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Electron diffusion under the influence of an electric field near absorbing boundaries

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Numerical solutions have been obtained for the electron distribution function as a function of both energy and position for the case of a uniform stream of electrons being absorbed at a collecting electrode under the influence of a uniform electric field. Solutions are obtained from the Boltzmann equation for various power-law dependences of the momentum-transfer cross section on electron energy. It is found that the electron density distribution obtained by integrating the distribution function over energy varies significantly from the conventional solution obtained by solving the electron continuity equation with drift velocities and either transverse or longitudinal diffusion coefficients taken to be independent of position. The average electron energy increases by $\sim 50\%$ near the boundary for a cross section increasing linearly with energy making the effective drift and diffusion coefficients a function of position.

I. INTRODUCTION

Electron drift and diffusion under the influence of a uniform electric field in a gas has been the study of extensive investigations.[1-3] For pressures of the order of 1 Torr or more each electron experiences approximately 10^9 collisions per second, and it has generally been believed that electron motion can be represented to a fair approximation by a drift velocity and a diffusion coefficient that are independent of position. The present investigation is concerned with examining this premise and concludes that for the large density gradients occurring near an absorbing boundary this approximation is significantly in error.

Theoretical analysis of such electron motion has been dominated by formulations using the Boltzmann transport equation,[1] although some workers have approached the problem using "mean-free-path" methods.[4-6] The present paper follows methods whereby the distribution function is represented by the first two terms in a spherical harmonics expansion in velocity space and on substitution in the Boltzmann transport equation one obtains after some manipulation a second-order differential equation in $f^0(\epsilon, z)$. The term f^0 is the first or spherically symmetric part of the distribution function, ϵ is the electron energy, and z is the axial position. The validity of the two term expansion has been shown[7] to be accurate, at least for low electric fields and where there are only elastic collisions. We assume the two term expansion is also accurate for the cases of large electron density gradients occurring in the examples discussed in the present paper. This assumption is reasonable because the electron mean free path λ is of the order of only 1% of the distance over which the electron density is appreciably perturbed. We must await investigations such as Monte Carlo[8] methods to test the validity of this expansion near a boundary.

Although $f^0(\epsilon, z)$ is a function of z and thus it would be expected that the diffusion coefficient would be a function of z, from the time of Langevin[9] in 1903 electron diffusion coefficients have been regarded as being accurately given by an average of $\lambda|\vec{v}|/3$ over the velocity distribution and to be independent of z. It was only in 1967 that it was found[10] that measurements of widths of electron pulses diffusing in an electric field indicated that the effective diffusion coefficient longitudinal to the electric field differed from the diffusion coefficient transverse to the electric field by about a factor of 2 for hydrogen, nitrogen, and helium and by up to a factor of 8 in argon. These measurements were explained quantitatively in two independent investigations as resulting from f^0 being dependent on z through terms involving spatial derivatives in the Boltzmann equation; Parker and Lowke[11,12] used an analysis specifically applied to diffusing electron pulses, and Skullerud[13] used an analysis similar to that of Wannier[14] to determine the effect to first order of spatial gradients in z.

Since these investigations, it has been assumed[15-19] that electron diffusion is represented by two diffusion coefficients D_T and D_L which, respectively, represent diffusion transverse and parallel to the electric field where D_T and D_L are independent of position. Such a representation successfully provides[18] an explanation for the validity of the partly empirical formula describing current ratios as a function of gas pressure in the Townsend-Huxley experiment. The difference between D_T and D_L arises because the average electron energy and thus the drift velocity W is a function of the electron density gradient in the direction of the electric field. A qualitative understanding of this difference can be given as follows. Representing W by $W_0 + W_1(\partial n/\partial z)/n$, the electron flux, $nW - \partial(nD)/\partial z$ becomes $nW_0 - (D - W_1)\,\partial n/\partial z$ if we omit the term in $\partial D/\partial z$; W and D are the local drift and diffusion coefficients which are a function of z and W_0 and W_1 are assumed to be independent of z. Thus the effective diffusion coefficient in the electric field direction is $D_L = D - W_1$ instead of D where generally $D_T = D$. In this formulation D_L is assumed to be independent of position and no account is taken of higher-order effects of density gradients that occur for example near an absorbing boundary.

In the present paper the theoretical approach in Sec. II is related to previous investigations using the Boltzmann equation. Numerical examples of solutions near an absorbing boundary for the case of a steady uniform stream of electrons impinging on an absorbing metal electrode are given in Sec. III. We show in Sec. IV that for the simple case of elastic collisions with a power-law dependence of the momentum-transfer cross section on energy, the electron density distribution $n(z)$ is independent of E/N provided the gas pressure is such that D_0/W_0 is constant; E/N is the ratio of electric field strength to gas density and D_0/W_0 is the ratio of the transverse diffusion coefficient to the drift velocity for no density gradients. In Sec. V we discuss effects of back diffusion to an absorbing plate.

II. THEORY

A. General

The Boltzmann equation for the distribution function $f(\vec{v},\vec{r})$ of electrons moving under the influence of a uniform electric field E is[1]

$$\vec{v}\cdot\frac{\partial f}{\partial\vec{r}} + \frac{\vec{E}e}{m}\cdot\frac{\partial f}{\partial\vec{v}} + \frac{\partial f}{\partial t} = \left(\frac{\partial f}{\partial t}\right)_{\rm col}; \tag{1}$$

where $\vec{v}$ is the velocity of electrons, $\vec{r}$ indicates the position of electrons, e is the electronic charge, m is the electron mass, and f is such that $f(\vec{v},\vec{r})\,d\vec{v}\,d\vec{r}$ is the number of electrons with velocity $\vec{v}$ in the range $d\vec{v}$, and at position $\vec{r}$ within $d\vec{r}$. $(\partial f/\partial t)_{\rm col}$ represents the rate of change of f in time due to elastic collisions. On representing f in Eq. (1) by the first two terms of a spherical harmonics expansion in velocity space, i.e.,

$$f(\vec{v},\vec{r}) = f^0(\epsilon, z) + f^1(\epsilon, z)\cos\theta, \tag{2}$$

where θ is the angle between $\vec{v}$ and $\vec{E}$, we obtain[1] two equations independent of θ by multiplying Eq. (1), respectively, by $2\pi\sin\theta\,d\theta$ and $2\pi\sin\theta\cos\theta\,d\theta$ and integrating over θ from 0 to π; ϵ is the electron energy $m|\vec{v}|^2/2$. These equations are[1]

$$\frac{4\pi}{m}\left(\frac{2}{m}\right)^{1/2}\epsilon^{1/2}\frac{\partial f^0}{\partial t} = \frac{16\pi N}{mM}\frac{\partial}{\partial\epsilon}\left[\epsilon^2 Q\left(f^0 + kT\frac{\partial f^0}{\partial\epsilon}\right)\right] - \frac{8\pi\epsilon}{3m^2}\frac{\partial f^1}{\partial z} - \frac{8\pi eE}{3m^2}\frac{\partial}{\partial\epsilon}(\epsilon f^1) \tag{3}$$

and

$$f^1 = -\frac{1}{NQ}\left(\frac{\partial f^0}{\partial z} + eE\frac{\partial f^0}{\partial\epsilon}\right), \tag{4}$$

where M is the mass of the gas molecules, N is the gas number density, k is Boltzmann's constant, e is the electronic charge, and Q is the momentum-transfer cross section.

On eliminating f^1 from Eqs. (3) and (4) we obtain the basic equation in f^0 for the first term or spherical part of the distribution function. This equation is

$$\frac{4\pi}{m}\left(\frac{2}{m}\epsilon\right)^{1/2}\frac{\partial f^0}{\partial t} = \frac{16\pi}{mM}\frac{\partial}{\partial\epsilon}\left[N\epsilon^2 Q\left(f^0 + kT\frac{\partial f^0}{\partial\epsilon}\right) + \frac{MeE}{6mN}\frac{\epsilon}{Q}\left(eE\frac{\partial f^0}{\partial\epsilon} + \frac{\partial f^0}{\partial z}\right)\right] + \frac{8\pi}{3m^2N}\frac{\partial}{\partial z}\left[\frac{\epsilon}{Q}\left(eE\frac{\partial f^0}{\partial\epsilon} + \frac{\partial f^0}{\partial z}\right)\right]. \tag{5}$$

The physical significance of the terms of Eq. (5) are as follows. Changes with time of the electron distribution function $f^0(\epsilon, z)$ can occur due to the electrons changing their energy, represented by the first group of terms within the square brackets of $\partial/\partial\epsilon$, or to the electrons changing their position represented by the second group of terms within $\partial/\partial z$. The terms in the first group represent, respectively, changes in electron energy due to (i) collisions with the gas atoms, (ii) collisions of the second kind due to the gas temperature T, and (iii) drift or (iv) diffusion in the electric field. The terms within the second group in $\partial/\partial z$ represent changes in position due to drift and diffusion.

Simple exact solutions of Eq. (5) that give f^0 as a function of the two variables ϵ and z are generally not possible. However the equation has been used extensively in two approximate forms. Firstly, all terms involving $\partial/\partial z$ are set at zero and numerical[20] and analytic solutions of $f^0(\epsilon)$ have been obtained for various forms of $Q(\epsilon)$. These solutions give the electron distribution as a function of energy for a uniform distribution of electrons in space.

Secondly, Eq. (5) has been used after integrating over energy. The first group of terms in $\partial/\partial\epsilon$ then all vanish as the terms in the square bracket are zero at infinite energy and also at zero energy, assuming $Q(0)$ is finite. The equation then becomes the electron continuity equation, i.e.,

$$\frac{\partial n}{\partial t}=-\frac{\partial}{\partial z}\left[nW-\frac{\partial(nD)}{\partial z}\right], \tag{6}$$

where

$$nW=-\frac{8\pi eE}{3m^2N}\int_0^\infty \frac{\epsilon}{Q}\frac{\partial f^0}{\partial\epsilon}d\epsilon, \tag{7}$$

$$nD=\frac{8\pi}{3m^2N}\int_0^\infty \frac{\epsilon f^0}{Q}d\epsilon, \tag{8}$$

$$n(z)=\frac{4\pi}{m}\left(\frac{2}{m}\right)^{1/2}\int_0^\infty \epsilon^{1/2}f^0(\epsilon,z)\,d\epsilon, \tag{9}$$

and n is the electron density. In Eq. (6) the drift velocity W and diffusion coefficient D are actually functions of z as W and D from Eqs. (7) and (8) are functions of f^0, and f^0 is a function of z. But in practice Eq. (6) is generally used with W and D assumed to be independent of z. Furthermore, the values of W and D are evaluated from Eqs. (7) and (8) using values of $f^0(\epsilon)$ obtained from Eq. (5) assuming that there are no density gradients.

In the present paper we obtain numerical solutions of $f^0(\epsilon,z)$ using Eq. (5) with all gradient terms present. As f^0 is obtained in detail as a function of both ϵ and z the electron density $n(z)$ can be obtained directly from Eq. (9). For steady-state solutions, the electron flux $\Gamma=nW-\partial(nD)/\partial z$ should be independent of z as follows from Eq. (6) where $W(z)$ and $D(z)$ are evaluated from Eqs. (7) and (8) using the derived solutions of $f^0(\epsilon,z)$. Tests as to whether $\Gamma(z)$ is a constant for all values of z provide a useful check on the validity of the solutions $f^0(\epsilon,z)$.

B. Boundary conditions

We consider the case of a uniform and steady stream of electrons moving towards an absorbing boundary or metal electrode at $z=0$. The function f^0 is represented as a two-dimensional array from $z=-h$, far from the boundary to $z=0$, at the absorbing boundary and from $\epsilon=0$ to $\bar{\epsilon}$, $\bar{\epsilon}$ being chosen to be so large that there is an insignificant number of electrons with energy greater than $\bar{\epsilon}$. Position h is chosen by trial so that f^0 at $z=-h$ is effectively unperturbed by the boundary and $\partial f^0/\partial z\simeq 0$.

As Eq. (5) is a second-order differential equation in both ϵ and z, we require boundary conditions for f^0 at both $\epsilon=0$ and $\bar{\epsilon}$ and also $z=-h$ and 0.

(1) At $z=-h$, $f^0(\epsilon,-h)$ is simply the conventional distribution function obtained by solving Eq. (5) omitting all terms in $\partial/\partial z$, i.e.,

$$f^0(\epsilon,-h)=A\exp\left[-\frac{6m}{M}\left(\frac{N}{E}\right)^2\int_0^\epsilon \frac{\epsilon Q^2\,d\epsilon}{1+(kT/e)(6m/M)(\epsilon Q^2N^2/E^2)}\right], \tag{10}$$

where ϵ is now in eV and E in V/cm. For convenience we have chosen the normalization constant $A=f^0(0,-h)$ to be 10^5.

(2) At $z=0$, which is the absorbing boundary, the effective boundary condition is $f^0(\epsilon,0)=0$. We require that electrons can pass into the surface, but that none come back from the surface, i.e., it is assumed that there is zero reflection of electrons at the surface. With a two term representation of f in spherical harmonics, it is impossible to make $f=f^0+f^1\cos\theta$ zero for all θ in the range $\frac{1}{2}\pi<\theta<\pi$ as is appropriate for a plane surface. As an approximation we set $f=0$ for $\theta=\pi$, i.e., $f^0(\epsilon,0)=-f^1(\epsilon,0)$ is the boundary condition from Eq. (2). Then using the expression for f^1 from Eq. (4) we obtain

$$f^0(\epsilon,0)=\frac{1}{NQ}\frac{\partial f^0}{\partial z}=\lambda\frac{\partial f^0}{\partial z},$$

where the term in $\partial f^0/\partial\epsilon$ of Eq. (4) has been omitted in comparison with the term in $\partial f^0/\partial z$ at the boundary; $\lambda=1/NQ$ is the mean free path of electrons. The boundary condition $f^0/\lambda=\partial f^0/\partial z$ implies that f^0 becomes zero at a distance λ behind the boundary. As λ is only of the order of 1% of the region over which f^0 is perturbed by the boundary we take as an effective boundary condition for our calculations $f^0(\epsilon,0)=0$.

(3) At $\epsilon=\bar{\epsilon}$ we set $f^0(\bar{\epsilon},z)=0$. It is assumed that there is a negligible number of electrons having energy greater than $\bar{\epsilon}$. The value of $\bar{\epsilon}$ is obtained by trial so that increasing $\bar{\epsilon}$ by say 50% produces a negligible change in the derived values of $f^0(\epsilon,z)$ and in particular of the electron density distribution $n(z)$.

(4) At $\epsilon=0$ we set $eE\,\partial f^0/\partial\epsilon=-\partial f^0/\partial z$. This boundary condition follows from the requirement that $\partial f^0/\partial t$ be finite at $\epsilon=0$. On dividing Eq. (5) by $\epsilon^{1/2}$ to obtain $\partial f^0\partial t$, all terms on the right-hand side are zero at $\epsilon=0$ except a term in $(eE\,\partial f^0/\partial\epsilon+\partial f^0/\partial z)/\epsilon^{1/2}$. To prevent this term becoming a singularity we require $eE\,\partial f^0/\partial\epsilon+\partial f^0/\partial z$ to be zero at $\epsilon=0$. It is assumed that near $\epsilon=0$, Q is finite and nonzero.

III. NUMERICAL SOLUTIONS

A. Distribution function

Numerical solutions of $f^0(\epsilon,z)$ have been obtained using two different methods of solution. In the first method, solutions were obtained using a simple explicit iterative scheme, whereby initial arbitrary values of $f^0(\epsilon, z)$ were modified by evaluating $\partial f^0/\partial t$ from Eq. (5). The time step for the modification of f^0 was kept less than the value determined by the Von Neumann criterion[21] for numerical stability for the terms $\partial^2 f^0/\partial\epsilon^2$ and $\partial^2 f^0/\partial z^2$.

A second method[22] using an implicit scheme with matrix inversion at each iteration was also developed. This method vastly reduced the computation time to obtain converged solutions and also enabled solutions to be obtained for $Q \propto \epsilon^{-3/4}$, where convergence was too slow to obtain a solution using the first method. Good agreement was obtained between results with each method for $Q \propto \epsilon$ and Q constant.

Results are shown in Figs. 1, 2, and 3 for $f^0(\epsilon, z)$ for the case of a uniform steady stream of electrons being absorbed by a boundary at $z = 0$. The dependence of the cross section on energy for the three figures is $Q = A_m \epsilon^m$, where $m = 1$, 0, and $-\frac{3}{4}$, respectively. The constant A_m was chosen such that in each case $Q = 6 \times 10^{-16}$ cm^2 at 0.1 eV. Values of E/N were chosen so that the characteristic energy D/μ for the three cases were approximately equal, i.e., E/N was 10^{-17}, 0.75×10^{-17}, and 0.185×10^{-17} V cm^2, and D/μ was 0.31, 0.36, and 0.38 V, respectively; μ is the electron mobility. The gas density or pressure was adjusted to make calculated values of D_0/W_0 to be 0.315 cm in each of

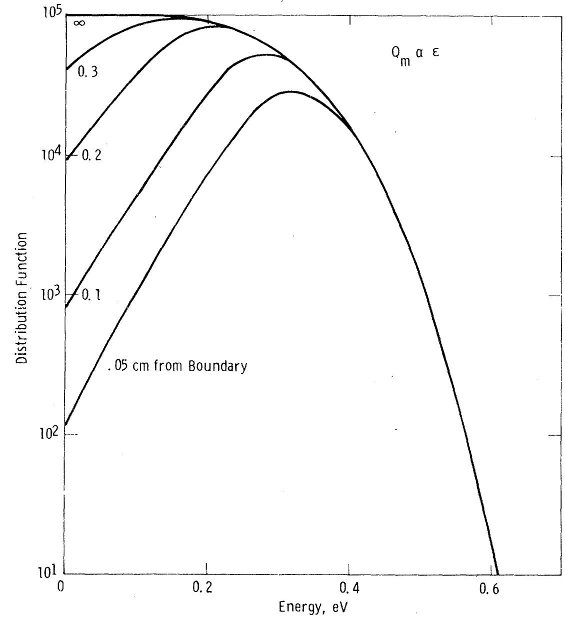

FIG. 1. Distribution function f^0(cm^{-6} sec^3) as a function of energy and position near an absorbing boundary; $Q = 60\epsilon$ cm^2, $E/N = 10^{-17}$ V cm^2, $p = 3$ Torr, and $D_0/W_0 = 0.315$ cm.

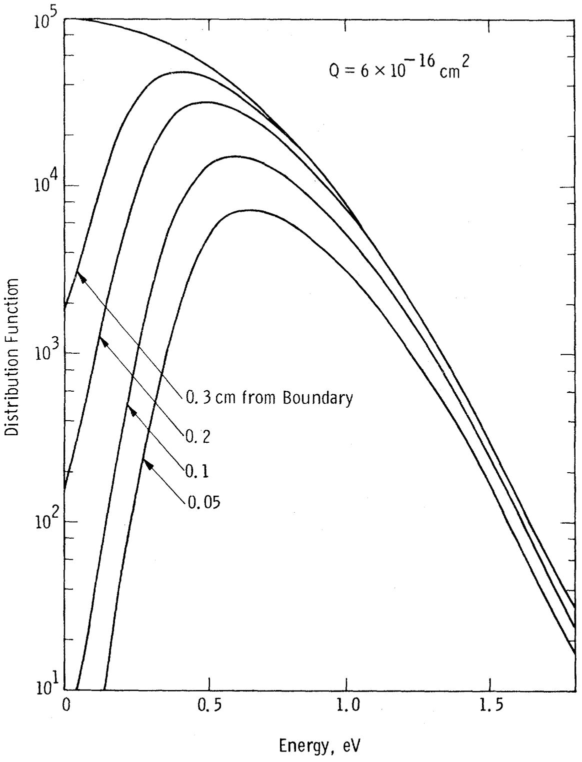

FIG. 2. Same as for Fig. 1 but with $Q = 6 \times 10^{-16}$ cm^2, $E/N = 0.75 \times 10^{-17}$ V cm^2, $p = 4.5$ Torr, and $D_0/W_0 = 0.315$ cm.

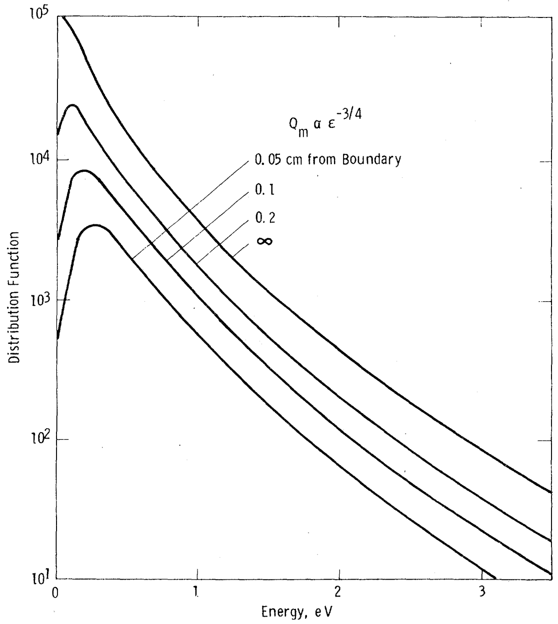

FIG. 3. Same as for Fig. 1 but with $Q \propto \epsilon^{-3/4}$, $E/N = 0.185 \times 10^{-17}$ V cm^2, $p = 19.6$ Torr, and $D_0/W_0 = 0.315$ cm.

the three cases, to within 2%. The resulting pressures for a temperature of 20 °C were 3, 4.5, and 19.6 Torr, respectively. The advantage of performing calculations at a constant value of D_0/W_0 is that the electron density predicted in the conventional way by solving the steady-state continuity Eq. (6), with $D=D_0$ and $W=W_0$ independent of position, is the same for all three cases, as is discussed later in relation to Fig. 7.

Results in Fig. 1 and Fig. 2 were obtained with the small effect of the term in kT of Eq. (5) effectively set at zero by putting $T=29.3$ °K. The derived values of W_0 and D_0 at $z=-h$ agree with values derived analytically to within 2% using Eqs. (7) and (8). For $Q \propto \epsilon^{-3/4}$ it is very difficult to obtain accurate solutions because f^0 is highly peaked around $\epsilon=0$, as shown in Fig. 3, and a fine mesh size in ϵ is required to accurately evaluate the integrations for W and D. Also f^0 falls off relatively slowly at high energies so that $\bar{\epsilon}$ must be made very large. To satisfy the requirement that Q be finite at $\epsilon=0$ we made $Q=4.41\times10^{-16}$ for $\epsilon \leq 0.15$ eV so that Q does not vary strictly as $\epsilon^{-3/4}$ over the whole energy range. Furthermore in order for the numerical method[22] to converge it was necessary for T to be 293 °K rather than 29.3 °K.

B. Average electron energy, drift velocity, and diffusion coefficient

The average energy, drift velocity, and diffusion coefficient are shown in Figs. 4–6 as a function of z corresponding to the cases of Figs. 1–3.

1. *Average energy*

From Figs. 1–3 it is clear that the shape of the distribution function and thus the average electron energy varies as a function of position because of the influence of the absorbing boundary. The average energy is calculated from

$$\langle\epsilon\rangle = \int \epsilon^{3/2} f_0(\epsilon, z)\, d\epsilon \Big/ \int \epsilon^{1/2} f_0\, d\epsilon \ .$$

It is seen from Figs. 4–6 that for all variations of Q with ϵ that have been investigated, the average energy increases due to the negative density gradients that occur near the absorbing boundary. This result is consistent with previous results[11,12] obtained for an electron pulse moving in an electric field where the electrons ahead of the centroid where $\partial n/\partial z < 0$ have an enhanced average energy.

There are three terms involving z in Eq. (5) which perturb the electron energy distribution from the zero density gradient distribution, and we label these terms, respectively, terms A, B, and C. Term A, involving $\partial f^0/\partial z$ in the first square bracket, is probably the most significant in that, because of the boundary condition $eE\,\partial f^0/\partial\epsilon + \partial f^0/\partial z = 0$, there is a reduction of electrons near $\epsilon=0$ if $\partial f^0/\partial z < 0$. Thus this term causes an increase in the average electron energy. This term

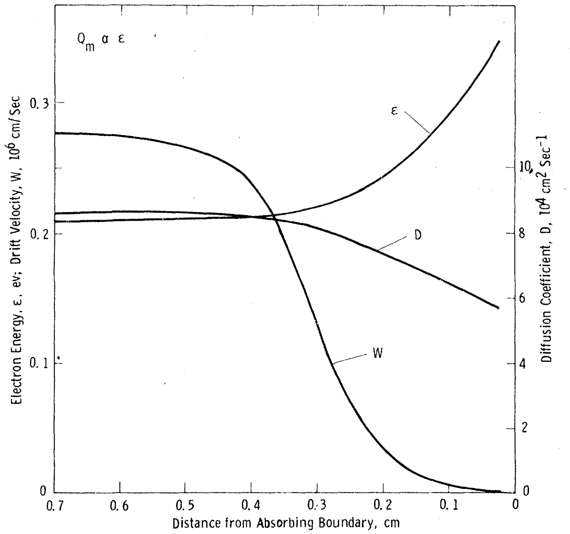

FIG. 4. Values of average energy ϵ, diffusion coefficient D, and drift velocity W derived from the distribution functions of Fig. 1.

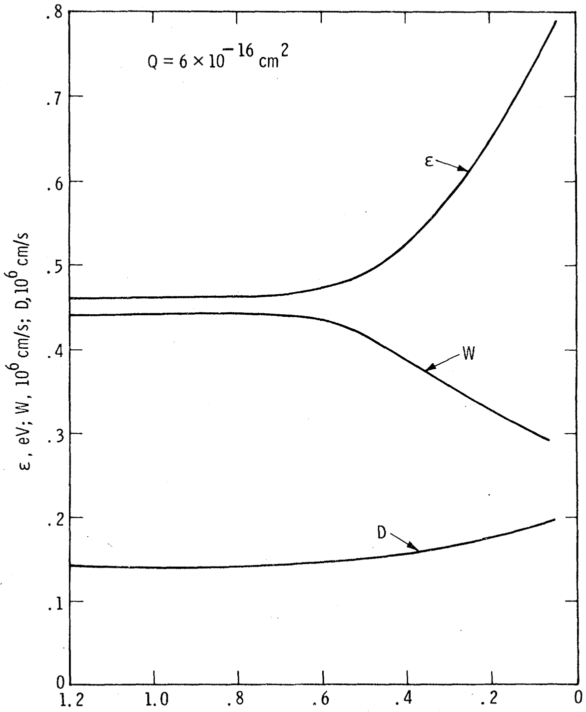

FIG. 5. Values of average energy ϵ, diffusion coefficient D, and drift velocity W derived from the distribution functions of Fig. 2.

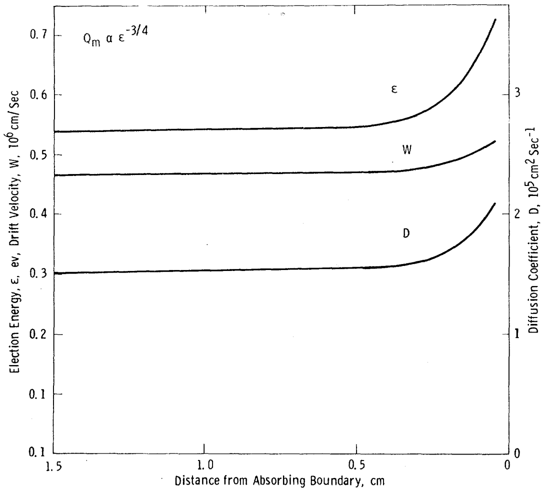

FIG. 6. Values of average energy ϵ, diffusion coefficient D, and drift velocity W derived from the distribution functions of Fig. 3.

can be given a physical interpretation in two equivalent ways. (a) The term accounts for a local increase in input power due to diffusion current as can be seen by multiplying the term by ϵ and integrating over all energy by parts to obtain $-Ee\,\partial(Dn)/\partial z$. Thus for $\partial n/\partial z < 0$ one obtains an increase in energy because of heating by the diffusion current. (b) An alternate physical picture indicating that the average electron energy is increased by the electric field in conjunction with a negative density gradient is seen by the following. For a uniform electron distribution about as many electrons enter a small volume after being scattered forwards as enter after being scattered in the backward direction. Those electrons scattered backwards are decelerated by the electric field and have a slightly lower electron energy. Those electrons scattered forward are accelerated and have a higher electron energy. However, if $\partial n/\partial z$ is negative, the number of electrons scattered backward is less than those scattered forwards and the effect is a slightly increased average electron energy.

The second and third terms involving z in Eq. (5) cause changes in electron energy because electrons of different energies drift at different rates and also diffuse at different rates. Unlike the effect of term A either a heating or a cooling effect can result, depending on the energy dependence of Q. For example for term C, if $Q \propto \epsilon$, as $D \sim \langle \epsilon^{1/2}/NQ \rangle$, electrons of high energy diffuse slower than electrons of low energy and the increase of electron energy resulting from term A is enhanced; symbols $\langle\,\rangle$ indicate averages over the distribution function. More usually, as for a constant cross section, term C results in "diffusion cooling."[23]

2. *Drift and diffusion coefficients*

The derived values of drift and diffusion coefficients are shown in Figs. 4–6 as a function of z, as obtained from Eqs. (7), (8), and (9). Of course the whole utility of these coefficients is destroyed if they become a function of position, as in the present examples. Then it is simplest to regard the distribution function f^0 as the fundamental quantity, without reference to drift and diffusion coefficients. Nevertheless it is of interest to derive these quantities, as there certainly are local coefficients which it is seen can either increase or decrease as one approaches the absorbing boundary. Whether W or D increases or decreases on approaching the boundary is consistent with the relations $D \sim \langle \epsilon^{1/2}/Q \rangle$ and $W \sim \langle \epsilon^{-1/2}\,\partial(\epsilon/Q)/\partial\epsilon \rangle$ from Eqs. (7) and (8) for the various dependences of Q and ϵ.

C. Electron density

Probably the most useful physical quantity is the electron density, which has been evaluated as a function of z using Eq. (9) and is shown by the solid curves of Fig. 7. For the three cases considered, D_0/W_0 is constant at 0.315 cm. Thus if the electron density is calculated in the conventional way by solving the steady-state electron continuity equation, Eq. (6), with $D = D_0$ and $W = W_0$ assumed to be independent of position, the analytic solution $n/n_0 = 1 - e^{zW_0/D_0}$ is obtained for all three cases where $n = 0$ at $z = 0$ and $n \to n_0$ as $z \to -\infty$. This solution is shown by the short dashed curve of Fig. 7. D_0/W_0 is thus a characteristic distance over which n varies by the fraction $1 - (1/e)$ of the density far from the boundary. It is seen that the more accurate treatment from the Boltzmann equation gives significantly different results, par-

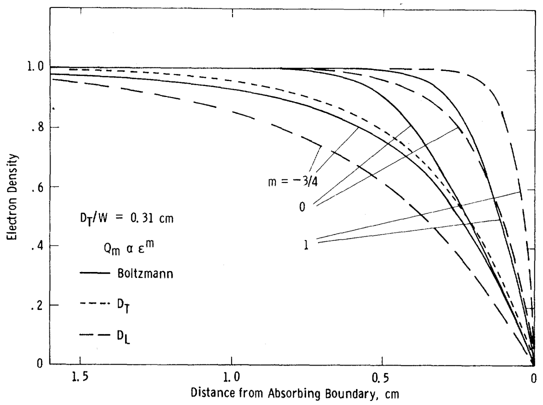

FIG. 7. Normalized electron density calculated from the distribution functions of Figs. 1–3. All curves are appropriate to $D_0/W_0 = 0.315$ cm.

ticularly for the cross section $Q \propto \epsilon$.

Also solutions of Eq. (6) are shown in Fig. 7 using longitudinal diffusion coefficients[11–13] D_L for D, and $W = W_0$, where D_L and W_0 are assumed to be independent of position. Values of D_L/D_0 for $Q \propto \epsilon^m$ where $m = 1$ and 0 are respectively taken[11] as 0.18 and 0.5. The value of D_L/D_0 for the cross section approximately corresponding to $m = -\frac{3}{4}$ was calculated[11] to be 1.65. It is seen from Fig. 7 that these solutions are also not good approximations to the Boltzmann solutions, the departure from the conventional solution using D_0 being overestimated. However from a more detailed analysis, our numerical solutions from the Boltzmann equation in the region where density gradients are small are consistent with the solutions obtained using D_L for the diffusion coefficient.

IV. SCALING PARAMETERS

From the previous sections it is seen that in general solutions of $f^0(\epsilon, z)$ and $n(z)$ will be dependent on two experimental variables, E/N and also N. However for the model situation where the momentum-transfer cross section Q varies as some power m of the electron energy, it is shown in this section that $n(z)$ is independent of E/N, provided that N is chosen such that D_0/W_0 is constant and that the term in kT is negligible. Thus $n(z)$ can be regarded as being just a function of D_0/W_0.

The solutions $f^0(\epsilon, z)$ will depend generally on both E/N and N. On substituting $Q = Q_0\epsilon^m$ and performing the integration for the distribution function when there are no density gradients we obtain from Eq. (10) if $T = 0$

$$f^0(\epsilon) = A\exp\left[-\frac{6mQ_0^2}{M(2m+2)}\left(\frac{\epsilon^{m+1}}{E/N}\right)^2\right].$$

This expression suggests using $\xi = \epsilon^{m+1}/(E/N)$ as a scaling parameter to eliminate E/N as a parameter. We substitute $\epsilon = (\xi E/N)^{1/(m+1)}$ and

$$d\epsilon = \left(\frac{E}{N}\right)^{1/(m+1)} \frac{\xi^{1/(m+1)-1}}{m+1}\, d\xi$$

in Eq. (5), to obtain for the steady-state equation, after dividing by E,

$$\frac{16\pi(m+1)}{mM\xi^{1/(m+1)-1}}\frac{\partial}{\partial\xi}\left[\xi^{1/(m+1)+1}Q_0 f^0 + \frac{Me}{6mQ_0\xi^{(m-1)/(m+1)}}\left(e\frac{\partial f^0}{\partial\xi}\frac{(m+1)}{\xi^{1/(m+1)-1}} + \frac{(E/N)^{1/(m+1)}}{E}\frac{\partial f^0}{\partial z}\right)\right]$$
$$+\frac{8\pi}{3m^2Q_0\xi^{(m-1)/(m+1)}}\frac{(E/N)^{1/(m+1)}}{E}\left(\frac{e(m+1)}{\xi^{1/(m+1)-1}}\frac{\partial^2 f^0}{\partial\xi\partial z} + \frac{(E/N)^{1/(m+1)}}{E}\frac{\partial^2 f^0}{\partial z^2}\right) = 0\ . \quad (11)$$

In Eq. (11), the same parameter $C = (E/N)^{1/(m+1)}/E$ appears in front of the three terms in z; E/N and N are otherwise eliminated from the equation. Thus solutions of $f^0(\xi, z)$ are dependent only on C, and provided N is chosen to keep C a constant, solutions are independent of E/N.

On substituting $Q = Q_0\epsilon^m$ and $\epsilon = (\xi E/N)^{1/(m+1)}$ in Eqs. (7) and (8) we obtain $D_0 \propto (1/E)(E/N)^{3/(m+1)}$ and $W_0 \propto (E/N)^{2/(m+1)}$. Thus $D_0/W_0 \propto (1/E)(E/N)^{1/(m+1)}$ and it is seen that keeping C constant is equivalent to keeping D_0/W_0 constant. The solutions $f^0(\xi, z)$ and $n(z)$ can thus be regarded as being a function of just D_0/W_0, provided that $Q \propto \epsilon^m$.

V. BACK DIFFUSION

To represent electron back diffusion to a metal electrode from which electrons are emitted is difficult. For a steady-state one-dimensional case, electrons will generally be emitted from the electrode with an energy not equal to the equilibrium energy of the applied uniform field, as in photoemission, or the electrons will be emitted in a region of highly nonuniform electric fields, if space charge effects occur.

Yet our solutions yield some insight into the processes of back diffusion for the idealized physical situation of a source of electrons introduced at the equilibrium energy at a distance h in front of the negative electrode. Such a source of electrons might be introduced by a beam of ionizing electrons parallel to the metal plate. Our model has some similarities to other physical situations. For example in the Townsend-Huxley experiment, electrons initially emitted thermionically from a filament, enter a cylindrical diffusion chamber through a small hole in the center of a plane metal plate. After strong radial diffusion, there is significant back diffusion of electrons against the electric field to the negative plate, where they are absorbed. In a further example, our model can represent the back diffusion to a plate for electrons emitted photoelectrically from the plate, where h is the distance at which the electrons attain the equilibrium energy appropriate to the applied E/N.

The solutions for back diffusion for the simple case of an electron source at distance h from the negative plate are directly related to the solutions for forward diffusion to an absorbing electrode. We represent the solutions for forward diffusion of the type given in Figs. 1–3 by $f_1^0(\epsilon, z)$ and the solution of the uniform density case by $f_0^0(\epsilon)$, as given by Eq. (10). Then $F(\epsilon,z)=f_0^0(\epsilon)-f_1^0(\epsilon,z)$ satisfies Eq. (5) and has boundary values such that $F(\epsilon,-h)=0$ and $F(\epsilon,0)=f_0^0(\epsilon)$. Thus $F(\epsilon,z)$ is the solution for back diffusion for our simple model that at $z=0$ there is a source of electrons with the equilibrium energy distribution and at $z=-h$ there is an absorbing boundary.

The values of electron density, average energy, drift velocity and diffusion coefficient appropriate to back diffusion, are then derivable from the values for forward diffusion. We denote values for back diffusion and forward diffusion with subscripts b and f, respectively. Thus the densities for back diffusion n_b are related with those for forward diffusion by

$$n_b=\frac{4\pi}{m}\left(\frac{2}{m}\right)^{1/2}\int(f_0^0-f_1^0)\epsilon^{1/2}d\epsilon=1-n_f, \qquad (12)$$

if it is assumed that the densities are normalized so that $n_b=1$ at $z=0$. The average electron energy is given by

$$\langle\epsilon\rangle_b=\frac{\int(f_0^0-f_1^0)\epsilon^{3/2}d\epsilon}{\int(f_0^0-f_1^0)\epsilon^{1/2}d\epsilon}$$

$$=\frac{\langle\epsilon\rangle_0-\langle\epsilon\rangle_f n_f}{1-n_f}\,; \qquad (13)$$

$\langle\epsilon\rangle_0$ is the average energy for the case of no density gradients. Similarly, using Eqs. (7) and (8)

$$W_b=(W_0-W_f n_f)/(1-n_f)\,, \qquad (14)$$

$$D_b=(D_0-D_f n_f)/(1-n_f)\,. \qquad (15)$$

In Fig. 8 are shown solutions of n, W, D, and $\langle\epsilon\rangle$ for the case of back diffusion to a plate 1.2 cm from the injected electrons where $Q=60\epsilon$ cm^2. Values of $E/N=10^{-17}$ V cm^2 and N are for a pressure of 3 Torr at 20 °C, as for the solution for forward diffusion of Fig. 1. It is seen that the average electron energy is reduced below the average energy for a uniform distribution. Such a reduction in energy is expected if the term representing the loss of energy due to diffusion against the electric field is dominant. The fall off in electron density n is greater than that predicted by the conventional solution using the continuity equation with the transverse diffusion coefficient $D_T=D_0$ as shown by the dashed curve in Fig. 8. The solid curves were obtained by the two numerical methods and are consistent with values that are predicted from Eqs. (12)–(15) using previous values derived for forward diffusion. However it is difficult to obtain high numerical accuracy using these equations when $1-n_f$ in the denominator is close to zero.

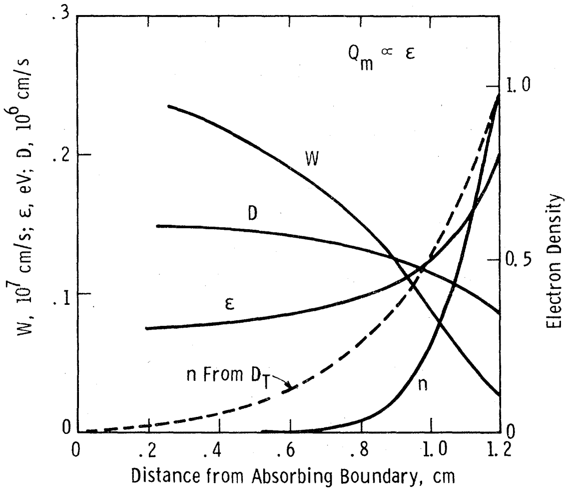

FIG. 8. Back diffusion from a source at 1.2 cm, $E/N=10^{-17}$ V cm^2, $p=3$ Torr for $Q=60\epsilon$ cm^2.

VI. CONCLUSIONS

(1) The conventional calculation of the electron density near an absorbing boundary by solving the electron continuity equation does not give accurate solutions. The shape of the electron distribution function as a function of energy varies with position near the boundary. As a consequence the diffusion coefficient, drift velocity and average electron energy become a function of position.

(2) Use of position independent, longitudinal diffusion coefficients give solutions of only slightly greater accuracy. Although longitudinal diffusion coefficients are valid for small density gradients[13] and also give good agreement[12] with the widths of diffusing electron pulses in an electric field, these diffusion coefficients cannot be used for all physical situations where there are large density gradients.

(3) The values of D_0/W_0, or more accurately D_L/W_0, are of value in giving an order of magnitude estimate in the distance over which an absorbing boundary disturbs the electron density where it

TABLE I. Characteristic perturbation distance. The distance over which the electron density varies by the fraction $(1-1/e)$ near the boundary as calculated by the Boltzmann equation (B.E.) and by the longitudinal diffusion coefficients (D_L/W) for $Q\propto\epsilon^m$. In all cases $D_0/W_0=0.315$ cm.

m	B.E. (cm)	D_L/W (cm)
1	0.16	0.06
0	0.30	0.16
−0.75	0.34	0.52

acts as a collector for a steady stream of electrons. Values of this fall-off distance, i.e., the distance over which the electron density varies by $1 - 1/e$ are compared in Table I.

Although the quantity D_0/W_0 cannot be used in the continuity equation to accurately predict electron densities, it can nevertheless be accurately measured in the Townsend-Huxley experiment. This conclusion is reached because for large chamber lengths and high gas pressures, results[3] are obtained to within 1%, independent of gas pressure. Then the values of D_0/W_0 can be used[3,20] together with measurements of W_0 to derive accurate momentum-transfer and collision cross sections using Eqs. (5), (7), and (8).

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