Radiative corrections to the atomic photoeffect*

James McEnnan and Mihai Gavrila†
Department of Physics and Astronomy, University of Pittsburgh, Pittsburgh, Pennsylvania 15260
and Faculty of Physics, University of Bucharest, Bucharest, Romania
(Received 22 November 1976)

The radiative corrections to the photoeffect are evaluated for the K-shell of hydrogenlike atoms to lowest order in the radiation field. The corrective matrix element is first discussed within the framework of the bound-state interaction (Furry) picture. We then employ the relativistic Born approximation for the Coulomb Dirac propagator and the final continuum wave function of the electron to obtain an expression which is correct to lowest order in αZ, provided $\alpha Z/\beta \ll 1$, where β is the velocity of the ejected electron. The renormalization program is carried out completely, and the lowest-order radiative corrections are given explicitly in terms of the familiar first-order invariant functions of QED. The matrix element which results is further analyzed in terms of invariant amplitudes which are expressed as sums of Feynman parameter integrals. Finally, we evaluate the differential cross section assuming the polarizations of the electron and photon are not observed. Infrared divergences are eliminated from this cross section by allowing for the possibility that an unobserved soft photon is emitted along with the photoelectron. Although in general a numerical evaluation is necessary, analytic expressions are given for the low- and high-energy limits of our final expression for the radiative corrections to the photoeffect. We find that, while the corrections are small at low energy, for incident photons in the range 1–10 MeV and for electrons emitted near the forward direction (finite momentum transfer), the radiative corrections tend to reduce the photoeffect differential cross section by 1.0 to 7.0%.

I. INTRODUCTION

The list of fundamental atomic processes for which radiative corrections have been evaluated is quite impressive. Bound-state energy calculations include the determination of the Lamb shift and the radiative corrections to hyperfine structure splittings. Scattering processes for which the radiative corrections have been obtained include Coulomb scattering, bremsstrahlung, pair production, and Compton effect. Indeed, almost the only basic process for which radiative corrections have not heretofore been determined is the atomic photoeffect.

It is not difficult to adduce reasons for the delay in calculating the radiative corrections to the photoeffect. Typically, for a given element, the photoeffect amplitude is large only at low photon energies. At higher energies, where radiative corrections might be expected to become significant, the photoeffect is considerably more difficult to observe. At the same time, the estimated errors in the theoretical calculations of the basic photoeffect cross section at high energy are of the order of magnitude expected for the radiative corrections. Hence, it has not been possible to systematically begin a search for possible radiative effects in atomic photoeffect. Moreover, the mathematical difficulties of a calculation of the radiative corrections to the photoeffect appear rather formidable. For processes such as pair production or bremsstrahlung, which involve only continuum state electrons, the radiative corrections can evidently be discussed completely within the framework of the Born approximation. There is, thus, a definite and very successful theoretical framework available. Similarly, radiative corrections involving only bound electrons, for example, the Lamb shift, can be evaluated by means of standard methods of perturbation theory. The photoeffect, however, is a scattering process involving both bound and continuum states. In this case, it is not immediately clear to what extent the Born approximation can be used or whether more sophisticated procedures are necessary.

There are, however, valid reasons for considering the radiative corrections to photoeffect at this time. With large computers, numerical calculations of the basic amplitude can be made with increasing accuracy. Advanced experimental techniques allow more precise determinations of the photoeffect cross section at high energies. The point is being approached at which radiative effects in atomic photoeffect at these energies should become apparent. It would thus be desirable to have a theoretical insight into the magnitude of these corrections in order to encourage their observation.

The current status of the theory of the atomic photoeffect, for high incident photon energies, is summarized in the review article by Pratt *et al.*[1] The basic cross section for photoeffect cannot be expressed in analytic closed form and numerical computations are needed to treat the general case.

Nevertheless, for light elements ($\alpha Z \ll 1$) and relatively high photoelectron velocities ($\alpha Z/\beta \ll 1$), Sauter[2] has derived a relativistic analytic formula for the cross section in a hydrogenlike atom which is correct to lowest order in αZ; i.e., $\alpha(\alpha Z)^5$. More recently, Sauter's result has been extended by Gavrila[3] (see also Nagel[4]), who evaluated the first Born correction. The Sauter-Gavrila formula, which has the correct energy dependence, has been combined with an expression derived by Pratt,[5] which gives the correct Z dependence at high energy, to obtain a composite formula which provides a basis for high energy predictions. At the same time, significant theoretical insight has been gained from the realization that, for energies well above threshold, the minimum possible momentum transfer to the nucleus, $q_{\min}$, is of the order of unity (in units of $m_e c$) so that the most important regions of configuration space for the photoeffect matrix element are of the order of the electron Compton wavelength. This observation leads to the normalization screening theory of atomic photoeffect at high energies and has important implications for the calculation of the radiative corrections as well. In addition to these analytic results, numerical evaluations of the total cross section, employing relativistic self-consistent-field methods, have been given for a wide range of photon energies and atomic numbers and are generally accurate to about 1%. These results are in agreement with experimental determinations of the cross section in this energy range which are of comparable accuracy.

In this work, we calculate the radiative corrections to K-shell atomic photoeffect assuming a point Coulomb atomic potential. Although in principle the bound-state interaction picture must be employed, we consider that the relativistic Born approximation for the electron propagator and the final continuum wave function can be used to obtain an expression which is correct to lowest order in αZ, where α is the fine structure constant and Z the nuclear charge. This expression for the corrective matrix element is then valid provided $\alpha Z/\beta \ll 1$, where $\beta = |\vec{p}|/E$ is the velocity of the ejected electron. This condition is necessary to insure that the remainder is, in fact, of relative order αZ and is the usual requirement for the convergence of the Born expansion. Thus, in this instance, our result for the radiative corrections is of order $\alpha^2(\alpha Z)^5$ and is directly related to the Sauter cross section.

In the following section, we will review the derivation of the Sauter approximation to the photoeffect matrix element. This will serve to introduce the procedure we will follow in evaluating the radiative corrections. In Sec. III, we will consider the Furry diagrams for the lowest-order (order α) radiative corrections to the photoeffect process. These include all effects of the atomic potential and are exact in αZ. Employing the Born approximation for the final continuum wave function and the electron propagator, an explicit expression for the lowest order (in αZ) radiative corrections to the photoeffect matrix element is given in Sec. IV. This result is then analyzed in terms of invariant amplitudes which are given as sums of Feynman-parameter integrals. Finally, in Sec. VI, we evaluate the differential cross section, assuming the polarization of the electron and photon is not observed, and the relevant trace calculations are performed. Infrared divergences are eliminated from the cross section by allowing for the possibility that an unobserved soft photon is emitted along with the final photoelectron and adding the corresponding cross section. Although in general a numerical evaluation is necessary, analytic expressions are given for the nonrelativistic and high-energy limits of the radiative corrections. In the following paper,[6] we will examine the connection between our results for photoeffect and the determination of the radiative corrections to the high-frequency limit of the bremsstrahlung spectrum.

II. SAUTER MATRIX ELEMENT

Although the Sauter formula for the K-shell photoeffect cross section in a point Coulomb field has been discussed in several places,[7] we will reproduce here a derivation of the basic matrix element in order to illustrate the general procedure which we use to evaluate the radiative corrections. Thus, following Gavrila,[3] we employ essentially the Born expansion to obtain the lowest order in αZ approximation to the photoeffect matrix element. This result then yields the Sauter formula directly.

The basic process of photoeffect can be described by the Furry diagram of Fig. 1. The transition matrix element corresponding to this dia-

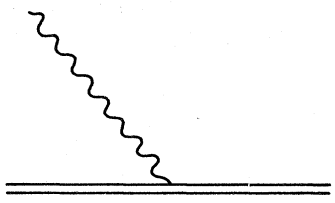

FIG. 1. Furry (bound-state interaction picture) diagram corresponding to the basic photoeffect process. The double line denotes an electron propagating in the atomic field, while the wavy line corresponds to the incoming photon.

gram is given in momentum space by

$$S_{fi} = e(2\pi)^3 \int d^3q \bar{\psi}_f(\vec{q}) \not{\epsilon} \psi_i(\vec{q} - \vec{k}) , \tag{2.1}$$

where the incident photon is specified by four-momentum $k^\mu = (\omega, \vec{k})$, four-polarization $\epsilon^\mu = (\epsilon^0, \vec{\epsilon})$, and $\not{\epsilon} = \epsilon \cdot \gamma = \epsilon_\mu \gamma^\mu = \epsilon^0 \gamma^0 - \vec{\epsilon} \cdot \vec{\gamma}$, where the γ^μ are Dirac γ matrices. (Our metric, γ matrices, and Dirac spinors are those of Bjorken and Drell.[8]) $\psi_i(\vec{q})$ is the initial bound-state wave function of energy E_i, given in momentum space and normalized such that $\int |\psi_i|^2 d^3q = (2\pi)^{-3}$. $\psi_f(\vec{q})$ is the final electron wave function, which is characterized asymptotically as a plane wave of four-momentum $p^\mu = (E_f, \vec{p})$, plus an incoming spherical wave. e is the magnitude of the charge of the electron in rationalized units. We also set $\hbar = c = m_e = 1$, so that distances are measured in electron Compton wavelengths, energies in units of the electron rest energy, and momenta in units of $m_e c$.

The amplitude (2.1) is subject to energy conservation,

$$E_f = E_i + \omega, \tag{2.2}$$

where $E_f = (1 - \beta^2)^{-1/2}$, with β the velocity of the emitted electron, and $E_i = (1 - a^2)^{-1/2}$, where $a = \alpha Z$. If the photon energy ω is much larger than the binding energy of the K-shell electron (which is approximately $E_B = -\frac{1}{2}a^2$ for low Z), then (2.2) reduces to

$$E_f = 1 + \omega \tag{2.3}$$

neglecting second and higher orders in αZ. For the derivation of the Sauter formula and for our lowest-order approximation to the radiative corrections, this form (2.3) of the energy-conservation relation is sufficient.

In order to evaluate the lowest-order contribution to (2.1), we use the Born expansion of the final continuum wave function. Thus, we write

$$\begin{aligned} \bar{\psi}_f(\vec{q}) &= \bar{u}_f(p) \left(\delta(\vec{p} - \vec{q}) + \not{V}(\vec{p} - \vec{q}) \frac{1}{\not{q} - 1 - i\epsilon} \right) \\ &= \bar{\psi}_f^{(0)}(\vec{q}) + \bar{\psi}_f^{(1)}(\vec{q}) , \end{aligned} \tag{2.4}$$

where $\bar{u}_f(p)$ is a free particle Dirac spinor normalized such that $\bar{u}_f(p) u_f(p) = 1$, and

$$\not{V}(\vec{q}) = \gamma^0 (-\alpha Z)/2\pi^2 |\vec{q}|^2 \tag{2.5}$$

is essentially the Fourier transform of the Coulomb potential. $\psi_f^{(0)}$ and $\psi_f^{(1)}$ are the zeroth and first Born terms, respectively. This approximation (2.4) for the continuum wave function is valid provided $\alpha Z/\beta \ll 1$.

For the bound-state wave function we use the Pauli approximation,

$$\psi_i(\vec{q}) = N \frac{1}{(|\vec{q}|^2 + a^2)^2} (1 + \tfrac{1}{2}\vec{q} \cdot \vec{\alpha}) u_i(l), \tag{2.6}$$

which describes the ground state for a Coulomb potential neglecting terms of higher order in αZ. In Eq. (2.6), $N = (a/\pi)^{5/2}$ is a normalization factor, $\vec{\alpha} = \gamma^0 \vec{\gamma}$ and $l^\mu = (1, \vec{0})$.

In order to simplify subsequent discussion, we note the following. Because of the particular analytic form of the ground-state wave function (2.6), there may occur a lowering by one unit of the order in αZ of an integral in which it appears. This can be seen by observing that one of the representations for the three-dimensional δ function is given by

$$\lim_{\lambda \to 0} \frac{1}{\pi^2} \frac{\lambda}{(|\vec{q}|^2 + \lambda^2)^2} = \delta(\vec{q}) . \tag{2.7}$$

There are, in fact, two possibilities. If the momentum $\vec{q}$ which appears in (2.6) is fixed at some nonzero value due to the action of an explicit δ function, then provided $|\vec{q}|^2 \gg a^2$ one may neglect the a^2 term in the denominator. In this case the order in αZ of the integral is its nominal order. On the other hand, if $\vec{q}$ is not fixed by an explicit δ function, but rather is integrated over all values, then, using (2.7), the bound-state wave function can be replaced to lowest order, by the form

$$\psi_i(\vec{q}) \simeq (\alpha Z)^{-1} \pi^2 N \delta(\vec{q}) u_i(l) . \tag{2.8}$$

In this case, there is an additional factor of $(\alpha Z)^{-1}$ which lowers the order of the integral in which $\psi_i(\vec{q})$ appears by one unit. In these circumstances, in order to evaluate the photoeffect matrix element to a given order in αZ, all other quantities must be expressed to at least one order higher. It is for this reason that we include the first Born correction $\psi_f^{(1)}$, in the final continuum wave function (2.4).

To proceed, we consider the contribution of each term in the Born expansion (2.4) of the final electron wave function to the matrix element (2.1). We define

$$S_{fi}^{(n)} = e(2\pi)^3 \int d^3q \, \bar{\psi}_f^{(n)}(\vec{q}) \not{\epsilon} \psi_i(\vec{q} - \vec{k}) , \tag{2.9}$$

where $\bar{\psi}_f^{(n)}(\vec{q})$ is defined by (2.4). The Sauter amplitude, which is the lowest-order approximation to (2.1), is then given by

$$S_{fi} = S_{fi}^{(0)} + S_{fi}^{(1)} . \tag{2.10}$$

The integration in $S_{fi}^{(0)}$ is immediate, due to the δ function in $\psi_f^{(0)}$. We find, neglecting higher-order corrections[9] in αZ, the result

$$S_{fi}^{(0)} = \frac{e(2\pi)^3 N}{|\vec{p} - \vec{k}|^4} \bar{u}_f(p) \not{\epsilon} [1 + \tfrac{1}{2}(\vec{p} - \vec{k}) \cdot \vec{\alpha}] u_i(l). \tag{2.11}$$

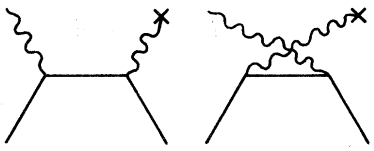

FIG. 2. Feynman diagrams describing the Sauter amplitude. (The cross denotes interaction with the potential.)

For the leading contribution to $S_{fi}^{(1)}$, we use the expression (2.8) for the bound-state wave function. Since $\psi_f^{(1)}(\vec{q})$ contains one power of the potential, only this approximation for $\psi_i(\vec{q})$ is needed. The result for $S_{fi}^{(1)}$ will then be of the same order as for $S_{fi}^{(0)}$. In this way,

$$S_{fi}^{(1)} = e(2\pi)^3(\alpha Z)^{-1}\pi^2 N \bar{u}_f(p) \not{V}(\vec{p}-\vec{k}) \frac{1}{\not{l}+\not{k}-1} \not{\epsilon} u_i(l) , \tag{2.12}$$

where we have used conservation of energy (2.3) to rewrite the propagator in covariant form. It is essentially trivial to verify that (2.11) and (2.12) represent all of the lowest-order contribution to the matrix element (2.1).

The expression (2.11) for $S_{fi}^{(0)}$ can be further simplified by employing the following relation, which is valid neglecting terms of relative order $(\alpha Z)^2$,

$$\frac{1}{|\vec{p}-\vec{k}|^4}\left[1+\tfrac{1}{2}(\vec{p}-\vec{k})\cdot\vec{\alpha}\right] u_i(l) = \pi^2(\alpha Z)^{-1} \frac{1}{\not{p}-\not{k}-1} \not{V}(\vec{p}-\vec{k}) u_i(l) . \tag{2.13}$$

With this result, $S_{fi}^{(0)}$ can be written in a form which is similar to (2.12), so that the Sauter matrix element can be expressed in the symmetric form,

$$S_{fi} = e(2\pi)^3(\alpha Z)^{-1}\pi^2 N \bar{u}_f(p) \times \left(\not{V}(\vec{p}-\vec{k}) \frac{1}{\not{l}+\not{k}-1} \not{\epsilon} + \not{\epsilon} \frac{1}{\not{p}-\not{k}-1} \not{V}(\vec{p}-\vec{k}) \right) u_i(l). \tag{2.14}$$

This result, (2.14), is especially attractive since the right-hand side is manifestly gauge invariant.

It is instructive to note that our final expression (2.14) for the Sauter amplitude is exactly that which would be obtained by evaluating the Feynman diagrams of Fig. 2 assuming the initial (bound) electron were described by the wave function,

$$\psi_i(\vec{q}) \simeq (\alpha Z)^{-1}\pi^2 N \delta(\vec{q}) u_i(l) . \tag{2.15}$$

This expression, aside from the normalization, corresponds (formally) to the first Born term for the wave function of a free electron at rest.[10] Some consequences of this point will be considered in the following paper.[6]

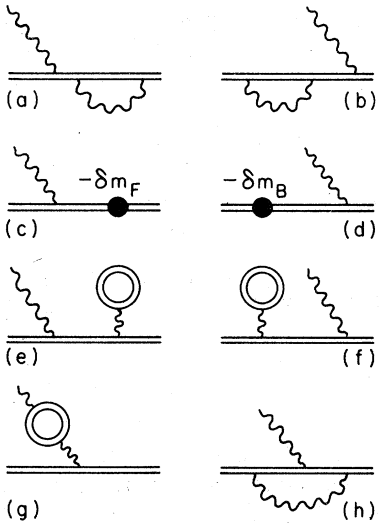

FIG. 3. Furry diagrams which describe the radiative corrections to photoeffect to lowest order in the radiation field (order α).

III. RADIATIVE CORRECTIONS IN THE FURRY PICTURE

We will now discuss the radiative corrections to photoeffect in the context of the Furry (bound-state interaction) picture. In this picture, the wave functions and electron propagators (Dirac Coulomb Green's functions) contain all effects of the potential so that the resulting matrix element is exact in αZ. Including self-energy counterterms, there are eight Furry diagrams which contribute to lowest order in the radiation field (order α). These diagrams are shown in Fig. 3. Writing explicitly the contribution of each diagram to the photoeffect transition matrix element, we have the following.

Diagrams (a) and (b) of Fig. 3 represent self-energy corrections. Their contributions, subject to energy conservation, are given by

$$R_{fi}^{(a)} = e(2\pi)^3(\tfrac{1}{2}\pi\alpha)(2\pi)^8 \int d^3p_1\, d^3p_2\, d^3p_3\, d^4q\, D(q)\, \bar{\psi}_f(\vec{p}_1)\gamma^\mu S(E_f - q^0;\ \vec{p}_1-\vec{q}, \vec{p}_2-\vec{q})\gamma_\mu S(E_f;\ \vec{p}_2, \vec{p}_3+\vec{k}) \not{\epsilon} \psi_i(\vec{p}_3) \tag{3.1}$$

and

$$R_{fi}^{(b)} = e(2\pi)^3(\tfrac{1}{2}\pi\alpha)(2\pi)^8 \int d^3p_1\, d^3p_2\, d^3p_3\, d^4q\, D(q)\ \bar{\psi}_f(\vec{p}_1)\not{\varepsilon} S(E_i;\ \vec{p}_1-\vec{k},\ \vec{p}_2)\ \gamma^\mu S(E_i-q^0;\ \vec{p}_2-\vec{q},\ \vec{p}_3-\vec{q})\gamma_\mu\psi_i(\vec{p}_3). \tag{3.2}$$

In these equations, $D(q)$ is the photon propagator, which has the form

$$D(q) = D(q^0;\vec{q}) = \frac{2i}{(2\pi)^4}\ \frac{1}{q^2+i\epsilon}\ , \tag{3.3}$$

and $S(E;\vec{p},\vec{p}')$ is the Dirac Coulomb Green's function. For our purposes the most convenient definition of $S(E;\vec{p},\vec{p}')$ is in terms of the following (iterated) integral equation[11]:

$$S(E;\vec{p},\vec{p}') = \frac{-2i}{(2\pi)^4}\left[\frac{\delta(\vec{p}-\vec{p}')}{\not{p}-1+i\epsilon} + \frac{1}{\not{p}-1+i\epsilon}\not{V}(\vec{p}-\vec{p}')\frac{1}{\not{p}'-1+i\epsilon}\right.$$
$$\left. + \frac{(2\pi)^4}{-2i}\ \frac{1}{\not{p}-1+i\epsilon}\left(\int d^3q_1 d^3q_2 \not{V}(\vec{q}_1)\, S(E;\vec{p}-\vec{q}_1,\vec{p}'+\vec{q}_2)\not{V}(\vec{q}_2)\right)\frac{1}{\not{p}'-1+i\epsilon}\right], \tag{3.4}$$

where $\not{V}(\vec{q})$ is defined by (2.5). However, we will also have occasion to refer to the eigenfunction expansion of $S(E;\ \vec{p},\vec{p}')$, which can be written

$$S(E;\vec{p},\vec{p}') = \frac{i}{\pi}\sum_{n,E\pm}\frac{\psi_n(\vec{p})\bar{\psi}_n(\vec{p}')}{E_n-E}\ . \tag{3.5}$$

Corresponding to each self-energy diagram, there is a mass counterterm [diagrams (c) and (d) of Fig. 3], where

$$R_{fi}^{(c)} = -ie(2\pi)^3(\tfrac{1}{2}\delta m_F)(2\pi)^4 \times \int d^3p_1\, d^3p_2 \bar{\psi}_f(\vec{p}_1)\, S(E_f;\vec{p}_1,\vec{p}_2+\vec{k})\not{\varepsilon}\psi_i(\vec{p}_2) \tag{3.6}$$

and

$$R_{fi}^{(d)} = -ie(2\pi)^3(\tfrac{1}{2}\delta m_B)(2\pi)^4 \times \int d^3p_1\, d^3p_2 \bar{\psi}_f(\vec{p}_1)\not{\varepsilon} S(E_i;\ \vec{p}_1-\vec{k},\ \vec{p}_2)\psi_i(\vec{p}_2). \tag{3.7}$$

The use of different mass counterterms for bound and free electrons in (3.6) and (3.7) requires some comment. For a free particle, the radiative corrections to the self-energy can only lead to a change in mass.[12] For a bound electron, however, interaction with the radiation field also produces a change in the binding energy. This correction is essentially the Lamb shift. For consistency, this additional contribution to the self-energy must be included in δm_B. Hence, we define

$$\delta m_B = -i\pi\alpha(2\pi)^7\int d^3p_1\, d^3p_2\, d^4q D(q)\bar{\psi}_i(\vec{p}_1)\gamma^\mu S(E_i-q^0;\ \vec{p}_1-\vec{q},\ \vec{p}_2-\vec{q})\gamma_\mu\psi_i(\vec{p}_2)$$
$$+ i\pi\alpha(2\pi)^7\int d^3p_1 d^4p_2 d^4q\,\delta(q^0)D(q)\bar{\psi}_i(\vec{p}_1)\gamma^\mu\psi_i(\vec{p}_1+\vec{q})\mathrm{Tr}[\gamma_\mu S(E_2;\vec{p}_2-\vec{q},\vec{p}_2)]$$
$$= \delta m_F + \Delta E_B, \tag{3.8}$$

where δm_F is the free-particle self-energy correction and ΔE_B is the radiative correction to the binding energy, including both self-energy and vacuum-polarization contributions. In a lowest-order evaluation of the radiative corrections, of course, this point is somewhat academic since the Lamb shift corrections are of higher order in αZ and may be neglected. It is, however, important for the general self-consistency of our formalism. This can be seen from an examination of the self-energy term, (3.2). [The following remarks also apply to the vacuum polarization term, see Eq. (3.10) below.] In this expression, we have written a factor, $S(E_i;p_1-k,p_2)$, where E_i is the energy of the initial bound state. However, from (3.5) we see that the Dirac Coulomb Green's function has poles at the bound-state energy eigenvalues E_n so that $S(E_i;\vec{p}_1-\vec{k},\vec{p}_2)$ is not, strictly, defined. [Note that this infinity is in addition to the divergence of δm_F.] This difficulty is resolved by the definition (3.8) of δm_B, since in this case the pole of the Green's function which appears in (3.2) [together with the pole in the vacuum-polarization term (3.10)] is canceled by the corresponding pole in the mass counterterm (3.7). Thus, the sum of these diagrams is finite. Effectively, then, for those diagrams which contain a factor $S(E_i;\vec{p}_1-\vec{k},\vec{p}_2)$, this Green's function may be replaced by a modified eigenfunction expansion of the form (3.5) in which the bound state with $n=i$ is omitted.

Diagrams (e)–(g) of Fig. 3 comprise the vacuum-polarization contributions to photoeffect. Explicitly, these yield

$$R_{fi}^{(e)} = -e(2\pi)^3(\tfrac{1}{2}\pi\alpha)(2\pi)^8\int d^3p_1\,d^3p_2\,d^4p_3\,d^4q\,\delta(q^0)D(q)\bar{\psi}_f(\vec{p}_1-\vec{q})\gamma^\mu S(E_f;\vec{p}_1,\vec{p}_2+\vec{k})\not{\epsilon}\psi_i(\vec{p}_2)\mathrm{Tr}[\gamma_\mu S(E_3;\vec{p}_3-\vec{q},\vec{p}_3)]\,, \tag{3.9}$$

$$R_{fi}^{(f)} = -e(2\pi)^3(\tfrac{1}{2}\pi\alpha)(2\pi)^8\int d^3p_1\,d^3p_2\,d^4p_3\,d^4q\,\delta(q^0)D(q)\bar{\psi}_f(\vec{p}_1)\not{\epsilon}S(E_i;\vec{p}_1-\vec{k},\vec{p}_2)\gamma^\mu\psi_i(\vec{p}_2+\vec{q})\mathrm{Tr}[\gamma_\mu S(E_3;\vec{p}_3-\vec{q},\vec{p}_3)]\,, \tag{3.10}$$

$$R_{fi}^{(g)} = -e(2\pi)^3(\tfrac{1}{2}\pi\alpha)(2\pi)^8\int d^3p_1\,d^3p_2\,d^3p_3\,d^4q\,D(\omega;\vec{p}_1-\vec{p}_2)\bar{\psi}_f(\vec{p}_1)\gamma^\mu\psi_i(\vec{p}_2)\mathrm{Tr}[\gamma_\mu S(q^0+\omega;\vec{p}_3,\vec{q}+\vec{k})\not{\epsilon}S(q^0;\vec{q},\vec{p}_3-\vec{p}_1+\vec{p}_2)]\,, \tag{3.11}$$

where, again, we note that, due to the definition, (3.8), of δm_B, only states with $n \neq i$ contribute to the Green's function, $S(E_i;\vec{p}_1-k,\vec{p}_2)$, which appears in (3.10).

Finally, the contribution of the vertex correction, diagram (h) of Fig. 3, can be written in the form,

$$R_{fi}^{(h)} = e(2\pi)^3(\tfrac{1}{2}\pi\alpha)(2\pi)^8\int d^3p_1\,d^3p_2\,d^3p_3\,d^4q\,D(q)\bar{\psi}_f(\vec{p}_1)\gamma^\mu S(E_f-q^0;\vec{p}_1-\vec{q},\vec{p}_2+\vec{k})\not{\epsilon}S(E_i-q^0;\vec{p}_2,\vec{p}_3-\vec{q})\gamma_\mu\psi_i(\vec{p}_3). \tag{3.12}$$

The integrals given in Eqs. (3.1), (3.2) and (3.6), (3.7), (3.9)–(3.12) represent the contributions of the Furry diagrams of Fig. 3 to the radiative corrections to photoeffect. The evaluation of these terms including all orders in αZ evidently presents formidable calculational difficulties. For a lowest-order calculation, however, it is sufficient to employ the Born approximation for the electron propagator and final continuum wave function. In this way, one can evaluate explicitly the contributions of these diagrams. In the next section, we give our result for the lowest-order radiative corrections to the photoeffect matrix element obtained in this manner.

We note that it should not be too surprising that Born-approximation techniques can be employed in this case, even though the electron is initially bound. We have already indicated that the principal contribution to the photoeffect matrix element comes from the region of configuration space on the order of the electron Compton wavelength. Thus, it is the small-distance behavior of the components of the matrix element which plays the fundamental role. In these circumstances, one can generally argue that the use of the Born expansion is appropriate. As an example, it may be remarked that the difference between the evaluation of the Lamb shift, where an expansion of the propagator in a Born series is not appropriate,[13] and the formally identical calculation of the radiative corrections to hyperfine structure, where the Born expansion can be employed,[14] is due to the fact that in the Lamb shift the typical distance is the Bohr radius, whereas in the case of the radiative corrections to hyperfine structure it is the electron Compton wavelength which sets the scale. In our calculation of the lowest-order radiative corrections to photoeffect we will see that the introduction of the Born expansion for the Dirac Coulomb Green's function and the final continuum wave function does not induce any pathologies which may indicate a failure of this approximation. Spurious lower-order terms or gauge noninvariant contributions, such as are encountered in the attempt to use the Born expansion for the evaluation of the Lamb shift,[14] simply do not arise.

IV. LOWEST-ORDER RADIATIVE CORRECTIONS

With the introduction of the Born approximation in these circumstances, the calculation of the lowest-order radiative corrections to the photoeffect matrix element is straightforward. As a preliminary step, we replace the exact Green's functions which appear in the contributions of the Furry diagrams (a)–(h) by the first two terms on the right-hand side of Eq. (3.4). That is, we set

$$S(E;\vec{p},\vec{p}') = \frac{-2i}{(2\pi)^4}\left(\frac{\delta(\vec{p}-\vec{p}')}{\not{p}-1+i\epsilon} + \frac{1}{\not{p}-1+i\epsilon}\not{V}(\vec{p}-\vec{p}')\frac{1}{\not{p}'-1+i\epsilon}\right). \tag{4.1}$$

Cross terms in the resulting product which involve more than one power of the potential and all higher-order corrections to the electron propagators are neglected. At this stage, however, no approximations are made for either $\psi_f(\vec{q})$ or $\psi_i(\vec{q})$; the exact wave functions are retained. Since a separation of zero and one potential terms from the many potential contribution to the propagator would have to be done in any case in order to effect the renormalization program, it is evident that at this point no essential approximations will have been made; those terms omitted can be considered at a later stage. Moreover, since all infinite renormalization effects are contained in the zero and one potential terms, the remainder will be finite except for possible infrared divergences.

At the same time, due to the δ function which appears in (4.1), a number of momentum integrations in the resulting expressions can be done explicitly. Also, the integration over the 4-momentum of the virtual photon can be accomplished by the usual Feynman techniques. Finally, the remaining integrations involving $\psi_f(\vec{q})$ and $\psi_i(\vec{q})$ can be effected by means of the procedure outlined in Sec. II. It should, of course, be apparent that an expansion of the transition operator in powers of the potential does not correspond directly to an expansion of the matrix element in αZ. This is due to the particular analytic structure of the bound-state wave function which was noted in the derivation of the Sauter amplitude. An integration involving $\psi_i(\vec{q})$ may involve a lowering by one unit of the order in αZ of the integral in which it appears, with the result that both the zero and one potential terms contribute to lowest order in αZ. However, since only one power of αZ is involved, it is relatively easy, in the case of the radiative corrections to photoeffect, to extract the lowest-order term in αZ from the Born expansion of the transition operator. Moreover, zeroth order in the potential only contributes to zeroth order in αZ so that there are no spurious lower-order terms. We may also remark that higher-order terms in this series will involve logarithmic contributions which come from higher-order terms in the expansion of the wave functions and the transition operator. These logarithmic terms, however, are not relevant to our discussion of the lowest-order radiative corrections.

With the substitution of the form, (4.1), for the exact Dirac-Coulomb-Green's functions and the elimination of explicit δ functions, the contribution of the zero and one potential terms to the diagrams (a)–(h) of Fig. 3, after renormalization, can be written in the form

$$R_{fi} = e(2\pi)^3 \left(\int d^3p_1\, \bar{\psi}_f(\vec{p}_1) M(p_1) \psi_i(\vec{p}_1 - \vec{k}) + \int d^3p_1\, d^3p_2\, \bar{\psi}_f(\vec{p}_1) N(p_1, p_2) \psi_i(\vec{p}_2) \right). \tag{4.2}$$

In this equation, $p_1^\mu = (E_f, \vec{p}_1)$, $p_2^\mu = (E_i, \vec{p}_2)$, and $k^\mu = (\omega, \vec{k})$ is the photon 4-momentum. The zero potential contribution $M(p_1)$ arises from the product of the zero-order terms in the expansion of $S(E; \vec{p}, \vec{p}')$, and is equivalent to the diagrams of Fig. 4. $N(p_1, p_2)$ comprises the one-potential contribution and originates in the cross terms between the zero- and the one-potential parts of the electron propagators. The diagrams corresponding to $N(p_1, p_2)$ are given in Fig. 5. The explicit results for these terms are given below.

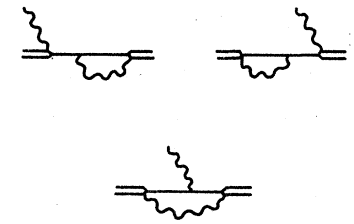

FIG. 4. Diagrams which correspond to the zero-potential contribution $M(p_1)$ of the electron propagators. Contributions due to the mass counterterms are not shown.

For $M(p_1)$, we have

$$M(p_1) = \Lambda(p_1, p_1 - k) \cdot \epsilon - C(p_1)\not{\epsilon} - \not{\epsilon} C(p_1 - k), \tag{4.3}$$

where $\Lambda^\mu(p, p')$ and $C(p)$ are the usual vertex and self-energy functions of QED. They are defined as follows (for a discussion of these functions and the vacuum polarization correction, see Bjorken and Drell, Ref. 8, Chap. 8):

$$\int \frac{d^4q}{q^2 - \lambda^2} \gamma^\mu \frac{1}{\not{p} - \not{q} - 1} \gamma_\mu = \frac{(2\pi)^4}{-4\pi\alpha i} \{\delta m_F - [Z_2^{-1} - 1 + C(p)](\not{p} - 1)\}, \tag{4.4}$$

where $\bar{u}(p)C(p) = C(p)u(p) = 0$, if $p^2 = 1$, and

$$\int \frac{d^4q}{q^2 - \lambda^2} \gamma^\nu \frac{1}{\not{p} - \not{q} - 1} \gamma^\mu \frac{1}{\not{p}' - \not{q} - 1} \gamma_\nu = \frac{(2\pi)^4}{-4\pi\alpha i} [(Z_1^{-1} - 1)\gamma^\mu + \Lambda^\mu(p, p')], \tag{4.5}$$

where $\bar{u}(p)\Lambda^\mu(p, p)u(p) = 0$, if $p^2 = 1$. Z_1 and Z_2 are the propagator and wave-function renormalization

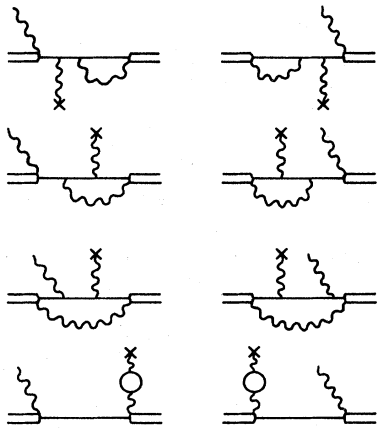

FIG. 5. Diagrams which correspond to the one potential contribution $N(p_1, p_2)$ of the electron propagators.

constants. By Ward's identity,[15] $Z_1 = Z_2$. λ is a small photon mass which is necessary to prevent infrared divergence.

The one potential contribution is given by

$$N(p_1, p_2) = C(p_1)\not{V}(\vec{p}_1 - \vec{p}_2 - \vec{k})\frac{1}{\not{p}_2 + \not{k} - 1}\not{\epsilon} + \not{\epsilon}\frac{1}{\not{p}_1 - \not{k} - 1}\not{V}(\vec{p}_1 - \vec{p}_2 - \vec{k})C(p_2)$$

$$+\Lambda(p_1, p_2 + k)\cdot V(\vec{p}_1 - \vec{p}_2 - \vec{k})\frac{1}{\not{p}_2 + \not{k} - 1}\not{\epsilon} + \not{\epsilon}\frac{1}{\not{p}_1 - \not{k} - 1}\Lambda(p_1 - k, p_2)\cdot V(\vec{p}_1 - \vec{p}_2 - \vec{k})$$

$$+V(\vec{p}_1 - \vec{p}_2 - \vec{k})\cdot T(p_1, p_2 + k, p_2)\cdot\epsilon + \epsilon\cdot T(p_1, p_1 - k, p_2)\cdot V(\vec{p}_1 - \vec{p}_2 - \vec{k})$$

$$-\Pi((p_1 - p_2 - k)^2)\left(\not{V}(\vec{p}_1 - \vec{p}_2 - \vec{k})\frac{1}{\not{p}_2 + \not{k} - 1}\not{\epsilon} + \not{\epsilon}\frac{1}{\not{p}_1 - \not{k} - 1}\not{V}(\vec{p}_1 - \vec{p}_2 - \vec{k})\right), \tag{4.6}$$

where $C(p)$ and $\Lambda^\mu(p, p')$ are defined by (4.4) and (4.5) and $\Pi(p^2)$, the vacuum polarization function, is defined by

$$\int d^4q\,\mathrm{Tr}\left(\gamma^\mu\frac{1}{\not{q} - \not{p} - 1}\gamma^\nu\frac{1}{\not{q} - 1}\right) = \frac{(2\pi)^4}{-4\pi\alpha i}(g^{\mu\nu}p^2 - p^\mu p^\nu)[1 - Z_3 + \Pi(p^2)], \tag{4.7}$$

with $\Pi(0) = 0$. Z_3 is the charge renormalization constant. The remaining function, $T^{\mu\nu}(p_1, p_2, p_3)$, is defined by

$$\int\frac{d^4q}{q^2 - \lambda^2}\gamma^\sigma\frac{1}{\not{p}_1 - \not{q} - 1}\gamma^\mu\frac{1}{\not{p}_2 - \not{q} - 1}\gamma^\nu\frac{1}{\not{p}_3 - \not{q} - 1}\gamma_\sigma = \frac{(2\pi)^4}{-4\pi\alpha i}T^{\mu\nu}(p_1, p_2, p_3), \tag{4.8}$$

and is discussed, in a different context, by Brown and Feynman.[16] We note that, because of Furry's theorem[17] and the fact that $k^2 = 0$, diagram (g) [Eq. (3.11)] only contributes to the renormalization of the electric charge in lowest order.

It can easily be seen that all infinite renormalization effects, expressed through the renormalization constants Z_1, Z_2, Z_3, and δm_F, are included in the zero and one potential terms. All remaining contributions are finite except for possible infrared divergences. Moreover, it is evident that these renormalization constants only appear implicitly in our expression (4.2) in terms of the definition of the physical charge and mass of the electron. Hence no further renormalization is necessary, to any order in αZ, for the calculation of the radiative corrections to photoeffect to lowest order in the radiation field (order α). In the appendix we give an explicit demonstration of the cancellation of ultraviolet divergences for this case.

The evaluation of the lowest order (in αZ) contribution to Eq. (4.2) is readily accomplished using the approach of Sec. II, so that our derivation of the lowest order radiative corrections parallels that of the Sauter amplitude. To proceed we consider first the zero-potential contribution. We define

$$M_{fi}^{(n)} = e(2\pi)^3\int d^3p_1\,\bar{\psi}_f^{(n)}(\vec{p}_1)M(p_1)\psi_i(\vec{p}_1 - \vec{k}), \tag{4.9}$$

where $\psi_f^{(n)}(\vec{p}_1)$, defined by Eq. (2.4), is the nth Born term in the expansion of the final continuum wave function. The lowest-order contribution of the zero-potential term is then given by

$$M_{fi} = M_{fi}^{(0)} + M_{fi}^{(1)}, \tag{4.10}$$

where only the lowest-order part of $M_{fi}^{(1)}$ is retained [cf. Eq. (2.10)]. Using the expression (2.6) for the ground-state wave function, $M_{fi}^{(0)}$ can be evaluated immediately. We have

$$M_{fi}^{(0)} = \frac{e(2\pi)^3N}{|\vec{p} - \vec{k}|^4}\bar{u}_f(p)[\Lambda(p, p - k)\cdot\epsilon - C(p)\not{\epsilon} - \not{\epsilon}C(p - k)] \times[1 + \tfrac{1}{2}(\vec{p} - \vec{k})\cdot\vec{\alpha}]u_i(l) \tag{4.11}$$

or, using the relation, (2.13),

$$M_{fi}^{(0)} = e(2\pi)^3(\alpha Z)^{-1}\pi^2N\bar{u}_f(p)[\Lambda(p, p - k)\cdot\epsilon - \not{\epsilon}C(p - k)] \times\frac{1}{\not{p} - \not{k} - 1}V(\vec{p} - \vec{k})u_i(l). \tag{4.12}$$

In (4.12), we have also used the fact that $\bar{u}_f(p)C(p) = 0$, to eliminate one of the self-energy functions.

The lowest-order contribution to $M_{fi}^{(1)}$ can be obtained by means of the approximation, (2.8), for the bound-state wave function. Because of the δ function, the integration is immediate. We find

$$M_{fi}^{(1)} = e(2\pi)^3(\alpha Z)^{-1}\pi^2N\bar{u}_f(p)\not{V}(\vec{p} - \vec{k})\frac{1}{\not{l} + \not{k} - 1} \times[\Lambda(l + k, l)\cdot\epsilon - C(l + k)\not{\epsilon}]u_i(l), \tag{4.13}$$

where we have used the fact that $C(l)u_i(l) = 0$.

Since $N(p_1, p_2)$ contains the potential once, we only need the zeroth Born term in the expansion

of the final continuum electron wave function. Moreover, to lowest order we can also use the form (2.8) for the bound-state wave function. Thus, both of the momentum integrations in (4.2) are determined by δ functions. We see immediately that the lowest-order contribution of the first two terms of (4.6) is zero, since $\bar{u}_f(p)C(p) = C(l)u_i(l) = 0$. The contribution of the last two terms of (4.6), the vacuum-polarization correction, is also simple since the result is proportional to the Sauter amplitude [cf. Eq. (2.14).] Explicitly, these terms yield

$$N_{fi}^{(e)} + N_{fi}^{(f)} = -\Pi((p-k-l)^2) S_{fi}\,. \tag{4.14}$$

This result could have been anticipated, since it is known[8] that the only effect of vacuum polarization, to lowest order, is to modify the potential by a multiplicative factor. As the Sauter amplitude is linear in $V(\vec{q})$, the form (4.14) follows immediately.

The remaining terms of $N(p_1, p_2)$, corresponding to the first-order self-energy and vertex corrections, are somewhat more complicated. After doing the integrations we have, to lowest order,

$$\begin{aligned} &N_{fi}^{(a)} + N_{fi}^{(b)} + N_{fi}^{(c)} + N_{fi}^{(d)} + N_{fi}^{(h)} \\ &= e(2\pi)^3(\alpha Z)^{-1}\pi^2 N \bar{u}_f(p)\Big\{\Lambda(p, l+k)\cdot V(\vec{p}-\vec{k})\frac{1}{\not l + \not k - 1}\not\epsilon + \not\epsilon\frac{1}{\not p - \not k - 1}\Lambda(p-k, l)\cdot V(\vec{p}-\vec{k}) \\ &\quad + V(\vec{p}-\vec{k})\cdot T(p, l+k, l)\cdot\epsilon + \epsilon\cdot T(p, p-k, l)\cdot V(\vec{p}-\vec{k})\Big\} u_i(l)\,. \end{aligned} \tag{4.15}$$

This form, however, while relatively simple in appearance, is not particularly convenient for later applications. Using the expression, (4.8), which defines $T^{\mu\nu}(p_1, p_2, p_3)$, it is trivial to show that

$$\bar{u}_f(p)V\cdot T(p, l+k, l)\cdot\epsilon u_i(l) = \bar{u}_f(p)\left[\frac{l\cdot\epsilon}{l\cdot k}\Lambda(p, l)\cdot V - \frac{l\cdot\epsilon}{l\cdot k}\Lambda(p, l+k)\cdot V + T(p, l+k, l)\cdot\left(\epsilon - \frac{l\cdot\epsilon}{l\cdot k}k\right)\right]u_i(l), \tag{4.16}$$

with a similar result for $\bar{u}_f(p)\epsilon\cdot T(p, p-k, l)\cdot V u_i(l)$. Inserting these expressions into (4.15), and rearranging terms, we can rewrite that equation in the form,

$$\begin{aligned} N_{fi}^{(a)} + N_{fi}^{(b)} + N_{fi}^{(c)} + N_{fi}^{(d)} + N_{fi}^{(h)} &= e(2\pi)^3(\alpha Z)^{-1}\pi^2 N\bar{u}_f(p)\Big\{\left(\frac{l\cdot\epsilon}{l\cdot k} - \frac{p\cdot\epsilon}{p\cdot k}\right)\Lambda(p, l)\cdot V(\vec{p}-\vec{k}) + \Lambda(p, l+k)\cdot V(\vec{p}-\vec{k})\frac{\not k\not\epsilon}{2l\cdot k} \\ &\quad + \frac{\not\epsilon\not k}{2p\cdot k}\Lambda(p-k, l)\cdot V(\vec{p}-\vec{k}) + V(\vec{p}-\vec{k})\cdot T(p, l+k, l)\cdot a(l) \\ &\quad + a(p)\cdot T(p, p-k, l)\cdot V(\vec{p}-\vec{k})\Big\} u_i(l)\,. \end{aligned} \tag{4.17}$$

This result (4.17) is somewhat easier to deal with analytically, since it is manifestly gauge invariant. Moreover, the introduction of the 4-vector, $a^\mu(q) = \epsilon^\mu - (q\cdot\epsilon/q\cdot k)k^\mu$, eliminates about one third of the terms present in $T^{\mu\nu}(p_1, p_2, p_3)$, since $k\cdot a(q) = q\cdot a(q) = 0$.

Collecting terms, we have the final result for the lowest-order radiative corrections to the photoeffect matrix element:

$$\begin{aligned} R_{fi} &= \bar{u}_f(p)R u_i(l) \\ &= M_{fi} + N_{fi} \\ &= e(2\pi)^3(\alpha Z)^{-1}\pi^2 N\bar{u}_f(p)\Big\{[\Lambda(p, p-k)\cdot\epsilon - \not\epsilon C(p-k)]\frac{1}{\not p - \not k - 1}\not V(\vec{p}-\vec{k}) \\ &\quad - \Pi((p-k-l)^2)\not\epsilon\frac{1}{\not p - \not k - 1}\not V(\vec{p}-\vec{k}) + \frac{l\cdot\epsilon}{l\cdot k}\Lambda(p, l)\cdot V(\vec{p}-\vec{k}) + \Lambda(p, l+k)\cdot V(\vec{p}-\vec{k})\frac{\not k\not\epsilon}{2l\cdot k} \\ &\quad + \not V(\vec{p}-\vec{k})\cdot T(p, l+k, l)\cdot a(l) + [p\leftarrow l, k\leftarrow -k]^\dagger\Big\} u_i(l)\,, \end{aligned} \tag{4.18}$$

where $[p \rightarrow l, k \rightarrow -k]^\dagger$ denotes the matrix adjoint of the preceding expression followed by the substitutions, $p \rightarrow l, k \rightarrow -k$. Thus the transition matrix R defined by (4.18) is Hermitian. We also note that the lowest-order radiative corrections defined by (4.18) are gauge invariant. That is, R_{fi} is invariant under the substitutions,

$$\epsilon^\mu \rightarrow \epsilon^\mu + \xi k^\mu \tag{4.19}$$

and

$$V^\mu(\vec{p}-\vec{k}) \rightarrow V^\mu(\vec{p}-\vec{k}) + \xi(p-k-l)^\mu , \tag{4.20}$$

where ζ and ξ are arbitrary.

At this point, it is convenient to display explicitly the infrared divergent part of R_{fi}. We note that each of the functions, $C(p)$, $\Lambda^\mu(p,p')$, and $T^{\mu\nu}(p_1,p_2,p_3)$ is infinite in the limit, $\lambda \rightarrow 0$. Summing the infrared-divergent terms,[8,16] we find that R_{fi} can be written in the form[18]

$$R_{fi} = \frac{\alpha}{2\pi}\left[-\left(\tfrac{5}{2}+\ln\lambda^2\right)+2\nu F_0(p,l)\right] S_{fi} + \bar{R}_{fi} , \tag{4.21}$$

where $\bar{R}_{fi}$ is independent of λ and S_{fi} is the Sauter amplitude (2.14). The function $F_0(p,l)$ is defined by

$$F_0(p,l) = -\frac{1}{2}\int_0^1 \frac{dx}{2p_x^2}\ln\left(\frac{p_x^2}{\lambda^2}\right) , \tag{4.22}$$

where $p_x^2 = 1-(2-\nu)x(1-x)$ and $\nu = 2l\cdot p$. This infrared divergence will be eliminated, in the final results, by the addition of the lowest-order Compton cross section, in which the incident photon produces a final real photon on interacting with the electron. In Sec. VI we will evaluate this contribution explicitly, assuming the energy of the emitted photon is small compared to the electron rest energy.

V. INVARIANT AMPLITUDES

The functions $C(p)$, $\Lambda^\mu(p,p')$, $T^{\mu\nu}(p_1,p_2,p_3)$, and $\Pi(q^2)$ which appear in our expression (4.18) for the lowest-order radiative corrections are rather complicated. Moreover, although these functions have been evaluated for certain kinematical situations, in general the kinematics appropriate to photoeffect have not been considered. In these circumstances it would be difficult to give a detailed derivation of the reduction of this expression to invariant amplitudes. Accordingly, we will only present here essentially the final results.

In order to simplify the evaluation of our result (4.18), we introduce a particular set of linearly independent spinor basis functions Y_n. In this basis the matrix element R_{fi} will have the form,

$$R_{fi} = \sum_{n=1}^{4} A_n(\omega,\cos\theta)\bar{u}_f(p)Y_n u_i(l) , \tag{5.1}$$

where $\cos\theta = \hat{p}\cdot\hat{k}$ is the cosine of the scattering angle. We note that, because of parity and time-reversal invariance, there are only four scalar amplitudes, A_n. The spinor basis functions which we choose are combinations of Dirac matrices rather than the usual linear combination of Pauli spin matrices. In this way we can maintain explicit relativistic covariance and gauge invariance. This not only simplifies the expression for the matrix element; it will also make the trace calculations for the cross section somewhat easier to perform. Thus we define

$$\begin{aligned} Y_1 &= 2\pi^2(\alpha Z)^{-1}\not{\epsilon}\not{k}\not{V}, \quad Y_2 = 2\pi^2(\alpha Z)^{-1}\not{V}\not{k}\not{\epsilon} , \\ Y_3 &= 2\pi^2(\alpha Z)^{-1}[(2l\cdot k)(2p\cdot\epsilon)-(2l\cdot\epsilon)(2p\cdot k)]\not{V} , \\ Y_4 &= 2\pi^2(\alpha Z)^{-1}[(2p\cdot k)\not{\epsilon}\not{V}-(2p\cdot\epsilon)\not{k}\not{V}] , \end{aligned} \tag{5.2}$$

where $\not{V} = \not{V}(\vec{p}-\vec{k})$ is given by Eq. (2.5). Although these basis functions are manifestly gauge invariant with respect to the incident photon, the particular forms (5.2) imply the choice of the Coulomb gauge for $V^\mu(\vec{q})$ in (4.18).

The set of spinor basis functions (5.2) is a natural set, in that reduction of our expression (4.18) to invariant amplitudes is achieved most easily with this basis. For completeness, however, we should indicate the relation between the Y_n and the usual Pauli matrices. In the Lorentz gauge, $\epsilon^\mu = (0,\vec{\epsilon})$, we have

$$\begin{aligned} \bar{u}_f(p)Y_1u_i(l) &= \frac{-[\frac{1}{2}(E_f+1)]^{1/2}}{|\vec{p}-\vec{k}|^2}\,\chi_f^\dagger\left(i\vec{\sigma}\cdot(\vec{k}\times\vec{\epsilon})-\frac{\omega}{E_f+1}[\vec{p}\cdot\vec{\epsilon}+i\vec{\sigma}\cdot(\vec{p}\times\vec{\epsilon})]\right)\chi_i , \\ \bar{u}_f(p)Y_2u_i(l) &= \frac{-[\frac{1}{2}(E_f+1)]^{1/2}}{|\vec{p}-\vec{k}|^2}\,\chi_f^\dagger\left(-i\vec{\sigma}\cdot(\vec{k}\times\vec{\epsilon})-\frac{\omega}{E_f+1}[\vec{p}\cdot\vec{\epsilon}+i\vec{\sigma}\cdot(\vec{p}\times\vec{\epsilon})]\right)\chi_i , \\ \bar{u}_f(p)Y_3u_i(l) &= \frac{-[\frac{1}{2}(E_f+1)]^{1/2}}{|\vec{p}-\vec{k}|^2}\,(2\omega)(-2\vec{p}\cdot\vec{\epsilon})\chi_f^\dagger\chi_i , \\ \bar{u}_f(p)Y_4u_i(l) &= \frac{-[\frac{1}{2}(E_f+1)]^{1/2}}{|\vec{p}-\vec{k}|^2}\,\frac{1}{E_f+1}\,\chi_f^\dagger\{(2\vec{p}\cdot\vec{\epsilon})i\vec{\sigma}\cdot(\vec{p}\times\vec{k})+2\omega(\vec{p}\cdot\vec{\epsilon})-(2p\cdot k)i\vec{\sigma}\times(\vec{p}\times\vec{\epsilon})\}\chi_i , \end{aligned} \tag{5.3}$$

where $\chi_1 = \binom{1}{0}$ and $\chi_2 = \binom{0}{1}$ are Pauli spinors describing the initial and final electron-spin states. Hence, the Y_n are equivalent to a description in terms of the four linearly independent Pauli basis functions, $\vec{p}\cdot\vec{\epsilon}$, $\vec{\sigma}\cdot(\vec{k}\times\vec{\epsilon})$, $\vec{\sigma}\cdot(\vec{p}\times\vec{\epsilon})$, and $\vec{p}\cdot\vec{\epsilon}$ $\vec{\sigma}\cdot(\vec{p}\times\vec{k})$. By choosing a particular coordinate system, these reduce to the spinor basis functions used in other calculations.[4]

In terms of the Y_n, the Sauter amplitude (2.14) has the form,

$$S_{fi} = \tfrac{1}{2}e(2\pi)^3 N\bar{u}_f(p)\left(\frac{1}{\kappa}Y_1 + \frac{1}{\tau}Y_2 - \frac{1}{\kappa\tau}Y_3\right)u_i(l), \tag{5.4}$$

where, we recall, $N = (\alpha Z/\pi)^{5/2}$. The invariants κ and τ are defined by

$$\kappa = 2p\cdot k = 2E_f\omega(1-\beta\cos\theta), \quad \tau = 2l\cdot k = 2\omega, \tag{5.5}$$

where $\beta = |\vec{p}|/E_f$ is the velocity of the emitted electron. Note that, neglecting quantities of order $(\alpha Z)^2$, κ is just the square of the momentum transferred to the nucleus,

$$\kappa = |\vec{p} - \vec{k}|^2. \tag{5.6}$$

The infrared-divergence-free part of the radiative corrections to the photoeffect, $\bar{R}_{fi}$, which is defined by Eq. (4.21), can be written in the form

$$\bar{R}_{fi} = \tfrac{1}{2}e(2\pi)^3 N\frac{\alpha}{2\pi}\sum_{n=1}^{4} R_n(\kappa,\tau)\bar{u}_f(p)Y_n u_i(l), \tag{5.7}$$

where the R_n are independent of α and αZ. For simplicity, we write R_n in the form

$$R_n = M_n + F_n + G_n + J_n + P_n\ , \tag{5.8}$$

where each term in (5.8) corresponds to one term in (4.18). Explicitly,

$$\frac{\alpha}{2\pi}\sum_{n=1}^{4} M_n\bar{u}_f(p)Y_n u_i(l) = 2\pi^2(\alpha Z)^{-1}\bar{u}_f(p)\left([\Lambda(p,p-k)\cdot\epsilon - \not{\epsilon}C(p-k)]\frac{1}{\not{p}-\not{k}-1}\not{V}(\vec{p}-\vec{k}) + [p\rightarrow l, k\rightarrow -k]^{\dagger}\right)u_i(l) \tag{5.9}$$

is the zero-potential contribution. F_n is defined by

$$\frac{\alpha}{2\pi}\sum_{n=1}^{4} F_n\bar{u}_f(p)Y_n u_i(l)$$

$$= 2\pi^2(\alpha Z)^{-1}\bar{u}_f(p)\left(\frac{l\cdot\epsilon}{l\cdot k}[\Lambda(p,l)\cdot V(\vec{p}-\vec{k}) + (\tfrac{5}{2}+\ln\lambda^2)\not{V}(\vec{p}-\vec{k}) - 2\nu F_0(p,l)\not{V}(\vec{p}-\vec{k})] + [p\rightarrow l, k\rightarrow -k]^{\dagger}\right)u_i(l), \tag{5.10}$$

where $F_0(p,l)$ is given by Eq. (4.22). G_n is defined by

$$\frac{\alpha}{2\pi}\sum_{n=1}^{4} G_n\bar{u}_f(p)Y_n u_i(l) = 2\pi^2(\alpha Z)^{-1}\bar{u}_f(p)\left([\Lambda(p,l+k)\cdot V(\vec{p}-\vec{k}) + (\tfrac{5}{2}+\ln\lambda^2)\not{V}(\vec{p}-\vec{k})]\frac{\not{k}\not{\epsilon}}{2l\cdot k} + [p\rightarrow l, k\rightarrow -k]^{\dagger}\right)u_i(l), \tag{5.11}$$

while J_n is determined by

$$\frac{\alpha}{2\pi}\sum_{n=1}^{4} J_n\bar{u}_f(p)Y_n u_i(l)$$

$$= 2\pi^2(\alpha Z)^{-1}\bar{u}_f(p)\left(\not{V}(\vec{p}-\vec{k})\cdot T(p,l+k,l)\cdot a(l) - 2\nu F_0(p,l)\not{V}(\vec{p}-\vec{k})\frac{\not{k}\not{\epsilon}}{2l\cdot k} + [p\rightarrow l, k\rightarrow -k]^{\dagger}\right)u_i(l). \tag{5.12}$$

Finally, P_n gives the vacuum polarization contribution,

$$\frac{\alpha}{2\pi}\sum_{n=1}^{4} P_n\bar{u}_f(p)Y_n u_i(l) = 2\pi^2(\alpha Z)^{-1}\bar{u}_f(p)\left(-\Pi((p-k-l)^2)\not{\epsilon}\frac{1}{\not{p}-\not{k}-1}\not{V}(\vec{p}-\vec{k}) + [p\rightarrow l, k\rightarrow -k]^{\dagger}\right)u_i(l). \tag{5.13}$$

Of these terms, the simplest is the vacuum-polarization contribution since it is proportional to the Sauter amplitude. Moreover, the integral (4.7) defining $\Pi(q^2)$ can be done analytically.[8] We have

$$-\Pi((p-k-l)^2) = -\Pi(-\kappa) = (\alpha/2\pi)P(\kappa), \tag{5.14}$$

where

$$P(\kappa) = -\frac{10}{9} + \frac{8}{3}\frac{1}{\kappa} + \frac{2}{3}\left(\frac{\kappa-2}{\kappa}\right)\left(\frac{\kappa+4}{\kappa}\right)^{1/2}\ln\left(\frac{(\kappa+4)^{1/2}+\kappa^{1/2}}{(\kappa+4)^{1/2}-\kappa^{1/2}}\right). \tag{5.15}$$

Hence, using (5.4) and (4.14),

$$\kappa P_1 = \tau P_2 = -\kappa\tau P_3 = P, \quad P_4 = 0. \tag{5.16}$$

The integrals which occur in the zero-potential terms M_n can similarly be evaluated analytically, since the square of the momentum transfer in $\Lambda^\mu(p, p-k)$ is $k^2 = 0$. Calculating $\Lambda^\mu(p, p-k)$ and $C(p)$ and simplifying, we have

$$\sum_{n=1}^{4} M_n \bar{u}_f(p) Y_n u_i(l) = 2\pi^2(\alpha Z)^{-1}\bar{u}_f(p)\left(\{I_1(1-\kappa)\not{k}\not{\epsilon} + I_2(1-\kappa)[(2p\cdot k)\not{\epsilon} - (2p\cdot\epsilon)\not{k}]\}\frac{1}{\not{p}-\not{k}-1}\gamma^0(\vec{p}-\vec{k}) + [p \rightarrow l, k \rightarrow -k]^\dagger\right)u_i(l), \tag{5.17}$$

where

$$I_1(1-\kappa) = \frac{1}{2(1-\kappa)}\left(1 + \frac{\kappa}{1-\kappa}\ln\kappa\right),$$

$$I_2(1-\kappa) = \frac{1}{\kappa^2}\left(\frac{\pi^2}{6} - L_2(1-\kappa)\right) - \frac{1}{2(1-\kappa)}\left(\frac{2-\kappa}{\kappa}\right) \times\left[1 - \ln\kappa + \left(\frac{\kappa}{1-\kappa}\right)\ln\kappa\right] \tag{5.18}$$

and $L_2(z)$ is the Euler dilogarithm,

$$L_2(z) = -\int_0^z \frac{dx}{x}\ln(1-x). \tag{5.19}$$

Expanding the right-hand side of (5.17) in terms of the Y_n, we find

$$M_1 = (2/\kappa)I_1(1-\kappa) + I_2(1-\kappa) - (1/\tau)I_1(1+\tau),$$
$$M_2 = (1/\tau)I_1(1+\tau) - I_2(1+\tau), \tag{5.20}$$
$$M_3 = 0, \quad M_4 = (1/\kappa)I_1(1-\kappa),$$

where $I_j(1+\tau)$ is obtained from (5.18) by the substitution, $\kappa \rightarrow -\tau$. We note that the functions $I_j(1+\tau)$ have nonzero imaginary parts. In order to fix the sign of $\mathrm{Im}I_j(1+\tau)$ unambiguously, we recall that according to the Feynman prescription, the masses of the electron and photon have infinitesimal negative imaginary parts. This implies that τ has an infinitesimal negative imaginary part, so that

$$\ln(-\tau) = \ln|\tau| + i\pi,$$
$$L_2(1+\tau) = -\int_0^{1+\tau}\frac{dx}{x}\ln|1-x| + i\pi\ln(1+\tau). \tag{5.21}$$

On the other hand, $I_j(1-\kappa)$ is real since $\kappa > 0$.

The expression for F_n, defined by Eq. (5.10), can likewise be evaluated analytically since, in the vertex function $\Lambda^\mu(p, l)$, both momenta are on the mass shell. We find simply

$$F_1 = F_2 = F_4 = 0,$$
$$F_3 = \frac{-1}{2\kappa\tau}\left[1 + 2\left(\frac{\tau+4}{\tau}\right)^{1/2}\ln\left(\frac{(\tau+4)^{1/2} + \tau^{1/2}}{(\tau+4)^{1/2} - \tau^{1/2}}\right)\right]. \tag{5.22}$$

The remaining terms, G_n and J_n, are quite complicated. Moreover, it is not feasible to evaluate the Feynman parameter integrals which occur analytically.[19] Accordingly, we will express these contributions to the invariant amplitudes in terms of linear combinations of certain definite integrals.

For G_n we write

$$G_n = \sum_{r=0}^{3}\sum_{s=1}^{2} G_n^{rs}(\kappa, \tau), \tag{5.23}$$

where the integrals G_n^{rs} have the form

$$G_n^{rs}(\kappa, \tau) = \int_0^1 dy\, \Gamma_n^{rs}(\kappa, \tau; y) g_{rs}(\kappa, \tau; y). \tag{5.24}$$

In (5.24), the Γ_n^{rs} are polynomials in y and the g_{rs} are logarithmic functions of their arguments. Of the 32 integrals indicated in Eq. (5.23), only 14 are nonzero. For these integrals, the explicit forms of Γ_n^{rs} and g_{rs} are given in Table I.

For J_n we define

$$J_n = \sum_{r=0}^{1}\sum_{s=1}^{2} H_n^{rs}(\kappa, \tau) + \sum_{r=0}^{3}\sum_{s=1}^{2} J_n^{rs}(\kappa, \tau), \tag{5.25}$$

where H_n^{rs} can be written

$$H_n^{rs}(\kappa, \tau) = \int_0^1 dx \int_0^1 dy\, \theta_n^{rs}(\kappa, \tau; x, y) h_{rs}(\kappa, \tau; x, y), \tag{5.26}$$

and J_n^{rs} has the form

$$J_n^{rs}(\kappa, \tau) = \int_0^1 dx \int_0^1 dy\, K_n^{rs}(\kappa, \tau; x, y) j_{rs}(\kappa, \tau; x, y). \tag{5.27}$$

θ_n^{rs} and K_n^{rs} are polynomials in x and y. h_{rs} and j_{rs} are logarithmic in κ, τ, x, and y. The arguments of the integral in Eq. (5.26) for ten nonzero H_n^{rs} are given in Table II, while the arguments of the integral in Eq. (5.27) for the 26 nonzero J_n^{rs} are given in Table III.

Each of the 50 nonzero Feynman parameter integrals G_n^{rs}, H_n^{rs}, and J_n^{rs} given in Tables I–III is is well defined[20] for all (finite) values of κ and τ. Moreover, the imaginary parts of G_n^{r1}, H_n^{r1} and J_n^{r1} are determined unambiguously from the usual

Feynman prescription which implies that τ has a small negative imaginary part. (G_n^{rs}, H_n^{rs}, and J_n^{rs} are real for physical values of κ and τ.) Thus, while an analytic treatment of these integrals is not feasible, they are amenable to numerical evaluation.

Although the number of integrals which must be considered in order to obtain a complete description of the radiative corrections to photoeffect, including all polarization phenomena, is rather large, some simplification results when we restrict the discussion to the unpolarized cross section. In this case, the number of distinct integrals which must be evaluated is reduced from 50 to 20 since the sum over polarizations results in the replacement of the spinor basis functions by scalar functions of the invariants κ and τ so that integrands for the same values of r and s can be combined to form a single integral. Moreover, only the real part contributes to the cross section, so that the imaginary parts can be ignored. In the following section we derive the expression for the lowest order radiative corrections to the photoeffect differential cross section.

VI. CROSS SECTION

With the normalization which we have adopted, the cross section for photoeffect can be written

$$\frac{d\sigma_{fi}}{d\Omega} = \frac{1}{(2\pi)^2}\,\frac{|\vec{p}|}{2\omega}\,|T_{fi}|^2\,, \tag{6.1}$$

where T_{fi} is the total amplitude. Including radiative corrections to lowest order in αZ, T_{fi} is given by

$$T_{fi} = S_{fi} + \frac{\alpha}{2\pi}\left[-\left(\tfrac{5}{2} + \ln\lambda^2\right) + 2\nu F_0(p,l)\right]S_{fi} + \overline{R}_{fi}\,, \tag{6.2}$$

TABLE I. The functions Γ_n^{rs} and g_{rs} given in Eq. (5.24) which comprise the integrands of the 14 nonzero integrals G_n^{rs}. In these functions, $p_y^2 = 1 + \tau y + \kappa y(1-y)$, $q_y^2 = 1 - \kappa y^2$, and $\nu = \tau + 2$.

G_n^{rs}	$\Gamma_n^{rs}(\kappa,\tau;y)$	$g_{rs}(\kappa,\tau;y)$
G_2^{01}	$1/\tau$	$-\frac{1}{2}\ln(p_y^2 - y\tau) + \frac{y\tau}{2p_y^2} + \frac{(y\tau)^2}{2(p_y^2)^2}\ln\frac{p_y^2 - y\tau}{-y\tau}$
G_1^{02}	$1/\kappa$	$-\frac{1}{2}\ln(q_y^2 + y\kappa) - \frac{y\kappa}{2q_y^2} + \frac{(y\kappa)^2}{2(q_y^2)^2}\ln\frac{q_y^2 + y\kappa}{y\kappa}$
G_1^{11}	$(\nu+1)/\tau$	$\frac{1}{p_y^2}\ln\frac{p_y^2 - y\tau}{-y\tau}$
G_2^{11}	$-(\kappa-1)/\tau$	
G_1^{12}	$-2/\kappa$	$\frac{1}{q_y^2}\ln\frac{q_y^2 + y\kappa}{y\kappa}$
G_4^{12}	$1/\kappa$	
G_1^{21}	$-(1+\nu y)/\tau$	$\frac{1}{p_y^2}\left(1 + \frac{y\tau}{p_y^2}\ln\frac{p_y^2 - y\tau}{-y\tau}\right)$
G_2^{21}	$(\kappa + 1 - 2y)/\tau$	
G_1^{22}	$(2-\kappa y)/\kappa$	$\frac{1}{q_y^2}\left(1 - \frac{y\kappa}{q_y^2}\ln\frac{q_y^2 + y\kappa}{y\kappa}\right)$
G_4^{22}	$-(1+2y)/\kappa$	
G_1^{31}	$-\nu(1-y)/\tau$	$\frac{1}{p_y^2}\left[\frac{1}{2} + \frac{y\tau}{p_y^2}\left(1 + \frac{y\tau}{p_y^2}\ln\frac{p_y^2 - y\tau}{-y\tau}\right)\right]$
G_2^{31}	$-[1-(2-\kappa)y - \kappa y^2]/\tau$	
G_1^{32}	$(1+\kappa y^2)/\kappa$	$\frac{1}{q_y^2}\left[\frac{1}{2} - \frac{y\kappa}{q_y^2}\left(1 - \frac{y\kappa}{q_y^2}\ln\frac{q_y^2 + y\kappa}{y\kappa}\right)\right]$
G_4^{32}	$2y/\kappa$	

where S_{fi} is the Sauter amplitude (2.14) and $\bar{R}_{fi}$ is defined by (4.21). Hence,

$$\frac{d\sigma_{fi}}{d\Omega}=\frac{d\sigma_{fi}^{(0)}}{d\Omega}+\frac{\alpha}{\pi}\left[-(\tfrac{5}{2}+\ln\lambda^2)+2\nu F_0(p,l)\right]\frac{d\sigma_{fi}^{(0)}}{d\Omega}$$
$$+\frac{1}{(2\pi)^2}\frac{|\vec{p}|}{2\omega}\left[\bar{R}_{fi}S_{fi}^{\dagger}+S_{fi}\bar{R}_{fi}^{\dagger}\right], \qquad (6.3)$$

where $d\sigma_{fi}^{(0)}/d\Omega$ is the analog of the Sauter cross section including polarization. The second term of (6.3), which is also proportional to the Sauter cross section, contains the infrared divergent part of $d\sigma_{fi}/d\Omega$.

In order to relate (6.3) to the physical cross section, it is necessary to take into account the fact that any experiment will have a finite energy resolution ΔE in regard to the possibility of emission of secondary photons; i.e., any number of photons of total energy less than ΔE could be emitted in the process. Then, for consistency to first order in in α, we must include the possibility that one photon of energy less than ΔE is also emitted along with the final electron. Thus, to obtain the physical cross section we must add to (6.3) the cross section for the inelastic scattering of a photon with final energy less than ΔE by the bound K-shell electron (Compton scattering). This addition of the Compton cross section will allow the elimination of the photon mass λ from our expression (6.3) and, hence, give a result which can be compared with experiment.[21]

The Compton amplitude for the scattering from a bound electron is described by the Furry diagrams of Fig. 6. For our purposes, however, we only need to evaluate these diagrams for the K-shell to lowest order in αZ. This can be accomplished without difficulty using the Born approximation for the intermediate electron propagators and essentially the same procedure as that outlined for the calculation of the radiative corrections. Hence, we will omit details of this calculation and simply present the final result for the lowest-order contribution to the Compton-effect transition matrix element G_{fi} assuming the initial and final electron states in this case are the same as in our expression (4.18) for the radiative corrections to photoeffect. We find

TABLE II. The functions Θ_n^{rs} and h_{rs} given in Eq. (5.26) which comprise the integrands of the 10 nonzero integrals H_n^{rs}. In these functions, $p_x^2=x\tau+[\tau y(1-y)-\kappa y](1-x)^2+\kappa y(1-x)+1$, $q_x^2=-x\kappa+[\tau y(1-y)-\kappa y](1-x)^2+\kappa y(1-x)+1$, and $\nu=\tau+2$.

H_n^{rs}	$\Theta_n^{rs}(\kappa,\tau;x,y)$	$h_{rs}(\kappa,\tau;x,y)$
H_2^{01}	$-(1-x)$	$\frac{1}{p_x^2}\left(1+\frac{x\tau}{p_x^2}\ln\frac{p_x^2-x\tau}{-x\tau}\right)$
H_1^{02}	$(1-x)$	$\frac{1}{q_x^2}\left(1-\frac{x\kappa}{q_x^2}\ln\frac{q_x^2+x\kappa}{x\kappa}\right)$
H_3^{02}	$(1-x)/\kappa$	
H_4^{02}	$(1-x)(\nu+2)/\kappa$	
H_1^{11}	$(1-x)[(\nu+2)(1-x)y-x\tau]/\tau$	$\frac{1}{p_x^2}\left[\frac{1}{2}+\frac{x\tau}{p_x^2}\left(1+\frac{x\tau}{p_x^2}\ln\frac{p_x^2-x\tau}{-x\tau}\right)\right]$
H_2^{11}	$(1-x)[(\nu+2)(1-x)y+x\tau]/\tau$	
H_1^{12}	$-x(1-x)$	$\frac{1}{q_x^2}\left[\frac{1}{2}-\frac{x\kappa}{q_x^2}\left(1-\frac{x\kappa}{q_x^2}ln\frac{q_x^2+x\kappa}{x\kappa}\right)\right]$
H_2^{12}	$x(1-x)$	
H_3^{12}	$-(1-x)/\kappa$	
H_4^{12}	$-(1-x)(\nu+2)[1-y+xy]/\kappa$	

TABLE III. The functions K_n^{rs} and j_{rs} given in Eq. (5.27) which comprise the integrands of the 26 nonzero integrals J_n^{rs}. In these functions, $p_x^2 = x\tau + [\tau y(1-y) - \kappa y](1-x)^2 + \kappa y(1-x) + 1$, $q_x^2 = -x\kappa + [\tau y(1-y) - \kappa y](1-x)^2 + \kappa y(1-x) + 1$, $k_y^2 = 1 + \tau y(1-y)$, and $\nu = \tau + 2$.

J_n^{rs}	$K_n^{rs}(\kappa,\tau;x,y)$	$j_{rs}(\kappa,\tau;x,y)$
J_2^{01}	$-\nu/\tau$	$\frac{1}{k_y^2}\left(\ln\left\|\frac{\tau}{k_y^2}\right\| + \frac{[\tau y(1-y) - y\kappa](1-x) - 1}{p_x^2 - x\tau}\right)$
J_1^{02}	$-\nu/\kappa$	$\frac{1}{k_y^2}\left(\ln\left\|\frac{\kappa}{k_y^2}\right\| + \frac{[\tau y(1-y) - y\kappa](1-x) - 1}{q_x^2 + x\kappa}\right)$
J_1^{11}	$-(1-x)[\nu^2 + 2 + 2\tau(1-x)y]/\tau$	$\frac{1}{p_x^2}\left(\frac{1}{p_x^2}\ln\frac{p_x^2 - x\tau}{-x\tau} - \frac{1}{p_x^2 - x\tau}\right)$
J_2^{11}	$-(1-x)[\nu^2 + 2 + 2\tau(1-x) + \tau\nu[y(1-x) + x]]/\tau$	
J_3^{11}	$(1-x)(\nu/\tau)[1 - y + xy]$	
J_4^{11}	$(1-x)[1 - y + xy]$	
J_1^{12}	$(1-x)[(\nu+1)(1-x)(1-y) + 2(1-x)y + 3x]$	$\frac{1}{q_x^2}\left(\frac{1}{q_x^2}\ln\frac{q_x^2 + x\kappa}{x\kappa} - \frac{1}{q_x^2 + x\kappa}\right)$
J_2^{12}	$(1-x)[1 - y + xy]$	
J_3^{12}	$-(1-x)[2(1 - y + xy) - x\kappa]/\kappa$	
J_4^{12}	$(1-x)[(\kappa - \tau)(1-x)y - \kappa\tau x - 2]/\kappa$	
J_1^{21}	$(1-x)[\tau^2(1-x)^2y(1-y) - \kappa(\nu+2)x(1-x)y + 4(\tau+1)(1-x)y + 2]/\tau$	$\frac{1}{(p_x^2)^2}\left(1 + \frac{2x\tau}{p_x^2}\ln\frac{p_x^2 - x\tau}{-x\tau} - \frac{x\tau}{p_x^2 - x\tau}\right)$
J_2^{21}	$(1-x)[\nu^2(x + y - xy) + \nu(1-2x) - \kappa(\nu+2)x(1-x)y]/\tau$	
J_3^{21}	$(1-x)^2y[-\tau(1-x)(1-y) + (\kappa+\tau)x]/\tau$	
J_4^{21}	$(1-x)^2y[1 - 2(1-x)(1-y) + \nu x]$	
J_1^{22}	$-(1-x)[\nu(1-x)(1-y)(1-y+xy) + (1-x)y(1+2x) + 2x]$	$\frac{1}{(q_x^2)^2}\left(1 - \frac{2x\kappa}{q_x^2}\ln\frac{q_x^2 + x\kappa}{x\kappa} + \frac{x\kappa}{q_x^2 + x\kappa}\right)$
J_2^{22}	$-(1-x)[(1-x)^2(1-y)(1-2y) + x(1-x)y - x^2]$	
J_3^{22}	$(1-x)\{(1-x)^2[(\kappa+\tau)(1-y)^2 - (\nu-1)(1-y) - y] + (\kappa-1)x(1-x) - x\}/\kappa$	
J_4^{22}	$(1-x)\{(1-x)^2(1-2y)[(\kappa+\tau)(1-y) - \nu] + \nu(\kappa-2)x(1-x)(1-y) + (\tau-\kappa)x(1-x)y + (\tau+1)(\kappa-2)x^2\}/\kappa$	
J_1^{31}	$x(1-x)^2y(\kappa/\tau)[-\tau(1-x)(1-y) + \nu]$	$\frac{1}{(p_x^2)^2}\left(\frac{1}{2} + \frac{2x\tau}{p_x^2} + \frac{3(x\tau)^2}{(p_x^2)^2}\ln\frac{p_x^2 - x\tau}{-x\tau} - \frac{(x\tau)^2}{p_x^2(p_x^2 - x\tau)}\right)$
J_2^{31}	$x(1-x)^2y(\kappa/\tau)[-\tau(1-x)(1-y) + \nu]$	
J_3^{31}	$(1-x)^2y\{(1-x)^2(1+\tau y^2) + x(1-x)[2 + (\tau-\kappa)y] + x^2\}/\tau$	
J_4^{31}	$x(1-x)^2y[\tau(1-x)(1-y) - \nu]$	
J_1^{32}	$x(1-x)^2y[\tau(1-x)(1-y) + 2]$	$\frac{1}{(q_x^2)^2}\left(\frac{1}{2} - \frac{2x\kappa}{q_x^2} + \frac{3(x\kappa)^2}{(q_x^2)^2}\ln\frac{q_x^2 + x\kappa}{x\kappa} - \frac{(x\kappa)^2}{q_x^2(q_x^2 + x\kappa)}\right)$
J_2^{32}	$x(1-x)^2y[\tau(1-x)(1-y) + 2]$	
J_3^{32}	$-(1-x)^2y\{1 + (1 - y + xy)[\tau(1-x)(1-y) + \kappa x]\}/\kappa$	
J_4^{32}	$-x(1-x)^2y(\tau/\kappa)[\tau(1-x)(1-y) + 2]$	

$$G_{fi}=e^2(2\pi)^3(\alpha Z)^{-1}\pi^2 N\bar{u}_f(p)\Big[\Big(\not\epsilon_f\frac{1}{\not p+\not k_f-1}\not\epsilon_i+\not\epsilon_i\frac{1}{\not p-\not k_i-1}\not\epsilon_f\Big)\frac{1}{\not p+\not k_f-\not k_i-1}V(\vec{\mathrm{p}}+\vec{\mathrm{k}}_f-\vec{\mathrm{k}}_i)$$
$$+V(\vec{\mathrm{p}}+\vec{\mathrm{k}}_f-\vec{\mathrm{k}}_i)\frac{1}{\not l+\not k_i-\not k_f-1}\Big(\not\epsilon_f\frac{1}{\not l+\not k_i-1}\not\epsilon_i+\not\epsilon_i\frac{1}{\not l-\not k_f-1}\not\epsilon_f\Big)$$
$$+\not\epsilon_f\frac{1}{\not p+\not k_f-1}V(\vec{\mathrm{p}}+\vec{\mathrm{k}}_f-\vec{\mathrm{k}}_i)\frac{1}{\not l+\not k_i-1}\not\epsilon_i+\not\epsilon_i\frac{1}{\not p-\not k_i-1}V(\vec{\mathrm{p}}+\vec{\mathrm{k}}_f-\vec{\mathrm{k}}_i)\frac{1}{\not l-\not k_f-1}\not\epsilon_f\Big]u_i(l)\,, \tag{6.4}$$

where k_i^μ is the 4-momentum of the incident photon, k_f^μ the 4-momentum of the outgoing photon, and ϵ_i^μ and ϵ_f^μ are the corresponding polarization 4-vectors.

If we assume the energy resolution is such that $\Delta E \ll 1$, then we need retain only those terms in (6.4) which dominate in the limit $k_f \to 0$ and thus consider just the infrared divergent part of the Compton amplitude.[22] In this limit we find

$$G_{fi}\to G_{fi}^{\mathrm{IRD}}=e\Big(\frac{p\cdot\epsilon_f}{p\cdot k_f}-\frac{l\cdot\epsilon_f}{l\cdot k_f}\Big)S_{fi}\,, \tag{6.5}$$

where S_{fi} is the Sauter amplitude. Using (6.5), we can write the cross section for Compton scattering, in which the final photon has energy less than $\Delta E \ll 1$, in the form,

$$\frac{d\sigma_{fi}^{\mathrm{IRD}}}{d\Omega}=\frac{\alpha}{\pi}\frac{d\sigma_{fi}^{(0)}}{d\Omega}\frac{1}{2\pi}\int_{k_f\le\Delta E}\frac{d^3k_f}{2\omega_f}\sum_{\mathrm{pol}}\Big(\frac{p\cdot\epsilon_f}{p\cdot k_f}-\frac{l\cdot\epsilon_f}{l\cdot k_f}\Big)^2\,, \tag{6.6}$$

where $d\sigma_{fi}^{(0)}/d\Omega$ is again the analog of the Sauter cross section including polarization which appears in Eq. (7.3), $\omega_f=(|k_f|^2+\lambda^2)^{1/2}$ and the sum in Eq. (6.6) is extended over the polarizations of the final (undetected) photon.

The integral in (6.6) has been discussed in many places.[23] For the kinematics which obtain in photoeffect it can be evaluated analytically. We find,

$$\frac{1}{2\pi}\int_{k_f\le\Delta E}\frac{d^3k_f}{2\omega_f}\sum_{\mathrm{pol}}\Big(\frac{p\cdot\epsilon f}{p\cdot k_f}-\frac{l\cdot\epsilon f}{l\cdot k_f}\Big)^2$$
$$=\tfrac{5}{2}+\ln\lambda^2-2\nu F_0(p,l)+\Lambda(\omega)\,, \tag{6.7}$$

where

$$\Lambda(\omega)=-\frac{3}{2}-2\ln 2\Delta E+\frac{1}{2}\Big(1+\ln\frac{2(\Delta E)^2}{E_f+1}\Big)\frac{1}{\beta}\ln\frac{1+\beta}{1-\beta}$$
$$+\frac{1}{\beta}[2L_2(\tfrac{1}{2}(1+\omega/|\vec{\mathrm{p}}|))-2L_2(\tfrac{1}{2}(1-\omega/|\vec{\mathrm{p}}|))$$
$$+L_2(-(|\vec{\mathrm{p}}|+\omega))-L_2(|\vec{\mathrm{p}}|-\omega)] \tag{6.8}$$

and $L_2(z)$ is the Euler dilogarithm. We note, from (6.8), that $\Lambda(\omega)$ is real.

Adding (6.6) to (6.3) and using (6.7), we have finally, for the physical photoeffect cross section,

$$\frac{d\sigma_{fi}}{d\Omega}=\frac{d\sigma_{fi}^{(0)}}{d\Omega}\Big(1+\frac{\alpha}{\pi}\Lambda\Big)+\frac{1}{(2\pi)^2}\frac{|\vec{\mathrm{p}}|}{2\omega}[S_{fi}\bar{R}_{fi}^*+\bar{R}_{fi}S_{fi}^*]\,, \tag{6.9}$$

where Λ is given by (6.8). We see explicitly that this expression, (6.9), is indeed free of infrared divergences.

The cross section (6.9) is completely general in that it includes all polarization effects. However, in the following we will consider only the unpolarized differential cross section for the K shell since it is this quantity which is of primary experimental interest. The unpolarized cross section is obtained by summing over all possible transitions from the K shell to the continuum final state of asymptotic momentum $\vec{\mathrm{p}}$ and averaging over the initial photon polarizations. For this purpose, we note that

$$\sum_{m_i,\,m_f=\pm1/2}\bar{u}_f(p)\,Y_n u_i(l)[\bar{u}_f(p)Y_k u_i(l)]^\dagger$$
$$=\tfrac{1}{4}\,\mathrm{Tr}(\not p+m)Y_n(\not l+m)\overline{Y}_k\equiv\tfrac{1}{4}(Y_n,\overline{Y}_k)\,, \tag{6.10}$$

where $\overline{Y}_k=\gamma_0 Y_k^\dagger\gamma_0$, and m_i and m_f are the magnetic quantum numbers of the electron. Evaluating the traces which result from summing over the electron-spin states in Eq. (6.9), we have

$$\tfrac{1}{4}(Y_1,\overline{Y}_1)=\tfrac{1}{2}(2\pi^2/\alpha Z)^2V^2(2l\cdot p)(2p\cdot k)\,,\quad \tfrac{1}{4}(Y_1,\overline{Y}_2)=\tfrac{1}{4}(Y_2,\overline{Y}_1)=-\tfrac{1}{2}(2\pi^2/\alpha Z)^2(2k\cdot V)^2\,,$$
$$\tfrac{1}{4}(Y_1,\overline{Y}_3)=\tfrac{1}{4}(Y_3,\overline{Y}_1)=\tfrac{1}{4}(2\pi^2/\alpha Z)^2V^2(2J\cdot\epsilon)^2\,,\quad \tfrac{1}{4}(Y_1,\overline{Y}_4)=\tfrac{1}{4}(Y_4,\overline{Y}_1)=\tfrac{1}{2}(2\pi^2/\alpha Z)^2V^2(2p\cdot k)(2p\cdot k-2l\cdot k)\,,$$
$$\tfrac{1}{4}(Y_2,\overline{Y}_2)=\tfrac{1}{2}(2\pi^2/\alpha Z)^2(2l\cdot k)\,[(2V\cdot k)(2V\cdot p)-V^2(2p\cdot k)]\,,\quad \tfrac{1}{4}(Y_2,\overline{Y}_3)=\tfrac{1}{4}(Y_3,\overline{Y}_2)=\tfrac{1}{4}(2\pi^2/\alpha Z)^2V^2(2J\cdot\epsilon)^2\,, \tag{6.11}$$
$$\tfrac{1}{4}(Y_2,\overline{Y}_4)=\tfrac{1}{4}(Y_4,\overline{Y}_2)=-\tfrac{1}{2}(2\pi^2/\alpha Z)^2\{2p\cdot k[V^2(2l\cdot k+2p\cdot k)-(2V\cdot k)(2V\cdot p)]+\tfrac{1}{2}V^2(2J\cdot\epsilon)^2\}\,,$$
$$\tfrac{1}{4}(Y_3,\overline{Y}_3)=\tfrac{1}{2}(2\pi^2/\alpha Z)^2V^2[2l\cdot p+2](2J\cdot\epsilon)^2\,,\quad \tfrac{1}{4}(Y_3,\overline{Y}_4)=\tfrac{1}{4}(Y_4,\overline{Y}_3)=-\tfrac{1}{2}(2\pi^2/\alpha Z)^2V^2(2J\cdot\epsilon)^2\,,$$

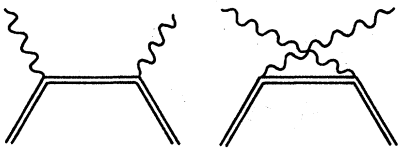

FIG. 6. Furry diagrams which correspond to the Compton effect from a bound atomic electron.

where $2J\cdot\epsilon = (2l\cdot k)(2p\cdot\epsilon) - (2p\cdot k)(2l\cdot\epsilon)$ and $V^\mu = (V(\vec{p}-\vec{k}), \vec{0})$. If we now average over the polarization states of the incident photon, then, due to the gauge invariance of the Y_n, we can make the replacement

$$\sum_{\lambda=\pm1} \epsilon^\mu(\lambda)\epsilon^\nu(\lambda) \to -g^{\mu\nu}, \tag{6.12}$$

so that

$$\tfrac{1}{2}(2J\cdot\epsilon)^2 \to -2J^2 = -2[\kappa^2+\tau^2-\kappa\tau\nu] = 8|\vec{p}|^2\omega^2\sin^2\theta\,. \tag{6.13}$$

Using the results (6.11) and (6.13), the radiative corrections to the unpolarized K-shell photoeffect differential cross section, to lowest order in αZ, can be written in the form,

$$\frac{d\sigma_K}{d\Omega} = \frac{d\sigma_K^{(0)}}{d\Omega}\left(1+\frac{\alpha}{\pi}\delta\right), \tag{6.14}$$

where[24]

$$\delta = \Lambda + \left(\frac{2-\tau}{\tau} - \frac{8}{\kappa\tau}\right)^{-1} \times \mathrm{Re}[R_1 + (1-\tau)R_2 - (\kappa-\tau-8)R_3 - 2R_4] \tag{6.15}$$

and $d\sigma_K^{(0)}/d\Omega$ is the Sauter formula,

$$\frac{d\sigma_K^{(0)}}{d\Omega} = \alpha(\alpha Z)^5 \frac{8|\vec{p}|^3\sin^2\theta}{\omega|\vec{p}-\vec{k}|^8}[4+(\omega-1)|\vec{p}-\vec{k}|^2]\,. \tag{6.16}$$

In Eq. (6.15), Λ is given by Eq. (6.8) while the R_n are defined by Eqs. (5.8), (5.14)–(5.16), (5.20), (5.22)–(5.27), and Tables I–III. (Note that only the real parts of the R_n contribute to δ.) As we have indicated, this expression is valid provided $\alpha Z/\beta \ll 1$.

It will be shown in the following paper[6] that our result for the K-shell photoeffect can be simply extended to the case of any $nS_{1/2}$ subshell. To the same order in αZ we find that the corresponding cross section is obtained by multiplying Eq. (6.14) by n^{-3} so that

$$\frac{d\sigma_{n1/2}}{d\Omega} = \frac{1}{n^3}\frac{d\sigma_K^{(0)}}{d\Omega}\left(1+\frac{\alpha}{\pi}\delta\right). \tag{6.17}$$

We note that, for consistency in the case of light elements, the basic Sauter cross section in Eq. (6.14) should be replaced by the Sauter-Gavrila formula[3,4] which includes the αZ corrections. For high-Z elements it is also possible that the accuracy of our expression (6.14) may be improved by substituting Pratt's modification[5] of the Sauter-Gavrila formula for $d\sigma_K^{(0)}/d\Omega$.

Even with the simplications which result when we consider the unpolarized differential cross section, the final analytic expression for the radiative corrections to photoeffect (6.15) is sufficiently complicated to require numerical evaluation for general values of κ and τ. In particular, one must determine the contributions of the 20 one- and two-parameter Feynman integrals of the type given in Tables I–III which remain after integrals with similar integrands are combined according to (6.15). Although we have made some preliminary estimates, the actual computation of these integrals has not yet been carried out. Nevertheless, we can make some quantitative statements concerning the magnitude of the radiative corrections to photoeffect since we have been able to obtain analytic results for the low- and high-energy limits of our expresssion (6.15).

VII. LOW- AND HIGH-ENERGY LIMITS

In the low-energy limit ($\omega \ll 1$) we expand the integrands of the integrals which define the R_n in Eq. (6.15) in powers of β, the electron velocity, and keep only the lowest-order terms. The resulting integrals can then be evaluated analytically. We find

$$\delta = -\tfrac{2}{3}[\beta^2(\tfrac{19}{30} - \ln 2\Delta E) + 2\vec{p}\cdot\vec{k}\ln 2\omega] + O(\beta^3) = -\tfrac{2}{3}\beta^2[\tfrac{19}{30} - \ln 2\Delta E + 2\beta\ln\beta\cos\theta] + O(\beta^3)\,. \tag{7.1}$$

We note that this expression (7.1) is of order β^2, which seems to be a characteristic feature of radiative corrections at low energies calculated in the Born approximation. It is also found in the results for the radiative corrections to elastic scattering,[25] Compton effect,[16] Møller scattering,[26] and bremsstrahlung.[27,28] However, it should be pointed out that these results cannot yield the limit $\beta \to 0$ since they are derived under the assumption $\alpha Z/\beta \ll 1$. In order to treat the case $\beta = 0$, then, another approach would be necessary.

As a check on our expression (7.1), we note that it agrees with the expression derived from Fomin's result[27] for the nonrelativistic limit of the radiative corrections to bremsstrahlung in the case all of the energy of the incident electron is carried away by the photon ($\vec{p}_2 = 0$). This will be explained in the following paper,[6] where we will explore more fully the connection between the radiative corrections to photoeffect and the high-frequency limit of bremsstrahlung.

Because δ is proportional to β^2 at low energy, the radiative corrections are not large in this region. Moreover, due to the additional factor of β multiplying $\cos\theta$, $\delta(\omega, \cos\theta)$ is nearly isotropic. At energies for which Eq. (7.1) is expected to be valid ($\omega \lesssim 5$ keV), we find $0 \leq -(\alpha/\pi)\delta \lesssim 10^{-4}$. (See Table I of the following paper.)

The high-energy limit of our expression for δ can also be obtained analytically using well-known techniques for evaluating the asymptotic behavior of Feynman-parameter integrals.[29] Because the basic photoeffect cross section is sharply peaked in the forward direction at high energy it is only observable in this instance for small angles; i.e., finite momentum transfers. Thus, we consider the case $\tau \gg 1$ and κ finite. (Note that, because of the kinematics particular to photoeffect, the minimum possible momentum transfer is of order 1 at energies well above threshold.) We find

$$\delta = 2(\ln\tau - 1)[\ln 2\Delta E - \tfrac{1}{2}\ln\tau] - \ln\kappa \ln\tau + O(1). \quad (7.2)$$

This logarithmic dependence of δ on ω and ΔE is characteristic of radiative corrections at high energy.[30] In Eq. (7.2) we see that δ is also weakly dependent on the momentum transfer.

As an estimate of the order of magnitude of the corrections in this region we take $\tau \simeq 20$ ($\omega \simeq 5$ MeV), $\kappa \simeq 5$ ($\theta \simeq 10°$). In this case

$$(\alpha/\pi)\delta \simeq 0.01(\ln 2\Delta E - 1). \quad (7.3)$$

For $\Delta E \simeq 0.10$ ($\simeq$50 keV), we find $(\alpha/\pi)\delta \simeq -0.03$ and for $\Delta E \simeq 0.01$, $(\alpha/\pi)\delta \simeq -0.05$. In general, for ω in the 1–10 MeV range, an energy resolution between 5 and 50 keV and for forward angles ($\lesssim 10°$), the radiative corrections $(\alpha/\pi)\delta$ range between -0.01 and -0.07. At sufficiently high energies these corrections may attain values of -0.10 to -0.20. (See Table II of following paper.) However, in this region the basic photoeffect cross section itself is very small, and consequently, difficult to observe.

We note, finally, that the simplicity of Eq. (7.2) allows an analytic derivation of the result for the high-energy behavior of the total cross section including radiative corrections. The dominant high-energy dependence can be obtained directly from the relation,

$$\sigma_K \simeq \frac{2\pi}{\omega^2}\int_1^\infty \left(\frac{d\sigma_K}{d\Omega}\right)\kappa\, d\kappa \quad (7.4)$$

where κ, the momentum transfer, is chosen as the integration variable. By writing

$$\sigma_K = \sigma_K^{(0)}(1 + \alpha\Delta/\pi), \quad (7.5)$$

we find that in the extreme relativistic limit,

$$\Delta = 2(\ln\tau - 1)[\ln 2\Delta E - \tfrac{1}{2}\ln\tau] - \tfrac{3}{2}\ln\tau + O(1), \quad (7.6)$$

where

$$\sigma_K^{(0)} = 4\pi\alpha(\alpha Z)^5/\omega \quad (7.7)$$

is just the asymptotic result for the Sauter cross section.

ACKNOWLEDGMENT

The authors would like to thank Professor R. H. Pratt for his hospitality at the University of Pittsburgh and for stimulating discussions.

APPENDIX

In this appendix, we give an explicit demonstration of the cancellation of ultraviolet divergences for the case of the lowest-order (in α) radiative corrections to photoeffect. Because we consider all orders in αZ (Furry picture), the renormalization program in this instance is somewhat more interesting than in the free-particle case.

The ultraviolet divergent parts $R_{fi}^{(p)}(\Lambda)$ of the contributions of the Furry diagrams (a)–(h) of Fig. 3 arise solely from the zero and one potential terms of the exact Coulomb Dirac electron propagators. Including all orders in αZ, we find the following divergent contributions from each diagram:

$$R_{fi}^{(a)}(\Lambda) = \frac{1}{2}ie(2\pi)^7\int d^3p_1\, d^2p_2\, \bar\psi_f(\vec p_1)\,[\delta m_F - (Z_2^{-1} - 1)(\not p_1 - 1)]\, S(E_f; \vec p_1, \vec p_2 + \vec k)\, \not\epsilon\, \psi_1(\vec p_2)$$

$$+ \frac{1}{2}ie(2\pi)^7\int d^3p_1\, d^3p_2\, d^3p_3\, (Z_1^{-1} - 1)\, \bar\psi_f(\vec p_1)\, \not V(\vec p_1 - \vec p_2)\, S(E_f; \vec p_2, \vec p_3 + \vec k)\, \not\epsilon\, \psi_1(\vec p_3),$$

$$R_{fi}^{(b)}(\Lambda) = \frac{1}{2}ie(2\pi)^7\int d^3p_1\, d^3p_2\, \bar\psi_f(\vec p_1)\, \not\epsilon\, S(E_i; \vec p_1 - \vec k, \vec p_2)\,[\delta m_F - (Z_2^{-1} - 1)(\not p_2 - 1)]\, \psi_i(\vec p_2)$$

$$+ \frac{1}{2}ie(2\pi)^7\int d^3p_1\, d^3p_2\, d^3p_3\, (Z_1^{-1} - 1)\, \bar\psi_f(\vec p_1)\, \not\epsilon\, S(E_i; \vec p_1 - \vec k, \vec p_2)\, \not V(\vec p_2 - \vec p_3)\, \psi_i(\vec p_3),$$

$$R_{fi}^{(c)}(\Lambda) = -\frac{1}{2}ie(2\pi)^7\,\delta m_F\int d^3p_1\, d^3p_2\, \bar\psi_f(\vec p_1)\, S(E_f; \vec p_1, \vec p_2 + \vec k)\, \not\epsilon\, \psi_i(\vec p_2),$$

$$R_{fi}^{(d)}(\Lambda) = -\frac{1}{2} ie(2\pi)^7 \delta m_F \int d^3p_1\, d^3p_2\, \bar{\psi}_f(\vec{p}_1) \not{S}(E_i; \vec{p}_1 - \vec{k}, \vec{p}_2)\, \psi_i(\vec{p}_2),$$

$$R_{fi}^{(e)}(\Lambda) = -\frac{1}{2} ie(2\pi)^7 \int d^3p_1\, d^3p_2\, d^3p_3\, (1 - Z_3)\, \bar{\psi}_f(\vec{p}_1) \not{V}(\vec{p}_1 - \vec{p}_2)\, S(E_f; \vec{p}_2, \vec{p}_3 + \vec{k})\, \not{\psi}_i(\vec{p}_3),$$

$$R_{fi}^{(f)}(\Lambda) = -\frac{1}{2} ie(2\pi)^7 \int d^3p_1\, d^3p_2\, d^3p_3\, (1 - Z_3)\, \bar{\psi}_f(\vec{p}_1) \not{S}(E_i; \vec{p}_1 - \vec{k}, \vec{p}_2) \not{V}(\vec{p}_2 - \vec{p}_3)\, \psi_i(\vec{p}_3), \tag{A1}$$

$$R_{fi}^{(g)}(\Lambda) = -e(2\pi)^3 (1 - Z_3) \int d^3p\, \bar{\psi}_f(\vec{p})\, \not{\psi}_i(\vec{p} - \vec{k}),$$

$$R_{fi}^{(h)}(\Lambda) = e(2\pi)^3 (Z_1^{-1} - 1) \int d^3p\, \bar{\psi}_f(\vec{p})\, \not{\psi}_i(\vec{p} - \vec{k}),$$

where, to order α

$$\delta m_F = \frac{3\alpha}{4\pi} (\ln\Lambda^2 + \tfrac{1}{2}), \quad Z_1^{-1} = Z_2^{-1} = 1 + \frac{\alpha}{4\pi} (\ln\Lambda^2 + \tfrac{9}{2} + 2\ln\lambda^2), \quad Z_3 = 1 - \frac{\alpha}{3\pi} \ln\Lambda^2 ,$$

and Λ is a high-energy cutoff which appears in the regularization procedure used to define the Feynman integrals for the invariant functions, $C(p)$, $\Lambda^\mu(p, p')$, and $\Pi(q^2)$. In order to show that these divergences cancel to order α when the matrix element for photoeffect is written in terms of the renormalized charge e_R, defined by $e_R = Z_3^{1/2} e$ [e is the unrenormalized charge which appears in the contributions (A1)], we note first that because of the inclusion of the mass counterterms [diagrams (c) and (d) of Fig. 3], the matrix element is already written in terms of the physical electron mass. Moreover, it is easily seen that in the sum of the contributions of diagrams (a)–(d), the mass counterterms cancel identically. Thus, the only divergences which must be considered are those of the renormalization constants Z_1, Z_2, and Z_3.

It is well known[8] that Z_1 and Z_2 only give rise to a spurious charge renormalization which, in fact, vanishes identically. To show this, we consider the sum $\sum'_{(1,2)}$ of the contributions of diagrams (a)–(d) and (h) of Figs. 3. Using the fact that $Z_1 = Z_2$ and that

$$(\not{p} - 1) S(E; \vec{p}, \vec{p}') + \int d^3q\, \not{V}(\vec{p} - \vec{q}) S(E; \vec{q}, \vec{p}') = [-2i/(2\pi)^4] \delta(\vec{p} - \vec{p}'), \tag{A2}$$

it is trivial to verify that

$$\begin{aligned} \sum'_{(1,2)} &\equiv R_{fi}^{(a)}(\Lambda) + R_{fi}^{(b)}(\Lambda) + R_{fi}^{(c)}(\Lambda) + R_{fi}^{(d)}(\Lambda) + R_{fi}^{(h)}(\Lambda) \\ &= [-2(Z_2^{-1} - 1) + (Z_1^{-1} - 1)] \\ &\quad \times e(2\pi)^3 \int d^3p\, \bar{\psi}_f(\vec{p})\, \not{\psi}_i(\vec{p} - \vec{k}) \\ &= [-2(Z_2^{-1} - 1) + (Z_1^{-1} - 1)] S_{fi} . \end{aligned} \tag{A3}$$

If we now add the basic photoeffect amplitude S_{fi} (of zeroth order in α, but exact in αZ) to the sum $\sum'_{(1,2)}$ and multiply the result by a factor $Z_2^{-1/2}$ for each external electron corresponding to the renormalization of the electron wave functions, then we find, to first order in α (again using $Z_1 = Z_2$),

$$\begin{aligned} \sum_{(1,2)} &\equiv (1/Z_2)(S_{fi} + \Sigma'_{(1,2)}) \\ &= (1/Z_2)[1 - 2(Z_2^{-1} - 1) + (Z_1^{-1} - 1)] S_{fi} \\ &= (1/Z_2)[1 + (Z_1^{-1} - 1)]^{-1} S_{fi} = S_{fi} . \end{aligned} \tag{A4}$$

Thus, the electron propagator and wave-function renormalization constants Z_1 and Z_2 cancel identically as they should.

Of the remaining contributions, $R_{fi}^{(e)}(\Lambda)$ and $R_{fi}^{(f)}(\Lambda)$ renormalize the electric charge contained implicitly in the wave functions and propagators due to the Coulomb potential, while $R_{fi}^{(g)}(\Lambda)$ renormalizes the explicit factor e which appears in the basic photoeffect amplitude S_{fi}. To show this we consider first $R_{fi}^{(e)}(\Lambda) + R_{fi}^{(f)}(\Lambda)$.

It is trivial to verify that, to order α, the difference between the wave function $\psi(\vec{p})$ satisfying the Dirac equation with unrenormalized Coulomb potential $V(\vec{q})$ and the wave function $\psi^R(\vec{p})$ satisfying the Dirac equation with renormalized Coulomb potential $V_R(\vec{q}) = Z_3 V(\vec{q})$ is given by

$$\psi(\vec{p}) - \psi^R(\vec{p}) = \frac{(2\pi)^4}{-2i} \int d^3p_1\, d^3p_2\, (1 - Z_3) \times S(E; \vec{p}, \vec{p}_1) \not{V}(\vec{p}_1 - \vec{p}_2) \psi(\vec{p}_2) . \tag{A5}$$

Similarly,

$$\bar{\psi}(\vec{p}) - \bar{\psi}^R(\vec{p}) = \frac{(2\pi)^4}{-2i} \int d^3p_1\, d^3p_2\, (1 - Z_3) \bar{\psi}(\vec{p}_1) \times \not{V}(\vec{p}_1 - \vec{p}_2) S(E; \vec{p}_2, \vec{p}) . \tag{A6}$$

Substituting the left-hand side of Eqs. (A5) and

(A6) in $R_{fi}^{(e)}(\Lambda)$ and $R_{fi}^{(f)}(\Lambda)$ and adding the basic photoeffect amplitude, we have, neglecting higher orders in α,

$$\begin{aligned}\sum_{(1,2,3)}{}' &\equiv S_{fi}+R_{fi}^{(f)}(\Lambda)+R_{fi}^{(e)}(\Lambda)\\ &= e(2\pi)^3\int d^3p\,\bar{\psi}_f(\vec{p})\not{\psi}_i(\vec{p}-\vec{k})-e(2\pi)^3\\ &\times\int d^3p\,[\bar{\psi}_f(\vec{p})-\bar{\psi}_f^R(\vec{p})]\not{\psi}_i(\vec{p}-\vec{k})\\ &-e(2\pi)^3\int d^3p\,\bar{\psi}_f(\vec{p})\not{\epsilon}[\psi_i(\vec{p}-\vec{k})-\psi_i^R(\vec{p}-\vec{k})]\\ &= e(2\pi)^3\int d^3p\,\bar{\psi}_f^R(\vec{p})\not{\epsilon}\psi_i^R(\vec{p}-\vec{k})\\ &= S_{fi}\big|_{V\to V_R}. \qquad \text{(A7)}\end{aligned}$$

Thus, the sum (A7) yields precisely the basic photoeffect matrix element corresponding to the correct renormalized Coulomb potential.

Finally, if we add to our previous result the remaining contribution, $R_{fi}^{(g)}(\Lambda)$ and multiply the sum by $Z_3^{-1/2}$ corresponding to the renormalization of the external photon line, then we have, to first order in α,

$$\begin{aligned}\sum_{(1,2,3)}{}' &= Z_3^{-1/2}Z_2^{-1}[S_{fi}+R_{fi}^{(a)}(\Lambda)+R_{fi}^{(b)}(\Lambda)\\ &+R_{fi}^{(c)}+R_{fi}^{(d)}(\Lambda)+R_{fi}^{(e)}(\Lambda)+R_{fi}^{(f)}(\Lambda)\\ &+R_{fi}^{(g)}(\Lambda)+R_{fi}^{(h)}(\Lambda)]=Z_3^{1/2}S_{fi}\big|_{V\to V_R}\\ &=S_{fi}^{(\text{renorm.})}. \qquad \text{(A8)}\end{aligned}$$

This completes the renormalization program to this order in α. We see explicitly that all divergent quantities sum up to give precisely the physical photoeffect cross section. Moreover, as we have indicated previously this result is exact in αZ.

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*Work supported in part by the National Science Foundation under Grant No. GF34985.

†Present address: FOM Institute for Atomic and Molecular Physics, Kruislaan 407, Amsterdam, The Netherlands.

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