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Radiative corrections to the high-frequency end of the bremsstrahlung spectrum*

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The connection between the high-frequency end of the bremsstrahlung spectrum (tip bremsstrahlung) and the photoeffect, which was first noted by Fano, is shown to be valid also when the lowest-order radiative corrections are included. By exploiting this relation and by using results for the photoeffect given in the preceding paper, we obtain an explicit expression for the corrected tip-bremsstrahlung cross section. Other expressions derived previously for the radiative corrections to bremsstrahlung are not valid at the spectrum tip. Limiting cases of low and high incident electron energy are presented and discussed.

I. INTRODUCTION

Some time ago Fano and collaborators[1,2] pointed out that the cross section for bremsstrahlung at the high-frequency end of the spectrum [tip bremsstrahlung], where virtually all of the kinetic energy of the incident electron is carried off by the emitted photon, is closely related to that of the photoeffect in the Sauter approximation. This connection was subsequently further considered by Pratt,[3] who showed it to be valid including also terms of relative order αZ. In the present work, we show that this relationship continues to hold when one includes radiative corrections to the cross section to lowest order in αZ. Thus, since we have already evaluated the radiative corrections to photoeffect to this order in the preceding paper[4] (hereinafter referred to as I), the connection between these processes allows us to write immediately the corresponding result for the corrected bremsstrahlung cross section at the spectrum tip.

Calculations of the radiative corrections to bremsstrahlung have previously been carried out by Fomin[5] and Mitra *et al.*[6] in relativistic Born approximation. The analytic expressions obtained are rather complicated and no numerical evaluation of these results is given. However, analytic formulas were derived in various limiting cases. The general expressions are valid provided $\alpha Z/\beta_1 \ll 1$ and $\alpha Z/\beta_2 \ll 1$, where β_1 and β_2 are the initial and final electron velocities respectively. Because of these restrictions, the results are not correct at the spectrum tip ($\beta_2 = 0$). In contrast to this, our present calculation of the radiative corrections expressly applies to the spectrum tip ($\beta_2 = 0$) and is restricted only by the condition $\alpha Z/\beta_1 \ll 1$.

In Sec. II, we will review the arguments for relating the high-frequency limit of bremsstrahlung to photoeffect and show that this relationship remains valid when the lowest-order radiative corrections are included. In Sec. III, explicit results are given for the corrected bremsstrahlung differential cross section in the neighborhood of the spectrum tip. Finally, we present and discuss the low- and high-energy limits of our expression for the radiative corrections. Throughout this work, the notation and units employed are those of I.

II. CONNECTION BETWEEN THE HIGH-FREQUENCY END OF THE BREMSSTRAHLUNG SPECTRUM AND PHOTOEFFECT

Fano and collaborators have given two different proofs of the connection between the basic cross sections for tip bremsstrahlung and photoeffect. The first[1] proceeds from the observation that at sufficiently high energies it is only the neighborhood of the origin which contributes in the integration over configuration space in the exact relativistic matrix elements of the two processes. In this region, aside from normalization, the zero-energy continuum wave function needed for tip bremsstrahlung has the same shape as the bound-$S_{1/2}$-state wave function needed for the photoeffect calculation. Thus, at sufficiently high energy ($\omega \gtrsim 1$) the two matrix elements are proportional. Moreover this result has been shown to hold not only to lowest order, but also including terms of the first order in αZ. It then follows that the cross section for tip bremsstrahlung is proportional to the cross section for photoeffect from the K shell if quantities of *order* $(\alpha Z)^2$ are neglected. Now, the lowest-order cross section for the K-shell photoeffect is given by the Sauter formula (see I, Sec. II). The corresponding cross section for tip bremsstrahlung turns out to be given by the limit as $\vec{p}_2 \to 0$ of the Bethe-Heitler formula[7] multiplied by a simple factor. This factor, which is proportional to $1/p_2$, is essential in order to prevent the tip bremsstrahlung cross section from vanishing.[8] Including terms of first order in αZ, a corrected expression for the K-shell photoeffect cross section was derived by Gavrila.[9,10] This expression yields in the same way a corrected cross section for tip bremsstrahlung. Beyond first order, the proportionality of the two cross sections breaks down. Unfortunately, this particular method of proof of the connection between photoeffect and tip bremsstrahlung is based on a detailed analysis of the integrands of the matrix elements and would be quite difficult to carry through for the case of the radiative corrections.

The second proof of the connection[2] employs the continuity theorem for differential oscillator strengths at the limits of continuous spectra. (Fano refers to this argument as "detailed balancing".) It is this argument which we will now extend to include the radiative corrections.

The continuity of differential oscillator strengths when passing from the continuum to the adjoining highly excited Rydberg states is a well-known and general property. It is based on the similarity of the wave functions for these bound states and those of the low-energy continuum. It holds for transitions originating in any initial (bound or continuum) state and for any transition operator (e.g. including radiative corrections). In our case the situation is particularly simple since we are dealing with a single electron in a purely Coulomb field, for which explicit analytic expressions are known for the wave functions.[11]

Let us consider radiative transitions from an initial relativistic continuum state of asymptotic momentum $\vec{p}$, energy E and polarization μ to a bound state (recombination) or continuum state (bremsstrahlung) of lower energy with the emission of a photon of momentum $\vec{k}$, energy ω and polarization $\vec{e}$. We will label the bound states by their relativistic quantum numbers n, j, l (angular momentum of the large component) and m. The continuum states will be described by ϵ, j, l and m. Let the corresponding cross section for the emission of a photon into the solid angle $d\Omega$ (summed over magnetic substates) be $d\sigma_{njl}/d\Omega$ and $d^2\sigma_{\epsilon jl}/d\omega d\Omega$.

The continuity theorem for the corresponding (generalized) differential oscillator strengths in this case takes the form

$$\lim_{n\to\infty} \frac{d\sigma_{njl}/d\Omega}{E_{njl}-E_{n-1jl}} = \lim_{\epsilon\to 1}\left(\frac{d^2\sigma_{\epsilon jl}}{d\omega d\Omega}\right). \tag{1}$$

This equation is exact, in that it holds to all orders in the external potential (i.e., αZ) and the radiation field (i.e., α), thus including all radiative corrections. From Eq. (1) it follows that, by summing over all j and l compatible with n and ϵ,

$$\lim_{n\to\infty} \sum_{j,l} \frac{d\sigma_{njl}/d\Omega}{E_{njl}-E_{n-1,jl}} = \lim_{\epsilon\to 1} \sum_{jl} \frac{d^2\sigma_{\epsilon jl}}{d\omega d\Omega} \equiv \frac{d^2\sigma^{\mathrm{TB}}}{d\omega d\Omega}, \tag{2}$$

where $d^2\sigma^{\mathrm{TB}}/d\omega d\Omega$ is just the tip-bremsstrahlung cross section. In the following, we will also assume an average over the initial spin projections has been performed.

Since recombination of an electron from the initial state of asymptotic momentum $\vec{p}$, energy E and polarization μ into the magnetic substates n, j, l is precisely inverse to photoeffect from these substates, it readily follows by detailed balancing that for cross sections we have

$$\frac{d\sigma_{njl}}{d\Omega} = \frac{\omega^2}{|\vec{p}|^2} \frac{d\sigma^{\mathrm{PE}}_{njl}}{d\Omega_e}, \tag{3}$$

where $d\sigma^{\mathrm{PE}}_{njl}/d\Omega_e$ is the photoeffect cross section.[12] Combining Eqs. (2) and (3), we obtain

$$\frac{d^2\sigma^{\mathrm{TB}}}{d\omega d\Omega} = \frac{\omega^2}{|\vec{p}|^2} \lim_{n\to\infty} \sum_{j,l} \frac{d\sigma^{\mathrm{PE}}_{njl}/d\Omega_e}{E_{njl}-E_{n-1jl}}. \tag{4}$$

This is essentially Eq. (1) of Ref. 2. Note that this Eq. (4) is still exact in the sense described above.

Let us now consider the evaluation of Eq. (4) to lowest orders in αZ, including radiative corrections. Not all j, l values will contribute in this case. It is known that for large photon energies $(\omega \gtrsim 1)$ $d\sigma^{\mathrm{PE}}_{njl}/d\Omega_e$ is of order $(\alpha Z)^{5+2l}$.[13] Consequently, the dominant contribution to Eq. (4) is given by the $j=\frac{1}{2}$, $l=0$ terms, the rest contributing to relative order $(\alpha Z)^2$ or higher. Further, it has been shown that, to first order in αZ inclusive, the following relation holds for the basic cross section (without radiative corrections)

$$\frac{d\sigma^{\mathrm{PE}}_{n\frac{1}{2}0}}{d\Omega_e} = \frac{1}{n^3}\frac{d\sigma^{\mathrm{PE}}_{1\frac{1}{2}0}}{d\Omega_e} = \frac{1}{n^3}\frac{d\sigma^{\mathrm{PE}}_{K}}{d\Omega_e}. \tag{5}$$

Here, $d\sigma^{\mathrm{PE}}_K/d\Omega_e$ is the K-shell photoeffect cross section correct to first order in αZ. Noting that for $n\to\infty$, $E_{njl}-E_{n-1jl} \to (\alpha Z)^2[n^{-3}+O((\alpha Z)^2)]$, we get from Eqs. (4) and (5),

$$\frac{d^2\sigma^{\mathrm{TB}}}{d\omega d\Omega} = (\alpha Z)^{-2}\frac{\omega^2}{|\vec{p}|^2}\frac{d\sigma^{\mathrm{PE}}_K}{d\Omega_e}. \tag{6}$$

This is the result of Fano and Pratt. Moreover, it is easy to check that Eq. (6) remains valid when the tip-bremsstrahlung cross section is summed over the polarization of the photon and averaged over those of the electron in the initial state, while the photoeffect cross section is averaged over the polarizations of the photon and summed over those of the electrons in the final state. In the following we shall consider that these polarization sums have been performed for the cross section and Eq. (6) should be interpreted accordingly.

By employing in Eq. (6) the explicit result for the Sauter cross section [see Eq. (6.16) of I], we find the lowest-order result for the tip-bremsstrahlung cross section averaged and summed over the electron and photon polarizations respectively,

$$\frac{d^2\sigma^{(0)}}{d\omega d\Omega}=(\alpha Z)^3\frac{8\omega|\vec{p}|\sin^2\theta}{|\vec{p}-\vec{k}|^8}\left[4+(\omega-1)|\vec{p}-\vec{k}|^2\right]. \tag{7}$$

We now want to include the virtual-photon part of the radiative corrections. These are of order α with respect to the basic cross section. For light elements they are of essentially the same order as the αZ corrections. Then, to order α and αZ inclusive, we again need consider only the contribution of the states $j=\frac{1}{2}, l=0$ in Eq. (4). It thus remains only to show that Eq. (5) is valid including also the lowest-order (order α) radiative corrections.

To lowest order in αZ, the radiative corrections to the photoeffect from the $n, j=\frac{1}{2},\ l=0$ states can be derived using the Pauli approximation for the initial state wave function.[4] This can be written in momentum space in the form

$$\psi_{n\frac{1}{2}0m}(\vec{q})=\Phi_{n0}(\vec{q})[1+\tfrac{1}{2}\vec{q}\cdot\vec{\alpha}]u_m, \tag{8}$$

where u_m is a Dirac spinor, $\vec{\alpha}$ are Dirac matrices, and $\Phi_{n0}(\vec{q})$ is the nonrelativistic Schrödinger wave function in momentum space for $l=0$.[14] It can be shown that for $\alpha Z\to 0$ the function $\Phi_{n0}(\vec{q})$ gives rise to a δ function according to[15]

$$\lim_{\alpha Z\to 0}\frac{1}{\pi}\left(\frac{\pi n}{\alpha Z}\right)^{3/2}\Phi_{n0}(\vec{q})=\delta(\vec{q}). \tag{9}$$

For $n=1$ this reduces to Eq. (2.8) of I.

Now, as in the calculation for the K-shell photoeffect [cf. Sec. II of I], two possibilities occur when evaluating the matrix elements of the radiative corrections due to virtual photons. If the momentum $\vec{q}$ which occurs in the wave function Eq. (8) is fixed at some nonzero value $|\vec{q}|^2\gg(\alpha Z)^2$ due to the action of an explicit δ function, then one can neglect $(\alpha Z/n)^2$ in comparison to $|\vec{q}|^2$ in $\Phi_{n0}(\vec{q})$. In this instance we have $\Phi_{n0}(\vec{q})\simeq n^{-3/2}\Phi_{10}(\vec{q})$ and, consequently, $\psi_{n\frac{1}{2}0m}(\vec{q})\simeq n^{-3/2}\psi_{1\frac{1}{2}0m}(\vec{q})$. On the other hand, if $\vec{q}$ is not fixed by the action of an explicit δ function, but rather, is integrated over, then using (9) the wave function $\psi_{n\frac{1}{2}0m}(\vec{q})$ can be replaced to lowest order in αZ by

$$\psi_{n\frac{1}{2}0m}(\vec{q})\to\pi\left(\frac{\alpha Z}{\pi n}\right)^{3/2}\delta(\vec{q})u_m. \tag{10}$$

In this case also we see that $\psi_{n\frac{1}{2}0m}(\vec{q})\simeq n^{-3/2}\psi_{1\frac{1}{2}0m}(\vec{q})$. Hence, to lowest order in αZ it is generally true that the matrix elements of the virtual-photon radiative corrections to photoeffect corresponding to transitions from the state $n, j=\frac{1}{2}$ differ from those corresponding to the ground state by a factor $n^{-3/2}$. This implies that the cross sections will be in the ratio n^{-3} and thus Eqs. (5) and (6) remain valid with the virtual photon radiative corrections included.

III. CORRECTED TIP-BREMSSTRAHLUNG CROSS SECTION

Following the discussion of Sec. II we may employ results derived in I for the radiative corrections to K-shell photoeffect to write the corrected tip-bremsstrahlung cross section in the form

$$\frac{d^2\sigma^{\mathrm{vpc}}}{d\omega d\Omega}=\frac{d^2\sigma^{(0)}}{d\omega d\Omega}\left(1+\frac{\alpha}{\pi}\delta_{\mathrm{vpc}}\right), \tag{11}$$

where $d^2\sigma^{(0)}/d\omega d\Omega$ is the Fano-Pratt result of Eq. (7) and δ_{vpc} is the virtual-photon contribution to the radiative corrections contained (implicitly) in Eq. (6.3) of I. We note that this expression (11) is not yet the physical cross section. In order to obtain a physically observable cross section we must add to (11) the real (soft) photon contribution, which will also eliminate the fictitious photon mass λ still contained in δ_{vpc}.

The physical process in which one additional photon is emitted is double bremsstrahlung. We will consider this process for the case in which the energy of the additional photon is less than the energy resolution ΔE. We assume for convenience that $\Delta E\ll 1$. The resulting contribution to the cross section can be derived in a manner similar to the discussion in Sec. VI of I. We find

$$\frac{d^2\sigma^{\mathrm{rpc}}}{d\omega d\Omega}=\frac{\alpha}{\pi}\frac{d^2\sigma^{(0)}}{d\omega d\Omega}\delta_{\mathrm{rpc}}, \tag{12}$$

with δ_{rpc} given by Eq. (6.7) of I.

Adding the real and virtual photon contributions (11) and (12), we obtain the corrected tip-bremsstrahlung cross section,

$$\frac{d^2\sigma}{d\omega d\Omega}=\frac{d^2\sigma^{(0)}}{d\omega d\Omega}\left(1+\frac{\alpha}{\pi}\delta\right), \tag{13}$$

where δ is given explicitly by Eq. (6.15) of I with the appropriate redefinition of symbols. Thus, the lowest-order (fractional) radiative corrections to photoeffect and tip bremsstrahlung are identical. We note that for consistency in the case of light elements we should also include the αZ corrections to the basic cross section in (7). However, for practical purposes it may prove more useful to use the results of exact numerical calculations in place of $d^2\sigma^{(0)}/d\omega d\Omega$. These results also include

TABLE I. Values of $-(\alpha/\pi)\delta\times10^4$ in the low-energy limit for various values of the photon energy ω = electron kinetic energy (in keV), scattering angle θ and energy resolution ΔE (in units of the electron rest energy $\simeq$511 keV). Note, $\delta(\omega, \cos\theta)\leq 0$.

ΔE	0.001			0.0001			0.000 01		
ω (keV) \ θ	0°	90°	180°	0°	90°	180°	0°	90°	180°
0.5	0.199	0.207	0.216	0.269	0.277	0.285	0.338	0.347	0.355
1.0	0.393	0.414	0.435	0.532	0.553	0.574	0.671	0.692	0.713
2.0	0.774	0.825	0.877	1.05	1.10	1.15	1.33	1.38	1.43
3.0	1.15	1.23	1.32	1.56	1.65	1.74	1.98	2.06	2.15
4.0	1.52	1.64	1.77	2.07	2.19	2.32	2.62	2.74	2.87
5.0	1.88	2.05	2.21	2.57	2.73	2.90	3.26	3.42	3.58

effects of electron screening which became important in the case of heavier atoms.[16]

Incidentally, let us remark that as a by-product of our derivation we find that the radiative corrections to photoeffect from any $nS_{1/2}$ state are described by the same expression δ as for the K shell. This is due to the fact that Eq. (5) holds with the radiative corrections due to virtual photons included and that the contribution of the real, soft photons δ_{rpc} is independent of the state n [cf. Eq. (6.6) of I].

The explicit expression for the radiative corrections is rather complicated so that a numerical evaluation of δ is necessary in general. Nevertheless, it is possible to extract analytic expressions for the low- and high-energy limits. For the low-energy limit ($E\simeq 1$) we find [cf. Eq. (7.1) of I]

$$\delta = -\tfrac{2}{3}\beta^2[\tfrac{19}{30} - \ln 2\Delta E + 2\beta\ln\beta\cos\theta] + O(\beta^3)\,, \qquad (14)$$

where β is the velocity of the incident electron and $\cos\theta = \hat{p}\cdot\hat{k}$. In the high-energy limit ($E\gg 1$) in the case of finite momentum transfers $|\vec{p}-\vec{k}|$, we find [cf. Eq. (7.2) of I]

$$\delta = 2(\ln 2\omega - 1)[\ln 2\Delta E - \tfrac{1}{2}\ln 2\omega] - |\vec{p}-\vec{k}|\ln 2\omega + O(1)\,. \qquad (15)$$

A brief summary of the behavior of δ in the low- and high-energy limits is given in Tables I and II.

IV. DISCUSSION

We can now discuss the relation between our results and those of other calculations of the radiative corrections to bremsstrahlung.[5,6] The work of Refs. 5 and 6 represents a determination of the radiative corrections in lowest-order Born approximation. This is equivalent to an evaluation of the Feynman diagrams of Figs. 1 and 2. Although these graphs are not directly relevant to the case of tip bremsstrahlung, they do offer the interesting possibility of checking our results.

TABLE II. Values of $-(\alpha/\pi)\delta$ in the high-energy limit for various values of the photon energy ω = incident electron kinetic energy (in MeV), scattering angle θ and energy resolution ΔE (in units of the electron rest energy $\simeq$511 keV). Note, $\delta(\omega, \cos\theta)\leq 0$.

ΔE	0.100			0.010			0.001		
ω (MeV) \ θ	0°	5°	10°	0°	5°	10°	0°	5°	10°
3.0	0.018	0.020	0.023	0.034	0.036	0.039	0.049	0.051	0.054
5.0	0.028	0.033	0.039	0.049	0.054	0.060	0.070	0.075	0.081
7.0	0.035	0.042	0.050	0.060	0.067	0.075	0.084	0.092	0.100
10.0	0.043	0.055	0.065	0.071	0.083	0.093	0.100	0.112	0.122
15.0	0.052	0.071	0.084	0.085	0.104	0.116	0.117	0.137	0.149
20.0	0.059	0.085	0.099	0.095	0.121	0.134	0.131	0.157	0.170

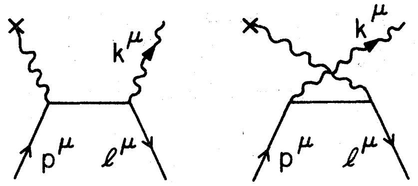

FIG. 1. Feynman diagrams which describe the lowest-order bremsstrahlung matrix element for the process in which the incident electron has 4-momentum p^μ, the emitted photon has 4-momentum k^μ, and the electron has final 4-momentum l^μ.

It was noted in Sec. II of I that the Sauter photoeffect matrix element, Eq. (2.14) of I, can be written in the form

$$S_{fi} = \pi(\alpha Z/\pi)^{3/2} U_{fi}^\dagger, \tag{16}$$

where U_{fi}, represented by the diagrams of Fig. 1, is the lowest-order Born-approximation matrix element for bremsstrahlung evaluated for the case in which the electron is left at rest in the final state ($\vec{l}=0$). Similarly, by examining our expression for the corrected (renormalized) photoeffect matrix element [Eq. (4.18) of I] one can readily see that it can be written in the form,

$$R_{fi} = \pi(\alpha Z/\pi)^{3/2} W_{fi}^\dagger, \tag{17}$$

where W_{fi}, represented by the diagrams of Fig. 2, is just the lowest-order Born approximation for the radiative corrections to bremsstrahlung evaluated for the case in which the final electron is left at rest.

Now, the corrected bremsstrahlung cross section can be related to its matrix element M^B by

$$2(2\pi)^5 \frac{|\vec{p}|}{\omega E_{\vec{l}}|\vec{l}\,|} \frac{d^3\sigma^B}{d\omega\, d\Omega\, d\Omega'} = \frac{1}{2}\sum |M^B|^2 , \tag{18}$$

where M^B is given to lowest order in αZ by the Feynman diagrams of Figs. (1) and (2) and the sum in (18) is taken over the polarizations of the electron in the initial and final states. $d\Omega'$ denotes the element of solid angle for the final electron. The normalization factors contained in M^B are fixed by the condition that, at the tip of the spectrum ($|\vec{l}\,|\to 0$), $M^B \to U+W$, with U and W defined by Eqs. (16) and (17). Since the right-hand side of Eq. (18) is finite at the tip, the left-hand side must also be finite. Taking into account Eqs. (16) and (17) and Eqs. (6.1) and (6.2), of I we can write the following relation, valid to lowest order in αZ, between the cross sections for photoeffect and bremsstrahlung, both including the radiative corrections due to the virtual photons to order α:

$$\frac{d\sigma_k}{d\Omega_e} = 8\pi^2(\alpha Z)^3 \frac{|\vec{p}|^2}{\omega^2} \lim_{|\vec{l}|\to 0} \frac{1}{|\vec{l}\,|} \frac{d^3\sigma^B}{d\omega\, d\Omega\, d\Omega'} . \tag{19}$$

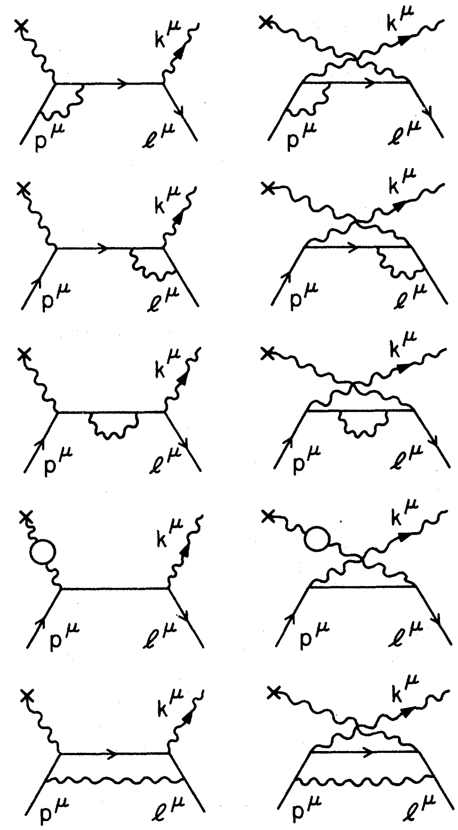

FIG. 2. Feynman diagrams which describe the lowest-order radiative corrections to bremsstrahlung. Diagrams which only contribute to the renormalization constants have been omitted.

Since a similar connection holds in this approximation between our cross section for Compton scattering from the K shell and double bremsstrahlung, Eq. (19) will also be valid for the physical corrected cross sections. We infer that the fractional radiative corrections δ to the photoeffect cross section can be obtained from the fractional radiative corrections to bremsstrahlung evaluated in the limit $|\vec{l}\,|\to 0$. Hence, we can directly compare our expression for δ with the ones derived in Refs. 4 and 5 when the final electron is taken at rest.

In the general case of arbitrary energy and momentum transfer, unfortunately, it is not possible to compare results due to the complexity of the respective expressions and the lack of definite numerical evaluations of δ. However, in the low-energy limit ($E \simeq 1$) we note that our expression (14) does agree with the one derived from Fomin's result [Eq. (57) of Ref. 4].[17] Our particular high-energy limit ($E \gg 1$ and finite momentum transfer) was not considered in those works.

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[3]R. H. Pratt, Phys. Rev. 120, 1717 (1960); R. J. Jabbur and R. H. Pratt, Phys. Rev. 129, 184 (1963); 133, B1090 (1964).

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[6]A. N. Mitra, P. Narayanaswami, and L. K. Pande, Nucl. Phys. 10, 629 (1959).

[7]See H. Bethe and E. Salpeter, *Encyclopedia of Physics* (Springer Verlag, Berlin, 1957), Vol. 35, Sec. 79.

[8]The peculiarity that the Bethe-Heitler formula, which is derived by a Born calculation ($\alpha Z/\beta_1 \ll 1$, $\alpha Z/\beta_2 \ll 1$), when multiplied by a factor (which tends to infinity for $\beta_2 \to 0$) can correctly describe the tip case originates in the fact that in the vicinity of the nucleus the exact form of a continuum wave function normalized in the energy scale is equal to its Born approximation times an appropriate factor (tending to infinity for $\beta_2 \to 0$), see Ref. 1.

[9]M. Gavrila, Phys. Rev. 113, 514 (1959).

[10]This formula can be applied only to very small Z since it turns out that the parameter involved in the expansion of the exact photoeffect cross section is not αZ but, instead, is of the order of $\pi\alpha Z$ (see Ref. 13).

[11]A first proof for a special case was given by Y. Sugiura, J. Phys. Radium 8, 113 (1927).

[12]See Ref. 7, Secs. 69 and 75.

[13]R. H. Pratt, A. Ron, and H. K. Tseng, Rev. Mod. Phys. 45, 273 (1973). See also R. H. Pratt, Phys. Rev. 119, 1619 (1960).

[14]See Ref. 7, Sec. 8.

[15]The integral of the function on the left-hand side of Eq. (9) is equal to 1 for any $\alpha Z \to 0$ it vanishes for any $\vec{p} \neq 0$, whereas for $\vec{p} = 0$, it behaves like $(\alpha Z)^{-3}$.

[16]R. H. Pratt and H. K. Tseng, Phys. Rev. A 11, 1797 (1975).

[17]The expressions for the low-energy limit given in Refs. 5 and 6 are in agreement. In the high-energy limit the expression of Fomin differs from the one of Mitra *et al.*, but the latter seems to be in error {see S. Ya. Guzenko and P. I. Fomin, Zh. Eksp. Teor. Fiz. 38, 513 (1960) [Sov. Phys. JETP 11, 372 (1960)]}.