PHYSICAL REVIEW A VOLUME 15, NUMBER 1 JANUARY 1977

X-ray absorption in atoms in the presence of an intense laser field*

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(Received 14 July 1976)

We consider the deep-level absorption of x rays in atoms in the presence of an intense laser field. The laser field is considered to be small relative to atomic fields but strongly interacting with the outgoing electron, represented here by a plane wave. The general expression for the x-ray absorption cross section as a function of the laser field is obtained using the two methods of power-series expansion and asymptotically large fields. We find that for low fields, the cross section increases as the square of the field, while for large fields it decreases asymptotically as the inverse power of the field.

I. INTRODUCTION

In recent years the problem of multiphoton processes has become an important spectroscopic tool.[1-3] In particular, multiphoton absorption in atoms and plasmas immediately comes to mind. One possible experiment involves the study of the change in the cross section of absorption of x rays in atoms due to the simultaneous irradiation of the sample by an intense optical laser. Such processes are also of interest in the study of laser-produced plasmas, where x rays are produced as a result of the interaction of the laser beam with matter.

In this paper, we consider an x-ray beam interacting with a deep level atomic electron and study the behavior of the x-ray absorption cross section in the presence of strong laser radiation. For the sake of simplicity, we assume that the electronic states can be represented by hydrogenic wave functions. Then the x-ray absorption cross section in the absence of the laser can be analytically obtained within the Born approximation.[4] However, in the presence of the laser field, the atomic wave functions are difficult to calculate. Nevertheless, as will be apparent later, there exists a wide range of laser intensities for which the modulation of the deep-level bound-electron wave function by the laser field is negligible while the laser modulation of the outgoing electron should be taken into account since the electron can interact with the intense laser field through multiphoton processes. Here again, analytic solution for a positive-energy electron in both the Coulomb and the laser fields is not available. However, if the energy of the outgoing electron is much larger than the binding energy, one can approximate it as a plane wave modulated by the strong laser field. Therefore, in the model problem of a plane-wave outgoing electron, one expects only qualitative agreement with experiment unless the x-ray energy is much larger than the binding energy.[5]

Considering the electromagnetic interaction between the x-ray photon and the electron to be the perturbation, transition probabilities are calculated between the unperturbed states. The electrons are treated quantum mechanically and nonrelativistically, while the laser field is treated classically since there are a large number of photons in the same state. Having obtained the transition probability, the x-ray absorption cross section is calculated using the two methods: a power-series expansion and asymptotic-field approximation. We plot the absorption cross section as a function of the normalized field strength for various values of x-ray photon energies. One finds a similar behavior at high laser fields as obtained in other processes involving multiple-photon events.[6,7]

We derive the parameters justifying the plane-wave approximation for the outgoing electron and the neglect of the initial Coulomb wave-function modulation by the laser field. It is shown that for a large range of values of the x ray, the laser, and the atom parameters, the present theory would be applicable for a qualitative estimate of the x-ray absorption cross section.

II. CALCULATIONS

For the calculation of the absorption cross section, we first determine the matrix element of the interaction Hamiltonian

$$H' = -(e/mc)\vec{A}_x \cdot \vec{p}$$

between the initial and final states of the electron. Here $\vec{A}_x$ represents the vector potential of the x-ray photon and $\vec{p}$ is the momentum operator. If x-ray radiation is taken to be plane polarized in the x direction,

$$\vec{A}_x(t) = A_{0x}\hat{x}\exp(i\omega_x t), \quad (1)$$

where ω_x is the x-ray frequency.

The initial electronic state ψ_i is considered to be the hydrogenlike 1s state unperturbed by the laser radiation and is given by

$$\psi_i(\vec{r}, t) = (\pi a^3)^{-1/2} \exp\left(-\frac{r}{a} - \frac{iE_b t}{\hbar}\right). \tag{2}$$

where $E_b = Z^2e^2/2a_0$ represents its binding energy and $a = a_0/Z$, with a_0 the Bohr radius and Z the nuclear charge of the atom;

$$\psi_f(\vec{r}, t) = \exp \frac{i}{\hbar}\left(\vec{p}\cdot\vec{r} + \frac{1}{2m}\int_0^t \left|\vec{p} - \frac{e}{c}\vec{A}(t')\right|^2 dt'\right) \tag{3}$$

is the final-state plane-wave function (normalized in a box of unit volume) modulated by the intense optical field represented by the vector potential $\vec{A}(t)$, which is taken to be spatially independent in the dipole approximation. If we assume the laser radiation to be a circularly polarized wave of frequency ω, propagating in the $\hat{z}$ direction, the vector potential can be written as

$$\hat{A}(t) = A_0(\hat{x}\cos\omega t + \hat{y}\sin\omega t). \tag{4}$$

Using first-order perturbation theory and Eqs. (1) and (4), the matrix element for a transition from the state ψ_i to the state ψ_f due to an interaction Hamiltonian H' is found to be

$$a(i\to f) = -\frac{i}{\hbar}\int_{-T/2}^{T/2} dt \int d\vec{r}\, \psi_f^*(\vec{r}, t) H' \psi_i(\vec{r}, t)$$

$$= -\frac{ie}{mc\hbar}\, p_x A_{x0}\left(\frac{1}{\pi a^3}\right)^{1/2} \frac{8\pi a^3}{(1 + a^2p^2/\hbar^2)^2}$$

$$\times\int_{-T/2}^{T/2} dt \exp\frac{i}{\hbar}\left(\Omega t + \frac{\lambda}{\omega}\sin(\omega t - \delta) - \frac{e^2A_0^2 t}{2mc^2}\right), \tag{5}$$

where

$$\Omega = -p^2/2m + \hbar\omega_x - E_b ,$$

$$\lambda = eA_0 p_\perp/mc = (eE_0/m\omega)p_\perp ,$$

$$\delta = \tan^{-1}(p_x/p_y) .$$

Here $p_\perp$ refers to the component of p in a direction perpendicular to the direction of propagation of the wave (the z direction). The transition probability per unit time is

$$\frac{|a(i\to f)|^2}{T} = \left(\frac{8\pi e A_{0x} p_x}{mc(1 + a^2p^2/\hbar^2)^2}\right)^2 \frac{2a^3}{\hbar} \times\sum_{n=-\infty}^{+\infty} J_n^2(\lambda/\hbar\omega)\delta(\Omega - n\hbar\omega) , \tag{6}$$

where J_n is the Bessel function of order n.

In obtaining Eq. (6), we have omitted the term $e^2A_0^2/2mc^2$ from the argument of the δ function. This term represents the average kinetic energy shift for an electron having a momentum $\vec{p}$, in the presence of the laser field. It should be noted that similarly the second-order energy shift of the ground state is given by

$$\Delta E_b \simeq \sum_s \left(\frac{\langle b|H|s\rangle\langle s|H|b\rangle}{E_s - E_b}\right),$$

where $(E_s - E_b) \gg \hbar\omega$. Here $|s\rangle$ and E_s are the excited-state eigenfunctions and eigenvalues, respectively, and

$$H = -(ie\hbar/mc)\nabla\cdot\vec{A} .$$

Using $[\vec{x}, H] = i\hbar\vec{p}/m$, we find that

$$\Delta E_b = \frac{e^2A_0^2}{2mc^2\hbar^2}\langle 0|[x, p_x]|0\rangle = \frac{e^2A_0^2}{2mc^2}$$

which equals the uniform energy shift of the final-state electron. We thus obtain that ΔE_b exactly cancels the A_0^2 term in the argument of the δ function. Hence the energy E_b is truly the unperturbed ground-state energy of the bound electron.

The presence of the multiphoton processes is evident from the energy conservation requirement represented by the argument of the δ function. The interpretation is that positive values of n correspond to the absorption of n photons, negative values to the emission of $|n|$ photons and summation over all n's must be carried out to add up the contribution of all such processes to the absorption cross section.

On dividing the transition probability per unit time by the incident flux of the x rays, the differential scattering cross section is obtained as

$$\frac{d\sigma}{d\Omega} = \frac{G4\pi^3e^2a^3p_x^2}{m^2c\omega_x(1 + a^2p^2/\hbar^2)^4} \times\sum_n J_n^2(\lambda/\hbar\omega)\delta(\Omega - n\hbar\omega) .$$

To obtain the total absorption cross section, one must integrate over all possible values of p. Thus the total cross section is

$$\sigma = \frac{1}{(2\pi\hbar)^3}\int d\vec{p}\,\frac{d\sigma}{d\Omega} = \frac{32\pi e^2a^3}{m^2c\omega_x\hbar^3}\sum_{n=-\infty}^{+\infty}\int_0^\infty dp\int_0^\pi d\theta\, \frac{p^4\sin^4\theta}{(1 + a^2p^2/\hbar^2)^4} J_n^2(\lambda/\hbar\omega)\delta(\Omega - n\hbar\omega) .$$

On carrying out the p integration using the δ function, the cross section becomes

$$\sigma = \frac{32\pi e^2}{mc\omega_x a^5}\left(\frac{\hbar^2}{2m(\hbar\omega_x - E_b)}\right)^{5/2}\sum_{n=-1/\beta}^{\infty}\frac{T^4(1 + n\beta)^{5/2}}{(1 + T + n\beta T)^4}\int_0^\pi d\theta\,\sin^3\theta\, J_n^2([\mathscr{E}(1 + n\beta)^{1/2}/\beta T^{1/2}]\sin\theta), \tag{7}$$

where $\beta = \hbar\omega/(\hbar\omega_x - E_b)$ and we have introduced an atom–x-ray dependent parameter $T = (\hbar\omega_x - E_b)/E_b$ and an atom-laser dependent parameter $\mathcal{E} = 2eE_0a/\hbar\omega$.

The field-independent absorption cross section σ_0 is contained in Eq. (7) for σ and is obtained by letting $A_0 \to 0$. Then, as expected, the only term that contributes is $n=0$ to give

$$\sigma_0 = \frac{128\pi e^2}{3mc\omega_x a^5}\left(\frac{\hbar^2}{2m(\hbar\omega_x - E_b)}\right)^{5/2}\frac{T^4}{(1+T)^4}, \tag{8}$$

and Eq. (7) is rewritten as

$$\frac{\sigma}{\sigma_0} = \frac{3}{4}(1+T)^4 \sum_{n=-1/\beta}^{\infty}\frac{(1+n\beta)^{3/2}}{(1+T+n\beta T)^4}\int_0^{\pi} d\theta \sin^3\theta\, J_n^2([\mathcal{E}(1+n\beta)^{1/2}/\beta T^{1/2}]\sin\theta) . \tag{9}$$

The integration over θ in Eq. (9) cannot be performed exactly to obtain an analytical result for σ/σ_0. Therefore we obtain approximate values for the cross section by using (i) power-series expansion for the J_n^2 functions and (ii) asymptotic approximation for J_n^2 functions with large arguments.

A. Power-series expansion

The J_n^2 function occurring in Eq. (9) can be replaced by a formal power-series expansion. Then the θ integration can be easily performed term by term. To the dominant order in β, the absorption cross section can be represented (Appendix A) by the series

$$\frac{\sigma}{\sigma_0} = \sum_{m=0}^{\infty}\frac{\mathcal{E}^{2m}}{T^m}\frac{3(m+1)}{(2m+3)(2m+1)}\sum_{p=0}^{2m}\frac{(2m+3)(2m+1)\cdots(5-2m+2p)(-1)^p 4\cdot 5\cdot 6\cdots(4+p-1)}{2^{2m-p}(2m-p)!\,p!\,(1+1/T)^p} . \tag{10}$$

The above series in powers of $\mathcal{E}^2/T$ is found to diverge for $\mathcal{E}T^{-1/2} \gtrsim 2$ (see Appendix A). Therefore, it is primarily useful for obtaining the cross section for low-field values with the low-field parameter given by $\mathcal{E}^2/T$. It is worthwhile to obtain the first few terms of the series in order to obtain an order-of-magnitude estimate of the correction to the cross section. To order $\mathcal{E}^6$, Eq. (10) becomes

$$\frac{\sigma}{\sigma_0} = 1 + \frac{3}{4}\frac{\mathcal{E}^2}{T}\left(1 - \frac{16T}{3(T+1)^2}\right) + \frac{9}{128}\frac{\mathcal{E}^4}{T^2}\left(1 - \frac{32T(T-1)^2}{(T+1)^4}\right) - \frac{1}{256}\frac{\mathcal{E}^6}{T^3}\left(1 - \frac{16T}{(T+1)^6}(3T^4 - 60T^3 + 130T^2 - 60T + 3)\right) . \tag{11}$$

Thus, to the lowest order, the correction to the cross section is of the order $\mathcal{E}^2/T$. This value is of the order $\beta^2 = [\hbar\omega/(\hbar\omega_x - E_b)]^2$ smaller than has been previously suggested in the literature.[2]

In order to discover the reason for this discrepancy, let us consider the term proportional to $\mathcal{E}^2$ in the power-series expansion in Eq. (9). The contributions to this order come from the values of $n=0$ and ± 1. Here the contribution from $n=1$ represents the process where only one photon is absorbed from the laser field, i.e., the kinetic energy of the outgoing electron is given by $\hbar\omega_x + \hbar\omega - E_b$. The correction to the cross section contributed by the $n=1$ term is proportional to $\mathcal{E}^2/T\beta^2$ to the dominant power in $\beta \ll 1$ and is in agreement with the result of Ref. 2. However, in order to get the correct $\mathcal{E}^2$ term, where no measurement on the outgoing electron is performed, one must also consider the other two contributions. One of them, given by $n=-1$, represents the process in which the outgoing electron, stimulated by the strong optical field, emits an optical photon. This term, to the dominant power in β, exactly equals the $n=1$ term. The other contribution, coming from $n=0$, arises from the interference between the field-independent matrix element and the matrix element of second order in the optical-field parameter. This second-order matrix element represents the process in which the outgoing electron first emits (absorbs) and then absorbs (emits) an optical photon, thus reaching the same final state as in the case of no electron–optical-photon interaction. This contribution to the cross section is negative and exactly cancels the positive contribution (to the dominant power in β) arising from the process in which the electron physically either absorbs or emits an optical photon. Therefore, the calculations must be performed to higher powers in β in order to obtain the term proportional to $\mathcal{E}^2$.

On summing up the contributions from $n=0$ and ± 1 terms, we find that the first nonzero terms are of the order $\mathcal{E}^2/T$, and not $\mathcal{E}^2/T\beta^2$. These results have been reported elsewhere.[8]

The parameter $\mathcal{E}^2/T = 2e^2E_0^2/m\omega^2(\hbar\omega_x - E_b)$ is essentially the ratio of the classical kinetic energy of the electron in the presence of the field and the kinetic energy imparted to it through absorption of the x-ray photon.

B. Asymptotic approximation

When the parameter $\mathcal{E}/\beta T^{1/2}$ occurring in the argument of the J_n^2 function is large, one can make use of the asymptotic approximation for J_n^2 in Eq. (9), and as shown in the Appendix B, the absorption cross section in this limit is given by

$$\frac{\sigma}{\sigma_0}=\frac{1}{16}\frac{(1+T)^4}{\mathcal{E}T^{3/2}}\left[f(Z_1)-f(Z_2)\right], \tag{12a}$$

where

$$f(Z)=\frac{1}{(1+T+TZ)^3}\times\{1+3T+3TZ+(2/\mathcal{E}^2)[3TZ(1+T+TZ)+(1+T^2)]\}, \tag{12b}$$

$$Z_1=\frac{\mathcal{E}}{T^{1/2}}\left[+\frac{1}{2}\frac{\mathcal{E}}{T^{1/2}}-\left(\frac{1}{4}\frac{\mathcal{E}^2}{T}+1\right)^{1/2}\right], \quad Z_2=\frac{\mathcal{E}}{T^{1/2}}\left[+\frac{1}{2}\frac{\mathcal{E}}{T^{1/2}}+\left(\frac{1}{4}\frac{\mathcal{E}^2}{T}+1\right)^{1/2}\right]. \tag{12c}$$

Here it must be pointed out that while the power-series expansion given by Eqs. (10) and (11) is valid for relatively low field values such that $\mathcal{E}T^{-1/2}<2$, the asymptotic approximation leading to the result given by Eqs. (12) holds good as long as $\mathcal{E}\beta^{-1}T^{-1/2}\gg 1$. Thus there is a large range of values of $\mathcal{E}$ for which the domains of the validity of the two approximations coincide, and either of them can be used to obtain the cross section which was evaluated numerically and is presented in Fig. 1.

It should be pointed out here that these results are valid only for values of $\beta\ll 1$, which is the usual situation for experiments to date. However a situation may arise for which β is not so small, for example, for coherent ultraviolet and soft x-ray sources. To obtain the results for such a case, for low fields, we extend our calculations and retain the next higher-order term in β in the power-series expansion given by Eq. (A1). One finds that for low fields, the cross section changes by a factor of $1+A\beta^2$ where A depends on the parameter T and is of the order unity or less.

III. RESULTS AND DISCUSSION

In the previous section, we have obtained the absorption cross section for the x rays in the low-field and asymptotic-field limits. The low-field result, represented by Eqs. (10) and (11), is of use mainly when processes involving only a few photons are being considered. With the present-day laser strengths available, the asymptotic-field-limit result, given by Eq. (12), is more likely to be of chief interest.

In Fig. 1, we plot σ/σ_0 obtained from Eq. (12) as a function of the field parameter $\mathcal{E}$ for various values of T. The cross section first increases as $\mathcal{E}^2$ as predicted by Eq. (11) for the low-field case. For increasing $\mathcal{E}$ values, σ/σ_0 reaches a peak and finally for higher fields, decreases asymptotically as $1/\mathcal{E}$. This behavior is reminiscent of other phenomena involving multiphoton processes such as "multiphoton ionization" in atomic systems[6] and high-field heating of electrons in plasmas[7] where one also encounters a drop in the "efficiency" of the laser at very large fields. Thus our finding is in sharp contrast with the results for large fields presented in Ref. 3 where the cross section seems to increase monotonically with the field. In addition, our results differ from Ref. 3 even at low fields, where our cross section, Eq. (11), is shown to depend explicitly on T, the ratio of the kinetic energy of the outgoing electron to the binding energy, whereas no such dependence is found in Ref. 3. We cannot make a detailed comparison of our results with those of Ref. 3, where only final results were quoted.

To explain these results, one can argue that for low fields, both absorption and emission of n photons are equally possible and the opening of extra "channels" for the final electronic state results in an initial increase in the absorption cross section. However, at intermediate fields, the emission of photons reaches a saturation since energy conservation puts an upper limit of $(\hbar\omega_x-E_b)/\hbar\omega$ on the maximum number of photons that can be emitted. Here the cross section tends to level off.

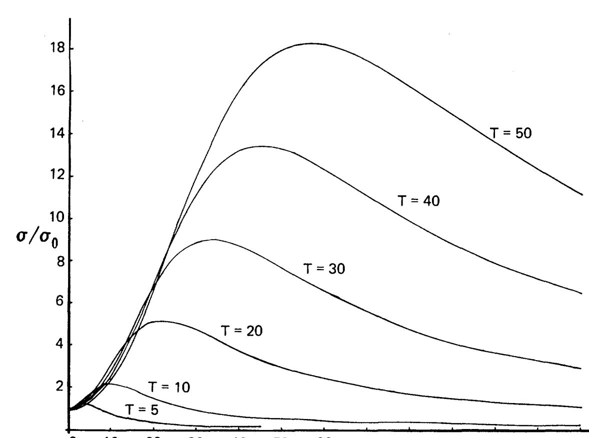

FIG. 1. Normalized absorption cross section σ/σ_0 as a function of the field parameter $\mathcal{E}=2eE_0a/\hbar\omega$ for various values of the parameter $T=(\hbar\omega_x-E_b)/E_b$.

For even higher fields, in addition to this limit, the electron is moving at very high velocities, and the effective coupling between the final electronic state and the initial Coulomb wave functions is very much reduced. This results in the reduction in cross section at very large fields.

The above reasoning is consistent with the behavior of the absorption-cross-section curves for increasing T values. T is essentially a measure of the energy of the incident x-ray photon. For a more energetic x-ray photon, the electron is ejected from the atom with higher kinetic energy and in turn can emit more photons itself through stimulated emission. Thus, an increase in the maximum number of emission photons results in a higher value of the laser field where the leveling off of the curve starts. Thus the peak value of the absorption cross section is higher.

This behavior of the cross-section dependence on the parameter $\mathcal{E}$ naturally leads one to look for the electric field at which the laser is most efficient in changing the absorption cross section. This field is obtained from $\mathcal{E}_{max}$, as the value of $\mathcal{E}$ for which σ/σ_0 is maximum. Using the curves in Fig. 1, one can read-off $\mathcal{E}_{max}$ for various values of the parameter T. When $\mathcal{E}_{max}$ is plotted as a function of T (see Fig. 2), one finds a linear relationship between the two parameters for $T>1$. This simple result is difficult to obtain directly from the rather involved formula given in Eq. (12). However, it leads to much simplification in choosing proper values for the various parameters for the process. One finds from Fig. 2 that for the range of values considered, $\mathcal{E}_{max}$ is not very far from being equal to T. Since we are essentially looking for the value of $\mathcal{E}$ around which the laser will be most efficient, we simply take $\mathcal{E}_{max} = T$.

The results obtained so far have been completely general with regard to the atom, laser, and the x-ray parameters. The main limitation here is the neglect of the modulation of the Coulomb wave function by the laser field. For this, one requires that the laser field, $E_0 = \hbar\omega\mathcal{E}/2ea$ be smaller compared to the atomic field $E_{atomic} = Z^3 e/a_0^2$. As discussed before, we can replace $\mathcal{E}$ by $\mathcal{E}_{max} = T$ for the most efficient laser field and obtain the condition as

$$54.4Z^2/\hbar\omega(\text{eV}) > T \ . \tag{13}$$

It is easily seen that there exists a wide range of values of the parameters ω, Z, and T for which the above condition can be satisfied. Consider, for example, a 100-keV x-ray photon ejecting a K-shell electron from an intermediate mass atom; i.e., we take $E_{atomic} \sim 8$ keV in the presence of a

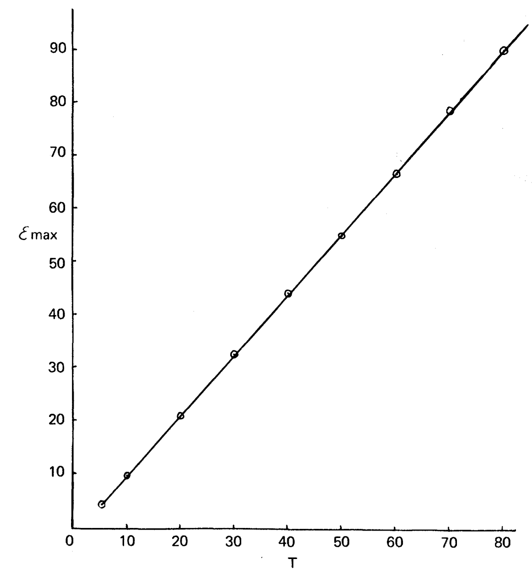

FIG. 2. Plot of $\mathcal{E}_{max}$ (the value of $\mathcal{E}$ at which the cross section is maximum) vs $T = (\hbar\omega_x - E_b)/E_b$.

Nd laser ($\omega = 1.8 \times 10^{15}$ sec^{-1}). These parameters result in a value about 4.5×10^4 for the left-hand side of inequality (13) while the right-hand side equals 11, thus implying $E_0 \ll E_{atomic}$. The value $\mathcal{E}_{max} = 11$ corresponds to a laser field intensity of about 4×10^{18} W/cm^2. This value is quite high, but is within the range of present day lasers. From Fig. 1, one obtains a value of $\sigma/\sigma_0 \sim 2$ for the parameters chosen above.

As another example, consider the case of a light atom like deuterium absorbing a 1-keV x ray in the presence of a CO_2 laser ($\hbar\omega \approx 0.1$ eV). Using these parameters, one obtains $T \approx 80$ while $54.4Z^2/\hbar\omega(\text{eV}) = 544$. Thus the condition represented by Eq. (13) holds. These values result in an absorption cross section $\sigma \sim 36\sigma_0$.

In conclusion, in this paper we have calculated the absorption of x rays in atoms in the presence of laser radiation and have obtained universal curves showing the large effect of the laser field on the absorption cross section.

APPENDIX A

In order to be able to perform the θ integration in Eq, (9), we use for the J_n^2 functions the power-series expansion given by[9]

$$J_n^2(y) = \sum_{k=0}^{\infty} \frac{(-1)^k (2|n|+2k)!\, y^{2|n|+2k}}{2^{2|n|+2k}[(|n|+k)!]^2 (2|n|+k)!\, k!}$$

and integrate term by term. The result is

$$\frac{\sigma}{\sigma_0}=(1+T)^4\sum_{n=-1/\beta}^{\infty}\sum_{k=0}^{\infty}\frac{(1+n\beta)^{|n|+k+3/2}}{(1+T+Tn\beta)^4}\left(\frac{\mathcal{E}}{\beta T^{1/2}}\right)^{2|n|+2k}\frac{3(|n|+k+1)(-1)^k}{(2|n|+2k+3)(2|n|+2k+1)(2|n|+k)!\,k!}. \tag{A1}$$

The above result is exact and can be employed to obtain a power-series expansion in powers of $\mathcal{E}$. However, after the first few terms, it becomes very cumbersome to obtain the coefficients. In what follows, we shall express the result in a more manageable form by expanding it in powers of β and keeping only the lowest-order terms. This approximation is justified since for a case of practical interest $\beta=\hbar\omega/(\hbar\omega_x-E_b)\ll 1$. In the same spirit, the lower limit in the n summation is allowed to approach $-\infty$. Thus, on combining the positive and the negative n terms, Eq. (A1) takes the form,

$$\frac{\sigma}{\sigma_0}=1+\sum_{k=1}^{\infty}\left(\frac{\mathcal{E}}{\beta T^{1/2}}\right)^{2k}\frac{3(k+1)(-1)^k}{(2k+3)(2k+1)(k!)^2}$$
$$+\sum_{n=1}^{\infty}\sum_{k=0}^{\infty}\left[\frac{(1+n\beta)^{n+k+3/2}}{[1+Tn\beta/(1+T)]^4}+\frac{(1-n\beta)^{n+k+3/2}}{[1-Tn\beta/(1+T)]^4}\right]\left(\frac{\mathcal{E}}{\beta T^{1/2}}\right)^{2n+2k}\frac{3(n+k+1)(-1)^k}{(2n+2k+3)(2n+2k+1)(2n+k)!\,k!}. \tag{A2}$$

Furthermore, the terms inside the large square brackets can also be expanded in powers of β to obtain the double summation,

$$\sum_{\substack{l=0\\ l+p=\text{even}}}^{\infty}\sum_{p=0}^{\infty}\frac{2(n+k+\frac{3}{2})(n+k+\frac{1}{2})\cdots(n+k+\frac{5}{2}-l)(-1)^p4\cdot5\cdot6\cdots(4+p-1)(n\beta)^{l+p}}{l!\,p!\,(1+1/T)^p}$$
$$=\sum_{j=0}^{\infty}\sum_{p=0}^{2j}\frac{(2n+2k+3)(2n+2k+1)\cdots(2n+2k+5-4j+2p)(-1)^p4\cdot5\cdot6\cdots(4+p-1)(n\beta)^{2j}}{2^{2j-p-1}(2j-p)!\,p!\,(1+1/T)^p}.$$

The $j=0$ term in the above expansion exactly cancels the second term (containing $\sum_{k=1}^{\infty}$) on the right-hand side of Eq. (A2). This is seen by making use of the identity

$$\sum_{n=1}^{m}\frac{(-1)^n n^{2l}}{(m+n)!\,(m-n)!}=\begin{cases}0, & 1\le l<m\\ \frac{1}{2}(-1)^m, & l=m\end{cases}. \tag{A3}$$

On further simplying Eq. (A2) by substituting $m=n+k$, one obtains

$$\frac{\sigma}{\sigma_0}=1+\sum_{m=1}^{\infty}\sum_{n=1}^{m}\left(\frac{\mathcal{E}^2}{T}\right)^m\beta^{-2m}\frac{3(m+1)(-1)^{m-n}}{(2m+3)(2m+1)(m+n)!\,(m-n)!}$$
$$\times\sum_{j=1}^{\infty}(n\beta)^{2j}\sum_{p=0}^{2j}\frac{(2m+3)(2m+1)\cdots(2m+5-4j+2p)(-1)^p4\cdot5\cdot6\cdots(4+p-1)}{2^{2j-p-1}(2j-p)!\,p!\,(1+1/T)^p}. \tag{A4}$$

An expansion of Eq. (A4) in powers of β shows that all terms containing negative powers of β vanish; i.e., the j summation actually starts contributing at $j=m$. As previously discussed, we keep only the lowest-order terms in powers of β. Hence the only term contributing significantly from the j summation is $j=m$ term and Eq. (A4) reduces to

$$\frac{\sigma}{\sigma_0}=1+\sum_{m=1}^{\infty}\left(\frac{\mathcal{E}^2}{T}\right)^m\frac{3(m+1)(-1)^m}{(2m+3)(2m+1)}\sum_{p=0}^{2m}\frac{(2m+3)(2m+1)\cdots(5-2m+2p)(-1)^p4\cdot5\cdots(4+p-1)}{2^{2m-p-1}(2m-p)!\,p!\,(1+1/T)^p}$$
$$\times\sum_{n=1}^{m}\frac{(-1)^n n^{2m}}{(m+n)!\,(m-n)!} \tag{A5}$$

Using Eq. (A3) in the above equation results in the power series given by Eq. (10). In Eq. (10), the leading term for each coefficient in the power series is given by $p=0$ term. A comparison of the $(m+1)$th term with the mth term as $m\to\infty$ shows that the series is convergent only for $\mathcal{E}/T^{1/2}<2$.

APPENDIX B

When $\mathcal{E}/\beta T^{1/2}\gg 1$, one can obtain an asymptotic approximation for Eq. (9) which is rewritten as

$$\frac{\sigma}{\sigma_0}=\frac{3}{4}(1+T)^4\sum_{n=-1/\beta}^{\infty}\frac{(1+n\beta)^{3/2}}{(1+T+Tn\beta)^4}I((\mathcal{E}/\beta T^{1/2})(1+n\beta)^{1/2}), \tag{B1}$$

where

$$I(t)=\int_0^{\pi} d\theta\ \sin^3\theta\, J_n^2(t\sin\theta)$$
$$=\frac{2}{t^3}\int_0^{t}\frac{y^3dy}{(t^2-y^2)^{1/2}}J_n^2(y)\ . \qquad \text{(B2)}$$

Now $J_n^2(y)$ is very small for $n \gg y$, while for $y \gg n$ it behaves like $1/\pi y$. Thus for $n > (\mathcal{E}/\beta T^{1/2})(1+n\beta)^{1/2}$, the nth-term contribution to the summation in (B1) is extremely small, and we can restrict the positive-n summation to M, the positive root of the equation,

$$n=(\mathcal{E}/\beta T^{1/2})(1+n\beta)^{1/2}\ . \qquad \text{(B3)}$$

For the same reason, the negative-n summation is restricted up to N, the negative root of Eq. (B2). These values are obtained as

$$M=\frac{\mathcal{E}}{\beta T^{1/2}}\left[+\frac{1}{2}\frac{\mathcal{E}}{T^{1/2}}+\left(\frac{\mathcal{E}^2}{4T}+1\right)^{1/2}\right],$$
$$N=\frac{\mathcal{E}}{\beta T^{1/2}}\left[+\frac{1}{2}\frac{\mathcal{E}}{T^{1/2}}-\left(\frac{\mathcal{E}^2}{4T}+1\right)^{1/2}\right]. \qquad \text{(B4)}$$

As a pictorial representation of this approximation we present in Fig. 3, a plot of $y_n=(\mathcal{E}/\beta T^{1/2})(1+n\beta)^{1/2}$ as a function of n. M and N are the values of n where this graph intersects the lines $y_n=n$ and $y_n=-n$ respectively. Essentially, the approximation made above is that instead of integrating over y and summing over n over all the area under the curve, we only take the shaded area ($y_n \geq |n|$) into consideration since contribution of the unshaded area to the cross section is very small. In addition, when $y>n$, a good approximation for integration purposes is

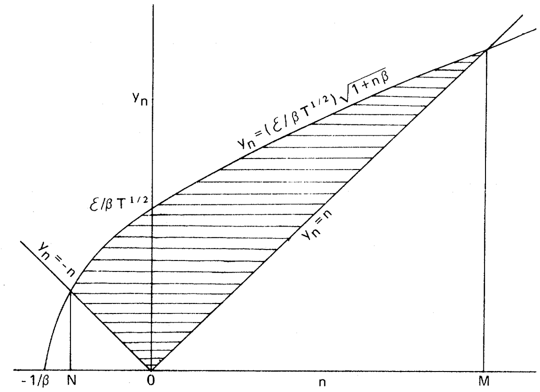

FIG. 3. Function $y_n=(\mathcal{E}/\beta T^{1/2})(1+n)^{1/2}\beta$ vs n.

$$J_n^2(y)=1/\pi(y^2-n^2)^{1/2}\ .$$

Thus we obtain

$$\frac{\sigma}{\sigma_0}=\frac{3}{4}(1+T)^4\sum_{n=N}^{M}\frac{(1+n\beta)^{3/2}}{(1+T+Tn\beta)^4}$$
$$\times I((\mathcal{E}/\beta T^{1/2})(1+n\beta)^{1/2})\ ,$$

where

$$I(t)=\frac{2}{\pi t^3}\int_n^t\frac{y^3\,dy}{(t^2-y^2)^{1/2}(y^2-n^2)^{1/2}}=\frac{t^2+n^2}{2t^3}\ .$$

Approximating the summation over n by an integral and using Eq. (B6) in Eq. (B5), one obtains

$$\frac{\sigma}{\sigma_0}=\frac{3}{8}(1+T)^4\frac{T^{1/2}}{\mathcal{E}}\int_{N\beta}^{M\beta}dZ\,\frac{1+Z+TZ^2/\mathcal{E}^2}{(1+T+TZ)^4}\ .$$

Performing this integral leads to the result given in Eq. (12).

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*Research supported in part by the City University Faculty Award Program.

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