the quantum fields of this class have a positive P representation and are strictly equivalent to classical fields. Moreover, when the field is assumed to be stationary, an interpretation of this condition is given which in particular makes precise the relations between photon-counting and light-intensity measurements. Finally it is shown that all the quantum fields without P representation cannot be consistent for modulation, and the condition of consistency for modulation appears as a characteristic property of fields strictly equivalent to classical ones.

I. INTRODUCTION

About ten years ago there appeared a large number of papers dealing with the relations between classical and quantum theory of coherence. The list of such papers is not of much importance for the following discussion. Since that time, the interest in such problems has been decreasing, even though many questions have not been completely clarified. We will start by very briefly summarizing the actual point of the situation, which seems characterized by two facts.

(a) The statistical properties of optical fields are completely described by a set of coherence functions which can be defined "classically" or "quantum mechanically." The quantum theory is more general than the classical one, because any classical coherence function can be associated with a quantum-mechanical coherence function, but the inverse is not true, and there are pure quantum fields without a classical equivalent in the sense of coherence functions. The best example of such a field is the k-photon field.

(b) Nevertheless, even with the development of very fast electronics or of new kinds of optical sources, such as the lasers, it is extremely difficult to perform experiments like interferences, photon counting, or coincidences of photons on fields which are not classically described. In particular, there are no results published concerning such experiments for a k-photon field.

From time to time, new papers concerning this kind of problem are published, but in several years no significant change has been reported in the situation briefly summarized by the two previous facts.

In this paper we try to present some clarifications of this problem and explain the difference between the pure quantum fields and the classical ones. For this precise purpose we introduce a condition of consistency for modulation which is implicitly assumed in almost all experiments of statistical optics on classical fields.

By "modulation of an optical field" we mean the possibility of varying the field's mean light intensity without changing its statistical properties defined by a set of coherence functions. This property is currently assumed, for example, in the use of a passive light attenuator. We do not study precisely the physical properties of light modulators, which is certainly an interesting problem.

More precisely, we will state that a given field is consistent for modulation if it is possible to construct theoretically another field with the same statistical properties and a different mean light intensity. But we do not discuss the problem of the modulation itself, which is the transformation of the initial field into the modulated field, and particularly the physical devices which allow this transformation.

After a precise definition of the condition of consistency for modulation, we show that it is always satisfied for classical fields, but not for quantum fields. Moreover, the theoretical discussion will show that this condition is a characteristic property of classical fields in the sense that any quantum field which cannot be represented classically also cannot be consistent for modulation. Before this discussion, we present a short summary of classical and quantum theories of coherence.

II. OPTICAL COHERENCE–CONSISTENCY CONDITION FOR MODULATION

In this section we begin with a short survey of standard results concerning the theoretical description of optical coherence. This survey is evidently not complete, and we present only the material necessary for understanding the following discussion.

It is well known that coherence properties of optical fields are well described by a set of *coherence functions*. These functions are higher-order correlation functions, and can be defined classically[1,2] or quantum mechanically.[3,4]

In the classical theory of optical coherence it is assumed that the electromagnetic field is a random field. For example, the electrical field $E(\vec{r},t)=E(x)$ is not a deterministic function, but rather a stochastic process $E(x,\omega)$, where ω is a point in some arbitrary probability space $(\Omega,\mathcal{F},\mathcal{P})$.[5] The process $E(x,\omega)$ is evidently real, but to define correctly the light intensity of a quasimonochromatic field it is convenient to associate with $E(x,\omega)$ its analytical signal $Z(x,\omega)$.[6] This function is directly obtained from $E(x,\omega)$ by a linear filtering, and there is the same information in E as in Z. Now $Z(x,\omega)$ is a complex stochastic process, and we introduce the coherence functions of the field defined by

$$\Gamma^{(n,m)}[\{x_i\}]=\langle Z^*(x_1)\cdots Z^*(x_n)Z(x_{n+1})\cdots Z(x_{n+m})\rangle, \tag{2.1}$$

in which the angular brackets mean an ensemble average on the probability space. The complete set of coherence functions $\Gamma^{(n,m)}[\{x_i\}]$ for every n, m, and $\{x_i\}$ gives a complete description of the statistical properties of the field, and particularly the properties of optical coherence.

In quantum theory the electric field $E(x)$ is an operator and the statistical properties of the field are introduced by means of an appropriate density matrix ρ. It is possible to introduce quantum coherence functions defined by

$$G^{(n,m)}[\{x_i\}]=\mathrm{Tr}[\rho E^-(x_1)\cdots E^-(x_n)\times E^+(x_{n+1})\cdots E^+(x_{n+m})], \tag{2.2}$$

in which $E^\pm(x)$ are obtained from $E(x)$ by using its positive and negative frequency parts. This procedure is very similar to the obtainment of the analytic signal of $E(x)$, but in quantum mechanics $E^+(x)$ and $E^-(x)$ are still operators, and more precisely are respectively annihilation and creation operators of photons.

Even if the physical backgrounds of classical and quantum theories are very different, the coherence properties are defined very similarly by a set of coherence functions. The structures of these functions are different, and $G^{(n,m)}$ is not necessarily a higher-order moment of an appropriate stochastic process.

Thus it is convenient to introduce the concept of *strict equivalence* between the two descriptions. A quantum field statistically defined by a density matrix is strictly equivalent to a classical one if there exists a set of classical coherence functions $\Gamma^{(n,m)}[\{x_i\}]$ identical to the quantum coherence function $G^{(n,m)}[\{x_i\}]$ for every n, m, and $\{x_i\}$. For such a field it is impossible to perform experiments which cannot be completely described classically. Again, if it is not possible to find a classical equivalent of a given field, we express this fact by saying that this field is a *pure quantum field*. There are good examples of such fields, particularly the k-photon field for which the coherence functions $G^{(n,n)}$ are equal to 0 if $n>k$.

Finally we note that in general the coherence functions $G^{(n,m)}$ are only used for $n=m$. This is the case, in particular, in the original papers of Glauber. For most problems this restriction is sufficient. Nevertheless, it is no longer possible to take $n=m$ in the following discussion if we hope to study, for example, the case of *nonstationary* monomode field which can appear in some interesting nonequilibrium systems. Thus, in all the following, we use the $G^{(n,m)}$ functions instead of the $G^{(n)}$ ones.

Now let us define the condition of *consistency for modulation*. This condition is only a precise presentation of assumptions implicitly introduced in some recent works concerning modulation of light beams.[7,8]

At first let us consider a classical field described by the stochastic process $E(x)$. We will say that the field described by $\lambda E(x)$ is obtained from the first one by a modulation with a rate λ. It is perfectly evident that the coherence functions of the two fields are connected by

$$\Gamma_\lambda^{(n,m)}[\{x_i\}]=\lambda^{n+m}\Gamma^{(n,m)}[\{x_i\}], \tag{2.3}$$

where $\Gamma_\lambda^{(n,m)}$ is the coherence function of the modulated field.

Now we will define the modulation for a quantum field defined by a density matrix ρ. This field is said to be modulable by a rate λ if it is possible to find a new field defined by ρ_λ, such that for every $\{x_i\}$, n, and m we have

$$G_\lambda^{(n,m)}[\{x_i\}]=\lambda^{n+m}G^{(n,m)}[\{x_i\}]. \tag{2.4}$$

Finally we will say that a quantum field is consistent for modulation if it is modulable at any rate λ. That means that for any λ it is possible to find a density matrix ρ_λ, such that Eq. (2.4) is valid for every $\{x_i\}$, n, and m.

It is clear that any classical field is consistent for modulation. For quantum fields, this property, which depends only on the density matrix ρ, is not necessarily true, and therefore will appear as a useful tool to compare the coherence properties of classical and quantum fields.

Before ending this section it is important to notice that our condition of consistency is a condition for a given entire field. We will not discuss the process of modulation, even if it is physically possible to transform the initial field into the modulated field. Then the condition of consistency is only defined as a particular property of the field, or of its density matrix. This point is particularly true for pure quantum fields, as for example the k-photon field, because, as noted previously, it is difficult to be realized rigorously in the laboratory. Thus its modulation is still more difficult to conceive. Nevertheless, when $\lambda < 1$, it is perfectly possible, as will be shown, to associate to its density matrix ρ the matrix ρ_λ for which Eq. (2.4) is satisfied, which means that this field is modulable at a rate $\lambda \leq 1$.

III. CONSISTENCY FOR SINGLE-MODE FIELDS WITH *P* REPRESENTATION

In this section we will apply the concepts previously introduced to the case of arbitrary single-mode fields. The extension to multimode fields which appears in the following only introduces some complications in the mathematical expressions.

A classical single-mode optical field is described by the complex stochastic process

$$Z(\vec{r},t;\omega)=Z(\omega)u_k(\vec{r})e^{-i\omega_k t},$$

written as

$$Z(x;\omega)=Z(\omega)u(x), \tag{3.1}$$

where $Z(\omega)$ is an appropriate random variable, and u a deterministic function. Thus the randomness of $Z(x;\omega)$ depends only on a random variable. In this case the coherence functions defined by Eq. (2.1) can be written

$$\Gamma^{(n,m)}[\{x_i\}]=\langle Z^{*n}Z^m\rangle u^{(n,m)}[\{x_i\}], \tag{3.2}$$

where $u^{(n,m)}[\{x_i\}]$ is a spatiotemporal function defined by

$$u^{(n,m)}[\{x_i\}]=u^*(x_1)\cdots u^*(x_n)u(x_{n+1})\cdots u(x_{n+m}). \tag{3.3}$$

It appears from Eq. (3.2) that the set of classical coherence functions is completely defined by the set of all the moments $\langle Z^{*n}Z^m\rangle$ of the random variable $Z(\omega)$, and by the deterministic functions $u^{(n,m)}[\{x_i\}]$.

Now let us consider a quantum single-mode field. The operators $E^+(x)$ can be expressed in terms of annihilation operators a of the single mode by[9]

$$E^{(+)}(x)=i(\tfrac{1}{2}\hbar\omega_k)^{1/2}u_k(\vec{r})e^{-i\omega_k t}a=v(x)a. \tag{3.4}$$

The coherence functions defined by Eq. (2.2) can be written

$$G^{(n,m)}[\{x_i\}]=\mathrm{Tr}[\rho a^{\dagger n}a^m]v^{(n,m)}[\{x_i\}], \tag{3.5}$$

and it is clear that the functions $v^{(n,m)}[\{x_i\}]$ are proportional to the functions $u^{(n,m)}[\{x_i\}]$.

Thus it appears that in the case of a single-mode field the comparison between classical and quantum theories of coherence is finally only a comparison between classical and quantum moments $\langle Z^{*n}Z^n\rangle$ and $\mathrm{Tr}[\rho a^{\dagger n}a^m]$.

For this purpose let us first suppose that the density matrix has a P representation in terms of coherent states defined by

$$\rho=\int P(\alpha)|\alpha\rangle\langle\alpha|\,d^2\alpha. \tag{3.6}$$

There are some conditions on $P(\alpha)$ in order to obtain a density matrix in Eq. (3.6). Particularly, $P(\alpha)$ must be real (but not necessarily positive), and normalized

$$\int P(\alpha)\,d^2\alpha=1. \tag{3.7}$$

Moreover, if the field is stationary, $P(\alpha)$ is only a function of $|\alpha|^2$, which is sometimes assumed in the following. This assumption is not very restrictive at optical frequencies. Indeed, as noticed by Glauber,[10] "at extremely high frequency we cannot be said to have any *a priori* knowledge of the time-dependent parameters". Moreover we have noticed in another context[11] that even if the field is nonstationary, the only distribution which can be obtained in many experiments is connected with the stationary equivalent field whose P representation is deduced from P by integration of the phase of α. Nevertheless, we will consider in the following the general nonstationary situation.

Finally we suppose that the function $P(\alpha)$ is sufficiently regular, and in particular that its singularities are integrable. We exclude from our discussion concerning $P(\alpha)$ singularities stronger than those of δ functions, as for example derivatives of δ functions.

If ρ has a P representation, the calculation of quantum mean values becomes very simple. Indeed, as coherent states are eigenvectors of the annihilation operator

$$a|\alpha\rangle=\alpha|\alpha\rangle, \tag{3.8}$$

we obtain directly

$$\mathrm{Tr}[\rho a^{\dagger n}a^m]=\int P(\alpha)\alpha^{*n}\alpha^m\,d^2\alpha. \tag{3.9}$$

This expression is very similar to the classical moment of a stationary mode

$$\langle Z^{*n} Z^m \rangle = \int P(z) z^{*n} z^m \, d^2 z. \tag{3.10}$$

With this background we can discuss in detail the problem of strict equivalence in the case of a single-mode field.

If $P(\alpha) > 0$, it can be considered as a probability distribution because its integral is 1. Evidently $P(\alpha)$ is not strictly a probability distribution defining in Eq. (3.6) a mixture of projectors $|\alpha\rangle\langle\alpha|$, because the coherent states are not orthogonal. But concerning the moments defined by Eq. (3.9), $P(\alpha)$ can be considered as a probability distribution. Moreover we can say that a stationary single mode with positive $P(\alpha)$ is strictly equivalent to a classical field, by using the definition given in the previous section. Indeed it is clear that if $P(z) = P(\alpha)$, all the classical and quantum moments are equal, and classical and quantum coherence functions can be identical because $u^{(n,m)}[\{x_i\}]$ and $v^{(n,m)}[\{x_i\}]$ are proportional.

In conclusion, the problem of strict equivalence can be reduced to the positivity of the P representation.

Now let us consider the problem of consistency for modulation of our single mode, as defined in the previous section. The function $G_\lambda^{(n,m)}[\{x_i\}]$ in Eq. (2.4) is obtained only by changing the density matrix ρ_λ. Indeed, the functions $v^{(n,m)}[\{x_i\}]$ in Eq. (3.5) are independent of λ because they are only spatiotemporal functions and independent of the modulation of the light intensity. Consequently, the condition of consistency for modulation given by Eq. (2.4) can be written

$$\mathrm{Tr}[\rho_\lambda a^{\dagger n} a^m] = \lambda^{n+m} \mathrm{Tr}[\rho a^{\dagger n} a^m]. \tag{3.11}$$

If we assume that ρ and ρ_λ have a P representation, this condition becomes

$$\int P_\lambda(\alpha) \alpha^{*n} \alpha^m \, d^2\alpha = \lambda^{n+m} \int P(\alpha) \alpha^{*n} \alpha^m \, d^2\alpha, \tag{3.12}$$

for every n, m, and λ. This equation can be written

$$\int [P_\lambda(\alpha) - \lambda^{n+m} P(\alpha)] \alpha^{*n} \alpha^m \, d^2\alpha = 0. \tag{3.13}$$

Let us introduce the function $P'(\alpha)$ defined by

$$P'(\alpha/\lambda) = \lambda^2 P_\lambda(\alpha). \tag{3.14}$$

By a simple change of variables we obtain the new form of Eq. (3.13)

$$\int [P'(\alpha) - P(\alpha)] \alpha^{*n} \alpha^m \, d^2\alpha = 0, \tag{3.15}$$

which must be valid for any n and m. Thus we deduce that $P'(\alpha) = P(\alpha)$, which gives

$$P_\lambda(\alpha) = (1/\lambda^2) P(\alpha/\lambda). \tag{3.16}$$

The corresponding density matrix can be expressed as

$$\rho_\lambda = \int (1/\lambda^2) P(\alpha/\lambda) |\alpha\rangle\langle\alpha| \, d^2\alpha. \tag{3.17}$$

For the following discussion it is interesting to write this equation in another form. For this purpose we calculate th matrix elements $\langle m|\rho_\lambda|n\rangle$ of ρ_λ in the n-photon basis. Starting from the standard relation[12]

$$\langle m|\alpha\rangle = e^{-|\alpha|^2/2} \frac{\alpha^m}{(m!)^{1/2}}, \tag{3.18}$$

we obtain directly from Eq. (3.17)

$$\langle m|\rho_\lambda|n\rangle = \int \frac{1}{\lambda^2} P\left(\frac{\alpha}{\lambda}\right) e^{-|\alpha|^2} \frac{\alpha^{*n}}{(n!)^{1/2}} \frac{\alpha^m}{(m!)^{1/2}} \, d^2\alpha, \tag{3.19}$$

By taking $\lambda\alpha' = \alpha$, this expression becomes

$$\langle m|\rho_\lambda|n\rangle = \int P(\alpha') e^{-\lambda^2|\alpha'|^2} \times \frac{(\lambda\alpha')^{*n}}{(n!)^{1/2}} \frac{(\lambda\alpha')^m}{(m!)^{1/2}} \, d^2\alpha',$$

which shows that ρ_λ can be written in the convenient form

$$\rho_\lambda = \int P(\alpha) |\lambda\alpha\rangle\langle\lambda\alpha| \, d^2\alpha, \tag{3.20}$$

where $|\lambda\alpha\rangle$ is the coherent state corresponding to the eigenvalue $\lambda\alpha$.

In conclusion, the optical single mode defined by $P(\alpha)$ is consistent for modulation if for every positive λ, the matrix ρ_λ defined by Eq. (3.20) is a density matrix.

We will now show that this condition of consistency for modulation is not necessarily true for all quantum fields, which means that there are fields which cannot be arbitrary modulated. This fact is directly connected with the properties of the matrix ρ_λ defined by Eq. (3.20), which is not necessarily a density matrix. Let us specify some properties of this matrix.

It is clear that ρ_λ is Hermitian because $P(\alpha)$ is a real function. Moreover, we have $\mathrm{Tr}\rho_\lambda = 1$, because

$$|\langle n|\lambda\alpha\rangle|^2 = \frac{\lambda^{2n}|\alpha|^{2n}}{n!} e^{-\lambda^2|\alpha|^2}. \tag{3.21}$$

Thus the only condition for ρ_λ in order for it to be a density matrix is to be positive definite, which means that for any vector $|f\rangle$ we have

$$\langle f|\rho_\lambda|f\rangle \geq 0. \tag{3.22}$$

By using Eq. (3.20) this condition can be written

$$\int P(\alpha)|\langle f|\lambda\alpha\rangle|^2 d^2\alpha \geq 0. \qquad (3.23)$$

If $P(\alpha)>0$, it is clear that this equation is valid. This fact can evidently be obtained without calculations because if $P(\alpha)\geq 0$, the field is strictly equivalent to a classical one, and we have shown that all classical fields are consistent for modulation.

In the next sections we will show the inverse property: If the field is consistent for modulation, then $P(\alpha)\geq 0$ and the field is strictly equivalent to a classical one.

Before leaving this section we can indicate the form of the problem when the single mode is assumed to be stationary. In this case, as noticed previously, $P(\alpha)$ is only a function of $|\alpha|^2$, and can be written $P(|\alpha|^2)$. Evidently the density matrix ρ is diagonal in the n-photon basis. The same result is true for ρ_λ defined by Eq. (3.20). Indeed, since

$$\langle m|\lambda\alpha\rangle = e^{-\lambda^2|\alpha|^2/2}(\lambda\alpha)^m/(m!)^{1/2}, \qquad (3.24)$$

it is clear that after integration over θ, where $\alpha = re^{i\theta}$, we obtain $\langle m|\rho_\lambda|n\rangle = 0$ if $m\neq n$.

Thus in this case ρ_λ is a density matrix if and only if $q_n = \langle n|\rho_\lambda|n\rangle \geq 0$ for every n. This quantity can be expressed in terms of $P(|\alpha|^2)$ by

$$q_\lambda(n) = \int P(|\alpha|^2)\frac{\lambda^{2n}|\alpha|^{2n}}{n!}e^{-\lambda^2|\alpha|^2}d^2\alpha. \qquad (3.25)$$

It is clear that if $P(|\alpha|^2)\geq 0$, then $q_n\geq 0$ and the field is consistent for modulation. Now we will discuss the inverse property.

IV. CONSISTENCY AND STRICT EQUIVALENCE FOR A STATIONARY SINGLE MODE WITH P REPRESENTATION

Let us consider a stationary single-mode optical field which is assumed to be consistent for modulation. We suppose that its density matrix has a P representation, which means that $P(|\alpha|^2)$ is such that for every λ the matrix ρ_λ defined by Eq. (3.20) is a density matrix. This assumption means also that the $q_\lambda(n)$ defined by Eq. (3.25) represent the probability $\mathcal{P}$ of a random variable N. This random variable represents the number of photons in the mode when the field is defined by the density matrix ρ_λ, because

$$q_\lambda(n) = \mathcal{P}(N=n) = \langle n|\rho_\lambda|n\rangle. \qquad (4.1)$$

Let us first calculate $q_\lambda(n)$ from Eq. (3.25). We begin by integration over the phase θ, and we obtain very simply

$$q_\beta(n) = \int_0^\infty p(m)e^{-\beta m}\frac{(\beta m)^n}{n!}dm, \qquad (4.2)$$

where $\beta=\lambda^2$, $m=|\alpha|^2$, and $p(m)=\pi P(m)$.

The assumption of consistency for modulation means that for every β, $q_\beta(n)$ is the probability of a random variable N_β, which is evidently a Poisson compound random variable.

Now let us discuss the properties of the family of random variables N_β.

First we suppose that $p(m)$ is a δ distribution, which implies that N_β has a pure Poisson probability distribution written as

$$q_\beta(n) = e^{-\beta m_0}(\beta m_0)^n/n!. \qquad (4.3)$$

Let us introduce the random variable X_β defined as

$$X_\beta = (1/\beta)N_\beta. \qquad (4.4)$$

Its mean value and variance are

$$E[X_\beta]=m_0,\quad \mathrm{Var}[X_\beta]=m_0/\beta, \qquad (4.5)$$

which implies that X_β converges in the quadratic mean sense to m_0 when $\beta\to\infty$, or

$$\lim_{\beta\to\infty}\mathrm{q.m.}X_\beta = m_0, \qquad (4.6)$$

where m_0 is a non-random positive number. This result is a very classical one and is often called the weak law of large numbers.[13]

Now let us introduce the domain A defined by $a<X\leq b$ and the probability

$$P_{X_\beta}(A,m_0)=\mathcal{P}(X_\beta\in A). \qquad (4.7)$$

Since the convergence in the quadratic mean defined by Eq. (4.6) implies the convergence in distribution,[14] we can write

$$\lim_{\beta\to\infty}P_{X_\beta}[A,m_0]=1,\quad \text{if } m_0\in A,$$
$$=0,\quad \text{if } m_0\notin A. \qquad (4.8)$$

Now let us consider the random variable X_β defined by Eq. (4.4) where the distribution of N_β is $q_\beta(n)$ from Eq. (4.2). It is clear that the probability $P_{X_\beta}(A)$ is obtained by the same kind of equation as $q_\beta(n)$, and we obtain

$$P_{X_\beta}(A)=\int_0^\infty p(m)P_{X_\beta}(A;m)\,dm, \qquad (4.9)$$

which is the extension to the distribution functions of Eq. (4.2). By taking the limit we easily obtain

$$\lim P_{X_\beta}(A)=\int_A p(m)\,dm. \qquad (4.10)$$

The limit of $P_{X_\beta}(A)$ is necessarily non-negative, as limit of non-negative functions, and the result is that for every A we have

$$\int_A p(m)\,dm>0. \qquad (4.11)$$

This fact means that $p(m)$ cannot have negative values on ensembles of nonzero measures. As $p(m)=\pi P(m)$, the conclusion is that $P(|\alpha|^2)>0$.

Thus the condition of consistency for modulation implies that the P representation is positive, which means that the quantum field is strictly equivalent to a classical one.

There is a simple physical interpretation of the previous proof. When β is increasing, which means that the mean light intensity of the field is also increasing, the random variable N_β defined by Eq. (4.2) is also increasing.

Nevertheless, the variable X_β defined by Eq. (4.4) remains finite and becomes a continuous random variable whose probability is precisely $p(m)$. This implies that this function is non-negative.

It is clear that the limit of X_β is the light intensity, which has a physical meaning in the case of a large number of photons. The same results appears in the study of the photoelectron shot noise in the case of detection of optical beams.[15] This result means that for pure quantum fields it is not possible to introduce a light intensity.

To conclude this discussion, we can say that our condition of consistency for modulation is a tool for the characterization of quantum fields that are strictly equivalent to classical ones.

Before leaving this section it is interesting to examine explicitly an example of a stationary pure quantum field with a P representation, which is not consistent for modulation. This example is a particular case of a more general class of pure quantum fields.[16] Let us suppose that

$$P(|\alpha|^2)=(c/\pi)(2c|\alpha|^2-1)e^{-c|\alpha|^2}, \tag{4.12}$$

which evidently is not always positive. The $q_\beta(n)$ of Eq. (4.2) are given by

$$q_\beta(n)=\int_0^\infty c(2cm-1)e^{-cm}e^{-\beta m}\frac{(\beta m)^n}{n!}\,dm. \tag{4.13}$$

This integral can be very easily calculated and we obtain

$$q_\beta(n)=\frac{c\beta^n}{(c+\beta)^{n+2}}(2cn+c-\beta). \tag{4.14}$$

These functions are positive for every n only if $\beta<c$, which shows very simply that the condition of consistency for modulation is not satisfied.

V. EXTENSION TO A NONSTATIONARY SINGLE MODE

As noticed previously, the single mode is consistent for modulation if and only if the Eq. (3.23) is satisfied for any vector $|f\rangle$ and value of λ. Now we will show that this condition implies that $P(\alpha)$ is positive, which means that the field is strictly equivalent to a classical one.

Let us take for vector $|f\rangle$ a coherent state $|z\rangle$. The first term of Eq. (3.23) can be written

$$\pi q_\lambda(z)=\int P(\alpha)|\langle z|\lambda\alpha\rangle|^2d^2\alpha, \tag{5.1}$$

and we must have

$$q_\lambda(z)\geq 0, \tag{5.2}$$

for any λ and z.

The scalar product of two coherent states is given by a Gaussian function,[17] and we have

$$|\langle z|\lambda\alpha\rangle|^2=\exp(-|z-\lambda\alpha|^2). \tag{5.3}$$

Thus $q_\lambda(z)$ can be written

$$q_\lambda(z)=\frac{1}{\pi}\int P(\alpha)e^{-|z-\lambda\alpha|^2}d^2\alpha. \tag{5.4}$$

It is easy to see that

$$\int q_\lambda(z)\,d^2z=1, \tag{5.5}$$

and Eqs. (5.2) and (5.5) mean that $q_\lambda(z)$ is a probability density of a random variable Z_λ. This random variable is complex, and we see on Eq. (5.4) that it is a compound Gaussian random variable.[18] It can be compared to N_β used in the previous section which was a compound Poisson variable. But there is a strong difference between these two random variables. We have seen that N_β has a physical meaning, and represents the number of photons in the mode. Conversely, Z_λ has no direct physical meaning and is only a theoretical means to develop our discussion which is very similar to the previous one.

Let us first suppose that the field is in a coherent state α_0. The P representation is evidently $\delta(\alpha-\alpha_0)$, and $q_\lambda(z)$ becomes

$$q_\lambda(z)=(1/\pi)e^{-|z-\lambda\alpha_0|^2}. \tag{5.6}$$

In this case, Z_λ is a pure complex Gaussian variable whose mean and variance are

$$E[Z_\lambda]=\lambda\alpha_0,\quad \mathrm{Var}[Z_\lambda]=1. \tag{5.7}$$

As previously, let us introduce the random variable W_λ defined by

$$W_\lambda=(1/\lambda)Z_\lambda. \tag{5.8}$$

Its mean and variance are evidently

$$E[W_\lambda]=\alpha_0,\quad \mathrm{Var}[W_\lambda]=1/\lambda, \tag{5.9}$$

which shows that

$$\lim_{\lambda\to\infty}\text{q.m. } W_\lambda=\alpha_0. \tag{5.10}$$

Now let us take an arbitrary domain D of the complex plane and consider the probability

$$P_\lambda(D,\alpha_0)=\mathcal{P}(W_\lambda\in D). \tag{5.11}$$

As W_λ converges in distribution to the non-random complex number α_0, we obtain

$$\lim_{\lambda\to\infty} P_\lambda(D, \alpha_0) = 1 \text{ if } \alpha_0 \in D$$
$$= 0 \text{ if } \alpha_0 \notin D. \tag{5.12}$$

If now we suppose that the field is defined by a P representation $P(\alpha)$, we deduce directly that the probability that $W_\lambda \in D$ is

$$P_\lambda(D) = \int P(\alpha) P_\lambda(D;\ \alpha)\, d^2\alpha, \tag{5.13}$$

and from Eq. (5.12)

$$\lim_{\lambda\to\infty} P_\lambda(D) = \int_D P(\alpha)\, d^2\alpha. \tag{5.14}$$

This limit is evidently positive, and as D is arbitrary, we must have $P(\alpha) \geq 0$.

Thus the condition (5.2) gives a non-negative P representation, and we obtain the same conclusion as in the last section, without assuming any stationarity for the field.

VI. EXTENSION TO MULTIMODE FIELDS

In this section we will extend the results previously established to the case of a multimode field. This extension does not make use of new ideas, but the mathematical expressions are unfortunately less tractable than in the single-mode case.

Let us first write the electric field appearing in the coherence functions. This field is obtained by superposition of single modes given by Eq. (3.4), so that

$$E^{(+)}(x) = \sum_{1}^{K}{}_k\, i(\tfrac{1}{2}\hbar\omega_k)^{1/2} u_k(\vec{r}) e^{-i\omega_k t} a_k$$
$$= \sum_{1}^{K}{}_k\, v_k(x) a_k. \tag{6.1}$$

To simplify our mathematical calculations, we suppose that the number of modes K is finite, and Eq. (6.1) is a sum and not a series. The function $v_k(x)$ is the same as in Eq. (2.4), and a_k is evidently the annihilation operator of photons in the mode k. We suppose that the field is consistent for modulation, which is expressed by Eq. (2.4), and we will study the implications of this assumption on the density matrix of the field. Thus the key of our problem is to calculate the higher-order coherence functions $G^{(n,m)}[\{x_i\}]$ appearing in Eq. (2.4).

For this purpose we suppose, as in Sec. III, that the density matrix has a P representation which can be written

$$\rho = \int P(\vec{\alpha}) |\vec{\alpha}\rangle\langle\vec{\alpha}|\, d^2\vec{\alpha}, \tag{6.2}$$

in which $\vec{\alpha}$ means the sequence $\alpha_1, \alpha_2, \cdots \alpha_K$, or a vector in a $2K$-dimensional space.

The coherence functions defined by Eq. (2.2) can be written

$$G^{(n,m)}(\vec{x}) = \mathrm{Tr}[\rho A(\vec{x})], \tag{6.3}$$

where $\vec{x}$ is the vector $(x_1, x_2, \ldots, x_{n+m})$, and $A(\vec{x})$ is obtained from Eq. (6.1) by

$$A(x) = \sum_{\vec{k}} v_{k_1}^* \cdots v_{k_n}^* v_{k_{n+1}} \cdots v_{k_{n+m}}$$
$$\times a_{k_1}^\dagger \cdots a_{k_n}^\dagger a_{k_{n+1}} \cdots a_{k_{n+m}}$$
$$= \sum_{\vec{k}} v^{(n,m)}(\vec{k}, \vec{x}) a_{k_1}^\dagger \cdots a_{k_{n+m}}. \tag{6.4}$$

The introduction of this expression into Eq. (6.3) leads to

$$G^{(n,m)}(\vec{x}) = \sum_{\vec{k}} v^{(n,m)}(\vec{k}, \vec{x}) \mathrm{Tr}[\rho B(\vec{k})], \tag{6.5}$$

where $B(\vec{k})$ is a product of normally ordered operators $a_i^\dagger$ and a_j. For the calculation of the trace it is particularly interesting to use the P representation of the density matrix given by Eq. (6.2), and we obtain, as in Eq. (3.9)

$$\langle B(\vec{k})\rangle = \mathrm{Tr}[\rho B(\vec{k})] = \int P(\alpha) \alpha_{k_1}^* \cdots \alpha_{k_n}^* \alpha_{k_{n+1}} \alpha_{k_{n+m}}\, d^2\vec{\alpha}. \tag{6.6}$$

Now let us apply the condition of consistency for modulation defined by Eq. (2.4). In this equation, $G_\lambda^{(n+m)}(\vec{x})$ is obtained from Eq. (6.5) by changing only ρ in ρ_λ. Thus the result can be written, as in Eq. (3.13),

$$\int [P_\lambda(\vec{\alpha}) - \lambda^{n+m} P(\vec{\alpha})] \alpha_{k_1}^* \cdots \alpha_{k_n}^* \alpha_{k_{n+1}} \cdots \alpha_{k_{n+m}}\, d^2\vec{\alpha} = 0. \tag{6.7}$$

This result must be valid for any values of n, m, and k. By using the same methods as in Sec. III, we easily obtain

$$\rho_\lambda = \int P(\vec{\alpha}) |\lambda\vec{\alpha}\rangle\langle\lambda\vec{\alpha}|\, d^2\vec{\alpha}. \tag{6.8}$$

As previously, we notice that this matrix is Hermitian and its trace is equal to 1. Thus the only condition to be a density matrix for every λ is that ρ_λ is positive definite. This can be written, as in Eqs. (3.22) and (3.23),

$$\int P(\vec{\alpha}) |\langle f|\lambda\vec{\alpha}\rangle|^2 d^2\vec{\alpha} \geq 0. \tag{6.9}$$

Now we can use the same kind of arguments as in Sec. V. We use as vector $|f\rangle$ a vectorial coherent state $|\vec{z}\rangle$, and we introduce $q_\lambda(\vec{z})$ by

$$\pi^K q_\lambda(z) = \int P(\vec{\alpha}) |\langle\vec{z}|\lambda\vec{\alpha}\rangle|^2 d^2\vec{\alpha}. \tag{6.10}$$

From Eq. (6.9) we deduce that we must have $q_\lambda(z) > 0$, and as $q_\lambda(z)$ is normalized it can be considered as the probability density of a complex K-dimensional random variable Z_λ. By extension of Eq. (5.3) the scalar product of two vectorial coherent states can be written

$$|\langle\vec{z}\,|\lambda\vec{\alpha}\rangle|^2 = \exp(-|\vec{z} - \lambda\vec{\alpha}|^2), \qquad (6.11)$$

where $|\vec{w}|^2$ means

$$\sum_{i=1}^{K} |w_i|^2 .$$

Thus, $q_\lambda(\vec{z})$ can be expressed

$$q_\lambda(\vec{z}) = \frac{1}{\pi^K} \int P(\vec{\alpha}) e^{-|\vec{z}-\lambda\alpha|^2} d^2\alpha , \qquad (6.12)$$

which is the probability density of a compound Gaussian vectorial random variable. From this point we can use exactly the same procedure as in the previous section to show that if $q_\lambda(z) \ge 0$ for every λ, then $P(\vec{\alpha}) \ge 0$.

Thus the equivalence between the quantum fields which are consistent for modulation and which can be described classically is complete in the multimode case and without the condition of stationarity. The only assumption used was the existence of a P representation of the density matrix. Now we will suppress this assumption.

VII. QUANTUM FIELDS WITHOUT *P* REPRESENTATION

To simplify the discussion of this section, we suppose now that the field is single mode and stationary. Extensions to multimode and nonstationary fields are possible with more complex analytical expressions.

We previously discussed the connection between classical and quantum fields with P representation by using the condition of consistency for modulation. We saw that this condition is necessary and sufficient for a quantum field with a P representation to be strictly equivalent to a classical field. Moreover we gave examples of fields with P representation which do not satisfy this condition and are pure quantum fields.

In this section we will consider the case of quantum fields without P representation. Many papers have discussed the generality of this representation; there are only a few examples of quantum fields without P representation. Nevertheless, that is the case of all the fields with a bounded number of photons, and that is the reason why we are obliged to consider this situation.

There are two ways to develop the density matrix in terms of coherent states without using a P representation: firstly we can use the R representation[19] which is always well defined, and secondly, we can use the regularization of the P representation introduced by Cahill.[20]

Let us first develop the connection between the consistency for modulation and the R representation of the density matrix. In the n-photon basis, the density matrix can be written

$$\rho = \sum_n p_n |n\rangle\langle n| , \qquad (7.1)$$

where p_n is the probability of obtaining n photons in the mode. Since the coherent states are complete we have

$$\frac{1}{\pi} \int |\alpha\rangle\langle\alpha|\, d^2\alpha = 1 , \qquad (7.2)$$

and the density matrix can be written

$$\rho = \frac{1}{\pi^2} \int\int |\alpha\rangle \left(\sum_n p_n \langle\alpha|n\rangle\langle n|\beta\rangle\right) \langle\beta|\, d^2\alpha\, d^2\beta .$$

By using the standard expressions for the scalar products $\langle\alpha|n\rangle$ and $\langle n|\beta\rangle$, we obtain

$$\rho = \frac{1}{\pi^2} \int\int R(\alpha^*\beta) \exp[-\tfrac{1}{2}(|\alpha|^2 + |\beta|^2)] \times |\alpha\rangle\langle\beta|\, d^2\alpha\, d^2\beta , \qquad (7.3)$$

with

$$R(\alpha^*\beta) = \sum_n p_n \frac{(\alpha^*\beta)^n}{n!} . \qquad (7.4)$$

Every density matrix has a R representation which is completely defined by Eq. (7.4). If moreover there is a P representation, the function R can be written

$$R(\alpha^*\beta) = \int P(|\gamma|^2) \exp(\alpha^*\gamma + \gamma^*\beta - |\alpha|^2)\, d^2\gamma . \qquad (7.5)$$

Conversely, the inversion of this equation [i.e., the obtainment of the P representation from $R(\alpha^*\beta)$] is not very simple, and we know that it is even impossible in those cases which correspond to a density matrix without a regular P representation.

It is clear that there are some conditions on $R(\alpha^*\beta)$ in order for it to be possible to obtain a R representation of a density matrix. In particular, Eq. (7.4) shows that the series expansion of R in terms of $(\alpha^*\beta)$ must have positive and normalized coefficients.

Let us now consider the problem of consistency for modulation. In Sec. III the basic equation was (3.11). Since we have assumed that the field is stationary, we can suppose $m = n$, which gives

$$\mathrm{Tr}[\rho_\lambda a^{\dagger n} a^n] = \lambda^{2n}\, \mathrm{Tr}[\rho a^{\dagger n} a^n] , \qquad (7.6)$$

where ρ is given by Eq. (7.3). Following the results of previous sections, we expand ρ_λ in the form

$$\rho_\lambda=\frac{1}{\pi^2}\int\int R(\alpha^*\beta)\exp[-\tfrac{1}{2}(|\alpha|^2+|\beta|^2)] \times f_\lambda(\alpha,\beta)|\lambda\alpha\rangle\langle\lambda\beta|\,d^2\alpha\,d^2\beta, \quad (7.7)$$

and by using Eq. (7.6) we easily find

$$f_\lambda(\alpha,\beta)=\exp[(1-\lambda^2)(\alpha\beta^*-\tfrac{1}{2}|\alpha|^2-\tfrac{1}{2}|\beta|^2)] =\frac{\langle\beta\,|\,\alpha\rangle}{\langle\lambda\beta\,|\,\lambda\alpha\rangle}. \quad (7.8)$$

If we introduce this expression into Eq. (7.7), we find a matrix ρ_λ which is Hermitian, with trace equal to 1, diagonal in the n-photon basis. Knowing whether or not this matrix is a density matrix is equivalent to knowing whether

$$q_\lambda(n)=\langle n|\rho_\lambda|n\rangle\geq 0. \quad (7.9)$$

By using the expressions for $\langle n|\lambda\alpha\rangle$ and $\langle\lambda\beta|n\rangle$, we find

$$q_\lambda(n)=\frac{1}{\pi^2}\int\int R(\alpha^*\beta)\exp[-(|\alpha|^2+|\beta|^2)] \times\exp[(1-\lambda^2)\alpha\beta^*]\frac{(\lambda^2\alpha\beta^*)^n}{n!}\,d^2\alpha\,d^2\beta, \quad (7.10)$$

and the question is whether or not the integral is positive for every λ.

As the density matrix is completely defined by the probabilities p_n appearing in Eq. (7.1), we will express $q_\lambda(n)$ in terms of p_n. For this purpose we use the expansion of $R(\alpha^*\beta)$ given by Eq. (7.4), and $q_\lambda(n)$ can be written

$$q_\lambda(n)=\frac{1}{\pi^2}\int\int \exp[-(|\alpha|^2+|\beta|^2)] \times\sum_{k,l}\frac{p_k}{k!}\frac{(1-\lambda^2)^l}{l!}\frac{\lambda^{2n}}{n!} \times(\alpha\beta^*)^{l+n}(\alpha^*\beta)^k\,d^2\alpha\,d^2\beta. \quad (7.11)$$

If we take $\alpha=re^{i\theta}$ and $\beta=\rho e^{i\phi}$, we see that integration in θ or ϕ gives $l+n-k=0$. By performing the integrations in r and ρ and taking $\beta=\lambda^2$, we easily find

$$q_\beta(n)=\sum_{l=0}^{\infty}p_{l+n}\frac{(l+n)!}{l!\,n!}(1-\beta)^l\beta^n. \quad (7.12)$$

This expression gives the probability of n photons in the mode for the field obtained by modulation of an initial field characterized by the probabilities p_n. Evidently we will say that the initial field is consistent for modulation if for every β the function $q_\beta(n)$ is positive.

First we will show that Eq. (4.2), obtained when the density matrix has a P representation, is a particular case of Eq. (7.12). Thus let us suppose that p_n is a Poisson compound probability

$$p_n=\int_0^\infty p(m)e^{-m}\frac{m^n}{n!}\,dm, \quad (7.13)$$

where $p(m)$ is not necessarily positive but has an integral equal to 1.

By introducing this expression into Eq. (7.12), we obtain

$$q_\beta(n)=\sum_l\int_0^\infty p(m)e^{-m}\frac{m^{l+n}}{(l+n)!}\frac{(l+n)!}{l!\,n!}(1-\beta)^l\beta^n\,dm =\int p(m)e^{-m}\frac{(\beta m)^n}{n!}\sum\frac{[(1-\beta)m]^l}{l!} =\int p(m)e^{-\beta m}\frac{(\beta m)^n}{n!}\,dm, \quad (7.14)$$

which is exactly Eq. (4.2).

Secondly let us now suppose that the initial field has exactly k photons, i.e., that $p_n=\delta_{nk}$. By introducing this into Eq. (7.12), we obtain for the modulated field

$$q_\beta(n)=\frac{k!}{n!\,(k-n)!}(1-\beta)^{k-n}\beta^n,\quad 0\leq n\leq k, \quad (7.15)$$

which is a binomial distribution. Evidently this implies that $\beta<1$, and thus the k-photon field is not consistent for modulation.

Now we will see that this is always the case if the number of photons in the initial field is bounded. If $p_n=0$ for $n>k$, the series in Eq. (7.12) is a sum. Moreover, the number of photons in the modulated field is also bounded because $q_\beta(n)=0$ if $n>k$. Let us calculate $q_\beta(k-1)$ from Eq. (7.12). We find

$$q_\beta(k-1)=\beta^{k-1}[p_{k-1}+p_k k(1-\beta)], \quad (7.16)$$

which is positive if

$$\beta<1+P_{k-1}/kp_k, \quad (7.17)$$

and shows that the field is not consistent for modulation.

Thus to characterize the fields consistent for modulation we are obliged to study fields with an *unbounded number of photons*, and $\{p_n\}$ appearing in Eq. (7.12) is necessarily an unbounded sequence of positive numbers. Two different cases can appear.

(i) If the sequence of probabilities p_n can be expanded as in Eq. (7.13) with a regular function $p(m)$, the condition of consistency modulation is equivalent to $p(m)>0$, which is the result of Sec. IV.

(ii) If the expansion (7.13) is not valid, even with an infinite sequence of probabilities p_n, we can use the second approach indicated previously.

Instead of using the R representation for a quantum field without a regular P representation, we can use its regularization which was extensively studied by Cahill.[20] In this work it was shown that any density matrix of a stationary field can be expanded as

$$\rho=\int|\alpha\rangle\langle\alpha|P_1(|\alpha|^2)d^2\alpha+\int|-\alpha\rangle\langle\alpha|P_2(|\alpha|^2)d^2\alpha, \tag{7.18}$$

where P_1 and P_2 are regular functions. The second term vanishes when the field has a regular P representation.

Let us now apply to Eq. (7.18) the condition of consistency for modulation

$$\mathrm{Tr}[\rho_\lambda a^{\dagger n}a^n]=\lambda^{2n}\mathrm{Tr}[\rho a^{\dagger n}a^n] \tag{7.19}$$

where ρ_λ has a regularized P representation defined by $P_{1\lambda}(|\alpha|^2)$ and $P_{2\lambda}(|\alpha|^2)$. By the same calculations used previously, we obtain the solution of Eq. (7.19), which is

$$P_{1\lambda}(|\alpha|^2)=\frac{1}{\lambda^2}P_1\left(\frac{|\alpha|^2}{\lambda^2}\right), \tag{7.20}$$

$$P_{2\lambda}(|\alpha|^2)=\frac{1}{\lambda^2}\exp\left[2|\alpha|^2\left(1-\frac{1}{\lambda^2}\right)\right]P_2\left(\frac{|\alpha|^2}{\lambda^2}\right). \tag{7.21}$$

From these expressions we obtain the probability of n quanta in the mode defined by $\langle n|\rho_\lambda|n\rangle$. This probability can be written

$$q_\beta(n)=q_{1,\beta}(n)+q_{2,\beta}(n), \tag{7.22}$$

where

$$q_{1,\beta}(n)=\int_0^\infty e^{-\beta m}\frac{(\beta m)^n}{n!}p_1(m)\,dm, \tag{7.23}$$

with $\beta=\lambda^2$ and $p_1(m)=\pi P_1(m)$. This part of the probability has evidently the same structure as $q_\beta(n)$ in Eq. (4.2). The second part is obtained from Eq. (7.21), and it is

$$q_{2,\beta}(n)=\int_{-\infty}^0 e^{-\beta m}\frac{(\beta m)^n}{n!}[e^{2m}p_2(-m)]\,dm, \tag{7.24}$$

which can also be written

$$q_{2,\beta}(n)=\int_0^\infty(-1)^n e^{\beta m}\frac{(\beta m)^n}{n!}[e^{-2m}p_2(m)]\,dm. \tag{7.25}$$

It is interesting to notice that $q_\beta(n)$ can be written in the form

$$q_\beta(n)=\int_{-\infty}^{+\infty}e^{-\beta m}\frac{(\beta m)^n}{n!}p(m)\,dm, \tag{7.26}$$

which is the most general extension of a Poisson compound distribution because $p(m)$ has not only possible negative values, but is also defined for negative values of m. Some examples of such distributions were already introduced in the study of point processes.[21] Evidently the Poisson kernel appearing in Eq. (7.26) cannot be interpreted as a conditional probability, even if $q_\beta(n)$ is a probability. But this fact was already true for Eq. (7.13) when $p(m)$ has negative values.

Now the problem of the positiveness of the density matrix ρ_λ is equivalent to the condition for $q_\beta(m)$ to be probabilities for every β. For that purpose we must assume that $0<q(0)\le 1$. This condition can be written

$$0<\int_0^\infty e^{-\beta m}p_1(m)\,dm+\int_0^\infty e^{\beta m}[p_2(m)e^{-2m}]\,dm\le 1. \tag{7.27}$$

By using a result due to Ehrenpreis,[22] it is possible to show that $p_1(m)$ and $p_2(m)e^{-2m}$ are both absolutely integrable functions. This result is due to the fact that the functions $p_i(m)$ can be written in the form

$$p_i(m)\,dm=f_i(m)\,d\mu_i(m),\quad i=1,2 \tag{7.28}$$

where the functions $f_i(m)$ are of bounded variation and the functions $f_i(m)$ are monotonic, continuous, and positive. Moreover, $f_1(m)$ goes to zero as $m\to\infty$ faster than any exponential of the form e^{-am}, and $f_2(m)\to 0$ for $m\to\infty$ faster than every inverse power of m.

In Eq. (7.27), the first integral is bounded for every β, but we must have $p_2(m)=0$ in order for the second integral to be bounded for every β. Indeed, let us consider the second integral in Eq. (7.27) and introduce the function of the complex variable z

$$I(z)=\int_0^\infty f(m)e^{zm}\,dm, \tag{7.29}$$

where $f(m)=p_2(m)e^{-2m}$. This function is evidently a holomorphic functions. If Eq. (7.27) is valid for every $\beta>0$, it follows that $I(z)$ is bounded in the complex plane. Therefore the Liouville theorem states that $I(z)$ is a constant. Thus it follows that $I(z)=I(-\infty)=0$, which shows that $f(m)=0$ and that $p_2(m)$ is vanishing.

Thus the condition of consistency for modulation requires that $p_2(m)=0$, which means that the field has a regular P representation. In this case we have evidently, as shown previously, $p_1(m) > 0$.

In conclusion, in all the cases considered in this paper the condition of consistency for modulation is a characteristic property of the quantum fields which are strictly equivalent to the classical ones.

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