Potential function and probability distribution of a nonequilibrium system: The ballast resistor

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The ballast resistor is a simple, one-dimensional device which has an instability far from equilibrium. The steady-state solutions of the energy-conservation equation show a first-order phase transition. We derive a potential function which is minimized in the steady state. The probability distribution of different states of the system is related to this potential.

In recent years there has been considerable interest in the spatial and temporal structures which arise in systems far from equilibrium.[1] These "dissipative structures," of which the most striking example is a living thing, maintain themselves by constantly producing entropy and characteristically decay when this dissipation ceases. They arise outside the linear range of irreversible thermodynamics, and their general properties are at present poorly understood.

Up to now, the most exhaustively studied dissipative structure has been the convection cell structure of a layer of fluid heated from below (the Bénard problem).[2] We suggest that the ballast resistor, a device which has been used as a practical current regulator for more than seventy years, provides a simple and more easily analyzed example which has the additional merit of being one dimensional. Although the basic mechanism of its operation was elucidated many years ago by Busch[3] and Jones[4] (its properties have also been discussed recently by Skocpol, Beasley, and Tinkham[5]), the ballast resistor has never been studied from the point of view of irreversible thermodynamics.

The typical ballast resistor is a straight, horizontal iron wire in a tube of gas. The outside of the tube is held at constant temperature. When a voltage is applied to the wire, its temperature increases. The wire will lose heat to the outside at a rate A (per unit length). Given the description of the gas in the tube, A is a function only of the local temperature T of the wire. The steady state of a wire constrained to be at uniform temperature is then described by

$$V = iR, \quad A = i^2R,$$

where R is the resistance of the wire per unit length, i is the current, and V is the voltage drop across the wire divided by its length. This can be expressed as

$$i = (A/R)^{1/2}, \quad V = (AR)^{1/2}. \tag{1}$$

To examine the operation of the ballast resistor, let us turn to a possible i-V characteristic (locus of $A = i^2R$) sketched in Fig. 1. Both A, R, and hence also V, are monotonically increasing functions of temperature T; thus T also increases along the characteristic in the direction of the arrow. If for some temperature range $\partial \ln R/\partial T > \partial \ln A/\partial T$, i will be a decreasing function of T, and the corresponding part of the characteristic will have $\partial i/\partial V < 0$. The large values of dR/dT required may be obtained near a ferromagnetic or superconducting phase transition.

For most of the i-V curve, V is specified for a given value of i. However, for $i_2 < i < i_1$ the characteristic shows three possible values of V. A simple argument shows that the negative-resistance part of the characteristic, like the analogous negative compressibility region of the equation of state for a Van der Waals fluid, is unstable. Assume a wire operating at some current i_0, $i_2 < i_0 < i_1$, and consider a portion operating at point c

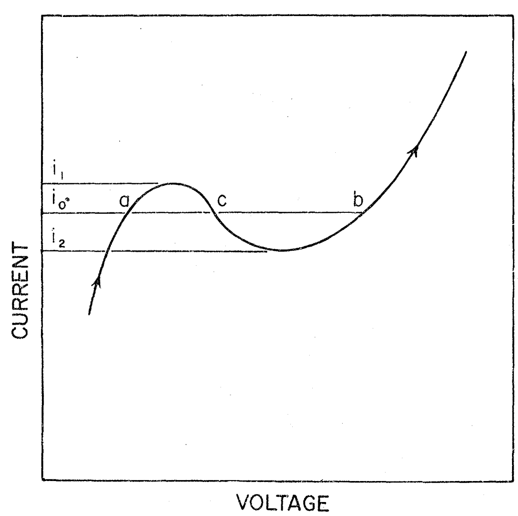

FIG. 1. Characteristic i-V curve discussed in text.

on the negative-resistance curve which fluctuates towards higher V. This is a fluctuation towards higher T and also into the region above the characteristic curve where $A < i^2R$. Thus the tendency will be for the temperature to rise even higher. A similar argument shows points a and b are stable, and so the wire will separate into regions of temperature at points a and b on the curve.

This separation into regions of different uniform temperatures constitutes a spatially ordered structure in some ways analogous to the Bénard instability where a thermal gradient serves as a source of energy dissipation and causes a spatially ordered velocity field. Here the applied voltage is the source of energy, and a spatially ordered, one-dimensional temperature field results.

When the wire is operating in the two-phase region, the current i_0 is constant, determined by the i-V characteristic curve, and the voltage drop across the total length depends upon the relative amounts of phases a and b. We should now like to see how the ballasting current i_0 can be determined from the i-V characteristic in a way analogous to the Maxwell construction for a Van der Waals gas. We start from the requirement that in each region of uniform temperature T_a and T_b the rate of heat conduction along the wire W must vanish. As will be seen from the final result, the derivation must start with the quantity W^2 in order to arrive at a formula that does not involve the unknown W. We therefore write

$$W^2(a) - W^2(b) = 0$$

or

$$\int_a^b W\,dW = 0\,.$$

We substitute $W = -\lambda\, dT/dx$, where $\lambda(T)$ is the thermal conductivity of the wire multiplied by its cross section. After an integration by parts, we obtain

$$-\int_a^b \lambda \frac{dW}{dx}\, dT = 0\,.$$

We now use the energy-conservation equation $dW/dx = i^2R - A$ to get our final result:

$$\int_{T_a}^{T_b} \lambda(A - i^2R)\, dT = 0\,. \qquad (2)$$

Since a and b must correspond to points of the characteristic with the same current, there is only one such pair of points which will satisfy this condition, and i will indeed be constant in the two-phase region. We will henceforth define T_a and T_b to be the temperatures which satisfy (2). It should be noted that if we plot the characteristic in terms of $i^2 = A/R$ versus $\int \lambda R\, dT$, we will have an equal-area construction like that of Maxwell.

One might ask whether this phase transition will exhibit a critical point. The critical point can in fact be reached by raising the temperature T_0 of the exterior of the tube. The two-phase region exists only for $\partial \ln R/\partial T > \partial \ln A/\partial T$; as T_0 increases with T constant, $\partial \ln R/\partial T$ will remain constant, but eventually $A \sim k(T - T_0)$ and $\partial \ln A/\partial T = k/A \to \infty$. At the critical point one will have $\partial \ln R/\partial T$ tangent to $\partial \ln A/\partial T$ at one point and $\partial \ln R/\partial T < \partial \ln A/\partial T$ elsewhere. The behavior could be interesting near a ferromagnetic phase transition where $\partial \ln R/\partial T$ diverges. In practice, the divergence will be rounded off so that a critical point will still occur for the ballast resistor; the detailed behavior could however be influenced by the nature of the divergence for $\partial \ln R/\partial T$.

The general form of a set of i-V characteristic curves for various values of T_0 or gas pressure will resemble pressure-volume isotherms for a liquid-vapor system, but with the pressure axis inverted. The inversion is explained by the fact that equilibrium systems become more ordered as they move towards zero temperature, while dissipative systems become more ordered as they move away from equilibrium. A similar inversion is noted for the Bénard instability; this system is invariant under a change in sign of the velocity field, and the upside down coexistence curve is therefore symmetric, like that of a ferromagnet. The ballast resistor, like the pure fluid system, does not possess this intrinsic symmetry.

To derive a potential function minimized by this system, we must start from the energy-conservation equation:

$$A(T) - i^2R(T) = \frac{d}{dx}\, \lambda \frac{dT}{dx}\,. \qquad (3)$$

This equation has the same form as Newton's second law, with $F \to A - i^2R$, $m \to 1$, $dx \to \lambda\, dT$, and $dt \to dx$. Boundary conditions of fixed temperature at the end points of the wire correspond to fixed locations for the beginning and end of an orbit. We can therefore spare ourselves the need to do a calculation and immediately write the mathematical analog of Hamilton's principle:

$$\mathfrak{L}\{T(x)\} \equiv \int_{\text{wire}} dx \left[\int^{T(x)} \lambda(T')[A(T') - i^2R(T')]\,dT' + \frac{1}{2}\left(\lambda(T)\frac{dT(x)}{dx}\right)^2\right]. \qquad (4)$$

This potential function is of the same form as the approximate free energy deduced by Van der Waals[6] and Cahn and Hilliard[7] for the interface between two fluid phases. As can be seen from the derivation of Eq. (2), the two terms in the potential become equal when the steady state is attained, like the two terms of the Van der Waals interfacial free energy at equilibrium.

Up to this point, our analysis has ignored fluctuations. If fluctuations are included, one finds that an ensemble of identically prepared systems is distributed over a range of accessible states.

Consider first a ballast resistor operating at constant current near a uniform steady state. To account for thermal fluctuations, we add a Langevin-type fluctuating external heat source $\tilde{Q}(x,t)$. Then the energy-conservation equation (3) becomes

$$c_v \frac{\partial \Delta T}{\partial t} = -\alpha \Delta T + \lambda \frac{\partial^2 \Delta T}{\partial x^2} + \tilde{Q}(x,t), \quad (5)$$

where

$$\alpha = \frac{dA}{dT} - i^2 \frac{dR}{dT}. \quad (6)$$

In the Appendix it is shown that if all energy transfers in the system can be described by the fluctuation-dissipation theorem, $\tilde{Q}$ is a Gaussian random process with spectral density

$$\langle \tilde{Q}^2(k,\omega)\rangle = \frac{2}{\pi} k_B T^2 \left(4\frac{A}{T} + \frac{dA}{dT} + \lambda k^2\right). \quad (7)$$

Equation (7) is valid for small temperature excursions about a locally stable state; unfortunately, it is probably not valid for the more interesting case of large fluctuations which could cause a transfer from metastable to stable states. If we Fourier-transform (5) and insert (7), we obtain

$$\langle \Delta T^2(k,\omega)\rangle = \frac{2k_B T^2}{\pi} \frac{4A/T + dA/dT + \lambda k^2}{(\alpha+\lambda k^2)^2 + \omega^2 c_v^2}. \quad (8)$$

Performing the inverse Fourier transform in the time domain yields

$$\langle \Delta T(k,0)\Delta T(k,\tau)\rangle = \frac{k_B T^2}{\pi c_v} \frac{4A/T + dA/dT + \lambda k^2}{\alpha + \lambda k^2} \times e^{-(\alpha+\lambda k^2)\tau}. \quad (9)$$

To obtain $\langle \Delta T^2(t)\rangle$ for a single spatial mode, Eq. (9) with $\tau = 0$ must be divided by the density of states L/π, where L is the length of the wire. As we will not be interested in fluctuations of wavelength shorter than $\lambda/(dA/dT)$, we will drop the λk^2 term in the numerator of Eq. (9).

Since ΔT is related to $\tilde{Q}$ by the linear relation (5), it is a Gaussian random process. Since we know the second moment (9) of $\Delta T(k,t)$, we can write its distribution function:

$$P[\Delta T(k)] = N \exp\left\{-\tfrac{1}{2}\Delta T^2(k)\left(\frac{k_B T^2}{L c_v} \frac{4A/T + dA/dT}{\alpha + \lambda k^2}\right)^{-1}\right\}. \quad (10)$$

But the increase in the potential (4) due to a sinusoidal temperature fluctuation $\Delta T \sin kx$ is

$$\Delta \mathcal{L} = \tfrac{1}{4} L\lambda\alpha \Delta T^2 + \tfrac{1}{4} L \lambda^2 k^2 \Delta T^2. \quad (11)$$

Therefore,

$$P[\Delta T(k)] = N \exp\left\{-\tfrac{1}{2}\Delta\mathcal{L}(k)\left[\frac{\lambda k_B T^2}{c_v}\left(\frac{A}{T} + \frac{1}{4}\frac{dA}{dT}\right)\right]^{-1}\right\}. \quad (12)$$

Since $\Delta\mathcal{L}(k)$ is proportional to $\Delta T^2(k)$, orthogonal modes will contribute independently to the potential. The probability of any temperature configuration of the entire wire will then be given by

$$P\{\Delta T(x)\} = N' \exp\left\{-\tfrac{1}{2}\Delta\mathcal{L}\{\Delta T(x)\} \times \left[\frac{\lambda k_B T^2}{c_v}\left(\frac{A}{T} + \frac{1}{4}\frac{dA}{dT}\right)\right]^{-1}\right\}. \quad (13)$$

Probability distributions of the same form as the Boltzmann factor have been found in other nonequilibrium cases as well.[1]

We have thus shown how, near the uniform steady state, the probability distribution for the temperature $T(x)$ of the wire depends on the functional $\mathcal{L}$. Haken[1] has shown that if the system satisfies the rule of detailed balance, it will have a potential function of a form resembling $\mathcal{L}$. The ballast resistor may not always satisfy the detailed-balance criterion[8]; nevertheless, it is tempting to conjecture that a distribution like (13) remains valid even away from the uniform steady state. This conjecture has a number of corollaries regarding the behavior of the system far away from the uniform steady state. To the extent that these corollaries can be tested, they appear to be correct. The testable corollaries are as follows:

(i) Absolute minima of (4) correspond to stable states, and relative minima correspond to metastable states. This is confirmed by the Maxwell construction (2).

(ii) Since, if fluctuations are ignored, systems evolve from less probable to more probable states, $\mathcal{L}$ should always decrease. This may be confirmed by differentiating (4) directly[9]:

$$\begin{aligned}\frac{d\mathcal{L}}{dt} &= \int_0^L dx \left\{\lambda(A - i^2 R)\frac{dT}{dt} + \lambda\frac{d\lambda}{dT}\frac{dT}{dt}\left(\frac{dT}{dx}\right)^2 + \lambda^2 \frac{dT}{dx}\frac{d}{dt}\left(\frac{dT}{dx}\right)\right\} \\ &= \int_0^L dx \left\{\lambda\frac{dT}{dt}\left[-c_v\frac{dT}{dt} + \frac{d}{dx}\left(\lambda\frac{dT}{dx}\right)\right] + \lambda\frac{dT}{dx}\left[\frac{d\lambda}{dx}\frac{dT}{dt} + \lambda\frac{d}{dx}\left(\frac{dT}{dt}\right)\right]\right\} \\ &= -\int_0^L \lambda c_v \left(\frac{dT}{dt}\right)^2 dx + \left[\lambda^2 \frac{dT}{dx}\frac{dT}{dt}\right]_0^L.\end{aligned}$$

c_v is the heat capacity per unit length of wire. The first term is always negative. The second term vanishes because the temperature is fixed at the boundaries, as was assumed in the derivation of (4).

(iii) Given the existence of a probability distribution like (13), the theorem of Landau and Lifschitz,[10] ruling out long-range order in one-dimensional systems, will apply. This implies that the most likely state, corresponding to the minimum of $\mathcal{L}$, which has long-range order, differs from the mean state. An argument can be constructed[11] which takes fluctuations into account while avoiding not only any assumptions like (13), but also weaker assumptions about relative probabilities of different states of the system which would suffice to prove the theorem of Landau and Lifschitz. This argument indicates that the ballast resistor does indeed lack long-range order.

We therefore conclude that the potential function deduced from the macroscopic state equation of the ballast resistor is intimately connected with its probability distribution. Whether this relationship can be extended into a general theorem, analogous to the H theorem of classical statistical mechanics, remains to be seen.

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APPENDIX

The spectrum of the noise source Q may be calculated by treating it as the sum of two sources of heat in the wire: random heat flows within the wire and between the wire and the gas, which we shall denote by $\tilde{Q}_h$, and fluctuations in electrical power dissipation due to Johnson noise in the wire $\tilde{Q}_e$. If we assume these are independent Gaussian noise sources, then we have

$$\langle \tilde{Q}(k,\omega)^2\rangle = \langle \tilde{Q}_e(k,\omega)^2\rangle + \langle \tilde{Q}_h(k,\omega)^2\rangle .$$

We consider first $\tilde{Q}_e$. We may expand the noise voltage, which may be a function of the applied current, about equilibrium as follows:

$$\langle V_n^2\rangle = \langle V_n^2\rangle^{(0)} + i\langle V_n^2\rangle^{(1)} + i^2\langle V_n^2\rangle^{(2)} + \cdots .$$

Bernard and Callen[12] compare this with the expansion of dc voltage in terms of the current:

$$V = iR^{(1)} + i^2R^{(2)} + i^3R^{(3)} + \cdots .$$

They show that there is a general relation (for $\hbar\omega \ll k_BT$):

$$\tfrac{1}{2}\pi\langle V_n^2\rangle^{(n)} \simeq k_BTR^{(n+1)} .$$

Since our current is small enough to give a constant resistance at constant temperature $i^nR^{(n+1)} \ll R^{(1)}$, and therefore the higher terms in the expansion of V_n^2 may also be neglected. We are left with the equilibrium fluctuations. The mean-square noise voltage across any length l of wire is therefore

$$\left\langle \left\{ \int_x^{x+l} E_n(x',\omega)\,dx' \right\}^2 \right\rangle = \frac{2}{\pi} k_BTRl , \tag{14}$$

where E_n is the noise in the electric field at any point, and $\int_0^l E_n\,dx$ is a Gaussian random variable. A theorem of Chandrasekhar[13] states that if (14) holds, then $V_n(k,\omega)$ is a Gaussian random variable with

$$\langle V_n(k,\omega)^2\rangle = (2/\pi)k_BTR ,$$

where $V_n(k,\omega)$ is defined by

$$V_n(k,\omega) = \int_{-L/2}^{L/2} E(x,\omega)\,e^{ikx}\,dx .$$

Now we can calculate $\tilde{Q}_e(k,\omega)$. Since the mean power delivered is i^2R, $\tilde{Q}_e$ is the difference between the instantaneous power delivered and i^2R (we take the limit $L\to\infty$):

$$\tilde{Q}_e(k,\omega) = \frac{1}{R}\int dx\, e^{-ikx}\int dt\, e^{-i\omega t} \times [2E_0E_n(x,t) + E_n^2(x,t)] ,$$

where E_0 is the mean electric field. E_n^2 may be neglected because $\langle E_n^2\rangle^{1/2} \ll E_0$. Then,

$$\langle \tilde{Q}_e(k,\omega)^2\rangle = (4E_0^2/R^2)\langle V_n(k,\omega)^2\rangle = \frac{8}{\pi}\frac{E_0^2}{R^2}k_BTR = \frac{8}{\pi}Ak_BT .$$

A and R are here the heat flow and resistance at the mean temperature. We will now calculate the remaining term $\tilde{Q}_h$. Consider a single mode of wave vector k of the wire. This mode will satisfy a Langevin equation of the form (near steady state):

$$c_v\frac{d}{dt}\Delta T(k,t) = -\left(\frac{\partial A}{\partial T} + \lambda k^2\right)\Delta T(k,T) + \tilde{Q}_h(k,t) .$$

The generalized fluctuation-dissipation relation[14] can be applied when the relationship between heat flows and temperature gradients is linear. It implies directly that

$$\langle \tilde{Q}_h(k,\omega)^2\rangle = \frac{2}{\pi} k_B T^2 \left(\frac{\partial A}{\partial T} + \lambda k^2 \right).$$

We therefore conclude that

$$\langle \tilde{Q}(k,\omega)^2\rangle = \frac{2}{\pi} k_B T^2 \left(\frac{4A}{T} + \frac{\partial A}{\partial T} + \lambda k^2 \right),$$

whenever the ballast resistor is operating in a regime where heat flows at any point are proportional to temperature gradients.

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