PHYSICAL REVIEW A VOLUME 15, NUMBER 1 JANUARY 1977

Polarization fractions in Glauber theory for electron impact excitation of the $n = 3$ levels of atomic hydrogen

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(Received 29 January 1976)

With the use of recently proposed analytic methods, Glauber scattering amplitudes for the excitation of the $3d$ substates of atomic hydrogen by electron impact are obtained in closed form. The individual $n = 3$ cross sections and the Balmer-α cross section predicted by the Glauber theory in the range $18 \leq E_i \leq 500$ eV are compared with experiments and with other calculations. The polarization fractions of $3p$-$2s$, $3d$-$2p$, and of the Balmer-α line are also calculated in the Glauber approximation, and comparison is made with the existing experimental data and with the Born predictions. The parameters (λ, χ) predicted by the Glauber approximation are also given.

I. INTRODUCTION

Although the scattering of a charged particle by a hydrogen atom has long been of interest to astrophysicists and atomic and molecular physicists, the data on the $3l$ excitation of atomic hydrogen by electron impact are rather scarce. To date, only two experimental measurements on the total cross section[1,2] and the total polarization fraction[1] have been reported for $n=3$ excitation. By modulating the exciting electron beam and separating the individual excitations on the basis of the different fluorescence decay rates of the three substates ($3s$, $3p$, and $3d$), Mahan, Gallagher, and Smith[3] recently measured the cross-section ratios $\sigma_{3s} : \sigma_{3p} : \sigma_{3d}$ as well as the cross section of the Balmer line σ_α. Furthermore, Eminyan *et al.*[4] have recently developed a delayed-coincidence technique to measure the angular correlations between the emitted photon and scattered electron in inelastic electron-atom collisions, and have reported results on e^--He collisions for $2\,^1P$ and $3\,^1P$ excitations. From the angular correlations they are able to deduce the ratio (λ) of the differential cross sections for exciting the degenerate magnetic sublevels of the substates and the relative phase (χ) of the corresponding excitation amplitudes. These collision parameters λ and χ, measured without the need of any normalization, have generally been hidden in most refined theoretical calculations of the collision and lost in experiments designed to measure cross sections and polarization fractions alone. Their measurement, therefore, is expected to provide a new and more conclusive test of electron-atom scattering theories. Such a detailed experimental study,[5] using the electron-photon coincidence technique, of the polarization and the excitation of the Balmer-α line is now under way at the University of Stirling. The present study is partly motivated by these experimental interests.

Previous Glauber calculations[6] for electron impact excitation of the $3l$ states of hydrogen atoms, using the direction perpendicular to $\vec{q}$ as the quantization z axis, have concentrated on the predicted differential and total cross sections.[7,8] However, for calculating the polarization fraction and the parameters (λ, χ) it is necessary to have the Glauber amplitudes calculated in the coordinate system,[6,9] quantized along the direction of the incident electron. To simplify the previous calculations[7,8] involving numerical evaluation of a relatively simple one-dimensional integral, we have used the recently proposed analytic methods[10,11] to obtain the closed-form Glauber amplitudes $[F_{3d,1s}(q, m_l)]$, which require no numerical integration. In Sec. II, we express the closed-form Glauber scattering amplitudes in terms of four generating functions. Two of these (for $1s$-$3s$ and $1s$-$3p$ excitation) are given by Thomas and Gerjuoy[11]; the detailed derivation of the other two (for $1s$-$3d$ excitation) is deferred to an appendix. In this appendix we also show that the two analytic methods[10,11] yield the same results. The expressions for polarization fraction and the parameters (λ, χ) are also given in this section. In Sec. III, we present and discuss the results of numerical calculations of the expressions obtained in Sec. II.

II. GLAUBER THEORY

The Glauber scattering amplitudes $F^{(\zeta)}_{3l,1s}(q, m_l)$ describing the excitation of the hydrogen atom from the ground state $\Psi_{1s}(\vec{r})$ to the final state $\Psi_{3lm}(\vec{r})$ by an incident charged particle $Z_i e$ with velocity v_i is given by

$$F^{(\zeta)}_{3l,1s}(q, m_l) = \frac{iK_i}{2\pi} \int \Psi^*_{3lm}(\vec{r})\Gamma(\vec{b}; \vec{r})\Psi_{1s}(\vec{r}) \times e^{i\vec{q}\cdot\vec{b}}\, d^2b\, d\vec{r}, \tag{1}$$

where

$$\Gamma(\vec{b};\vec{r}) = 1 - (|\vec{b} - \vec{s}|/b)^{2i\eta} \quad (2)$$

and

$$\eta \equiv -Z_i/v_i \text{ (in atomic units).}$$

In Eqs. (1) and (2), $\vec{b}$ and $\vec{s}$ are the respective projections of the position vectors of the incident particle and the bound electron onto the plane perpendicular to the direction of the Glauber path. The superscript (ζ) represents the $\vec{q}$-dependent coordinate system $C^g(\hat{\xi})$, whose z axis lies along $\hat{\xi}$ and is perpendicular to $\vec{q}$, in which the Glauber amplitudes $F^{(\zeta)}_{3l,1s}(q,m_l)$ are readily computable.

For radiation lines emitted by the hydrogen atom following electron excitation to the $3l$ states, the polarization fraction

$$P_l(E_i) = (I_\parallel - I_\perp)/(I_\parallel + I_\perp)\,, \quad (3)$$

according to the theory of Percival and Seaton,[12] is related to $Q_{lm}(E_i)$. In Eq. (3), $I_\parallel$ and $I_\perp$ are the intensities, observed at 90° to the incident-electron-beam direction, of the respective lines having electric vectors parallel and perpendicular to the incident-electron-beam direction. The quantity E_i is the incident-electron energy; $Q_{lm}(E_i)$ is the total cross section for exciting the hydrogen atom from the ground state to the $3lm$ sublevels. It was pointed out by Gerjuoy, Thomas, and Sheorey[9] (GTS) that the total cross sections Q_{lm}, which appear in the expressions for P_l, are computed from the Glauber scattering amplitudes $F^{(i)}_{3l,1s}(q,m_l)$, quantized along the direction $\vec{K}_i$ of the incident electron [which we denote by $C(\vec{K}_i)$]. The connection between these two sets of Glauber amplitudes is found by the following transformation[9,13]:

$$F^{(i)}_{3l,1s}(q,m_l) = \sum_{m_l'} D^{(l)}_{m_l m_l'}(\alpha,\beta,\gamma) F^{(\zeta)}_{3l,1s}(q,m_l')\,. \quad (4)$$

In Eq. (4), $D^{(l)}_{mm'}$, is the usual representation of $Rg\,[\equiv Rg(\alpha,\beta,\gamma)]$ on the space spanned by eigenvectors of L^2 with angular momentum number l. The representation $D^{(l)}_{mm'}(\alpha,\beta,\gamma)$ is related to the matrix $d^{(l)}_{mm'}$ by[14]

$$D^{(l)}_{mm'}(\alpha,\beta,\gamma) = e^{im\gamma} d^{(l)}_{mm'}(\beta) e^{im'\alpha}\,, \quad (5)$$

where the Euler angles[9] $\alpha = \phi_q$, $\beta = (\theta_q - \frac{1}{2}\pi)$, and $\gamma = -\phi_q$ [θ_q and ϕ_q are the angular coordinates of $\vec{q}$ in $C(\vec{K}_i)$]. Using Eq. (4.1.15) of Ref. 14 one can easily find the matrix $d^{(l)}_{mm'}(\beta)$, and hence the $F^{(i)}_{3l,1s}(q,m_l)$.

A. 1s-3s excitation

The 1s-3s Glauber amplitude, evaluated in $C^g(\hat{\xi})$, is given by TG equations (19a) and (19b) with $n=3$,

$$F^{(\zeta)}_{3s,1s}(\vec{q}) = -iK_i \frac{2}{\sqrt{27}}\left(\frac{\partial I_0(\lambda,q)}{\partial\lambda} + \frac{2}{3}\frac{\partial^2 I_0(\lambda,q)}{\partial\lambda^2} + \frac{2}{27}\frac{\partial^3 I_0(\lambda,q)}{\partial\lambda^3}\right)\Bigg|_{\lambda=4/3}. \quad (6)$$

In Eq. (6), $I_0(\lambda,q)$ is defined and given in TG equations (9) and (19c),

$$I_0(\lambda,q) = \frac{1}{(2\pi)^2}\int_0^\infty b\,db \int_0^\infty s\,ds \int_{-\infty}^{+\infty} dz \frac{e^{-\lambda(s^2+z^2)^{1/2}}}{(s^2+z^2)^{1/2}} \int_0^{2\pi} d\varphi_b\, e^{iqb\cos\varphi_b} \int_0^{2\pi} d\varphi_s \left[1 - \left(\frac{b^2+s^2-2bs\cos\varphi_s}{b^2}\right)^{i\eta}\right]$$
$$= -4i\eta\Gamma(1+i\eta)\Gamma(1-i\eta)\lambda^{-2-2i\eta}q^{-2+2i\eta}\,{}_2F_1(1-i\eta,1-i\eta;1;-\lambda^2/q^2)\,, \quad (7)$$

where Γ and ${}_2F_1$ are the usual gamma and hypergeometric functions, respectively. Since the initial and final states are s states, at any given $\vec{q}$ the absolute square of the scattering amplitude will be independent of the choice of quantization axis. Dropping the superscript (ζ) in the amplitude, the differential and total cross sections for the 1s-3s excitation are obtained from the scattering amplitude in the usual way; namely,

$$\left(\frac{d\sigma}{d\Omega}\right)^{(i)}_{1s-3s} = \left(\frac{d\sigma}{d\Omega}\right)^{(\zeta)}_{1s-3s} = \frac{K_f}{K_i}|F_{1s-3s}(q)|^2 \quad (8)$$

and

$$\sigma^{(i)}_{1s-3s}(E_i) = \frac{K_f}{K_i}\int |F_{1s-3s}(q)|^2 d\Omega\,. \quad (9)$$

The polarization fraction of the resulting radiation from 3s to 2p state is

$$P_s(E_i) = 0\,, \quad (10)$$

since the upper level is a S state.[12]

B. 1s-3p excitation

The 1s-3p Glauber amplitudes, evaluated in $C^g(\hat{\xi})$, have also been obtained in closed form by TG and are given by TG equation (23a) with $n=3$,

$$F^{(\zeta)}_{3p,1s}(q,m_l=0) = 0 \quad (11)$$

and

$$F^{(\zeta)}_{3p,1s}(q,m_l=\pm1) = \pm e^{\mp i\phi_q}\, iK_i \frac{8}{27}\left(\frac{\partial I_1(\lambda,q)}{\partial\lambda} + \frac{1}{6}\frac{\partial^2 I_1(\lambda,q)}{\partial\lambda^2}\right)\Bigg|_{\lambda=4/3} = \pm e^{\mp i\phi_q} h_1(q)\,. \quad (12)$$

In Eq. (12), $I_1(\lambda, q)$ is defined and given in TG equations (23b) and (27c),

$$I_1(\lambda,q)=\frac{1}{(2\pi)^2}\int_0^\infty b\,db\int_0^\infty s^2 ds\int_0^\infty dz\,\frac{e^{-\lambda(s^2+z^2)^{1/2}}}{(s^2+z^2)^{1/2}}\int_0^{2\pi} d\varphi_b\, e^{i(qb\cos\varphi_b\mp\varphi_b)}$$
$$\times\int_0^{2\pi} d\varphi_s\, e^{\mp i\varphi_s}\left[1-\left(\frac{b^2+s^2-2bs\cos\varphi_s}{b^2}\right)^{i\eta}\right]$$
$$=i4\Gamma(1+i\eta)\Gamma(2-i\eta)(i\eta)\lambda^{-2-2i\eta}q^{-3+2i\eta}[-{}_2F_1(2-i\eta,1-i\eta;1;-\lambda^2/q^2)$$
$$+(1+i\eta){}_2F_1(2-i\eta,1-i\eta;2;-\lambda^2/q^2)]. \tag{13}$$

By using $d^{(1)}_{mm'}(\beta)$,[14] and Eqs. (4), (5), (11), and (12), we obtain

$$F^{(i)}_{3p,1s}(q,m_l=0)=\sqrt{2}\,\cos\theta_q\cdot h_1(q) \tag{14}$$

and

$$F^{(i)}_{3p,1s}(q,m_l=\pm1)=\pm e^{\mp i\phi_q}\sin\theta_q\cdot h_1(q), \tag{15}$$

where $h_1(q)$ is defined in Eq. (12). Equations (12), (14), and (15) differ slightly from the expressions given by TG, since we have used the spherical harmonics defined by Edmonds.[15] From Eqs. (11)–(15), one immediately gets

$$\left(\frac{d\sigma}{d\Omega}\right)^{(\zeta)}_{1s-3p}=\frac{K_f}{K_i}\,2|F^{(\zeta)}_{3p,1s}(q,m_l=1)|^2$$
$$=2\frac{K_f}{K_i}|h_1(q)|^2, \tag{16}$$

$$\left(\frac{d\sigma}{d\Omega}\right)^{(i)}_{1s-3p}=\frac{K_f}{K_i}[|F^{(i)}_{3p,1s}(q,m_l=0)|^2+2|F^{(i)}_{3p,1s}(q,m_l=1)|^2]$$
$$=2\frac{K_f}{K_i}|h_1(q)|^2=\left(\frac{d\sigma}{d\Omega}\right)^{(\zeta)}_{1s-3p}, \tag{17}$$

and

$$\sigma^{(i)}_{3p,1s}(E_i)=\sigma^{(\zeta)}_{3p,1s}=2\frac{K_f}{K_i}\int|h_1(q)|^2 d\Omega. \tag{18}$$

Thus, the Glauber cross sections are independent of whether the quantization axis is chosen along $\vec{K}_i$ or along an axis perpendicular to $\vec{q}$.[9] However, Eqs. (14) and (15) are necessary in calculating the polarization fraction and the parameters (λ, χ) in connection with experimental investigations of inelastic electron-photon angular correlations.[4,5] The parameter $\lambda_0(3p)$ is defined via

$$\lambda_0(3p)=\frac{|F^{(i)}_{3p,1s}(q,m_l=0)|^2}{|F^{(i)}_{3p,1s}(q,m_l=0)|^2+2|F^{(i)}_{3p,1s}(q,m_l=1)|^2}. \tag{19}$$

By Eqs. (14) and (15), we obtain

$$\lambda_0(3p)=\cos^2\theta_q. \tag{20}$$

Indeed as pointed out in Ref. 4, that the first Born approximation, or any theory which results in a $\Delta M_L=0$ selection rule along the momentum transfer direction in the excitation, implies that $\lambda_0=\cos^2\theta_q$, and $\chi=0$.

The polarization fraction of the resulting radiation from $3p$ to $2s$, according to the theory of Percival and Seaton,[12] is given by

$$P_p(E_i)=3(Q_{p0}-Q_{p1})/(7Q_{p0}+11Q_{p1}). \tag{21}$$

In Eq. (21), Q_{pm_l} ($m_l=0$ and 1) are the total cross sections for exciting the hydrogen atom from ground state to $3pm_l$ sublevels and are found via Eqs. (14) and (15),

$$Q_{pm_l}(E_i)=\frac{K_f}{K_i}\int|F^{(i)}_{3p,1s}(q,m_l)|^2 d\Omega. \tag{22}$$

C. 1s-3d excitation

Substituting the ground-state wave function

$$\Psi_{1s}(\vec{r})=(\sqrt{\pi})^{-1}e^{-r}, \tag{23}$$

and the possible final-state wave functions[15,16]

$$\Psi_{3dm_l}(\vec{r})=R_{32}(r)Y_{2m_l}(\theta,\phi)$$
$$=(4/81\sqrt{30})r^2e^{-r/3}Y_{2m_l}(\theta,\phi) \tag{24}$$

into Eq. (1), we find that

$$F^{(\zeta)}_{3d,1s}(q,m_l=0)=i\frac{2K_i}{81\sqrt{6}}\times\left(-3\frac{\partial I_z(\lambda,q)}{\partial\lambda}+\frac{\partial^3 I_0(\lambda,q)}{\partial\lambda^3}\right)\bigg|_{\lambda=4/3}$$
$$=h_0(q). \tag{25}$$

In Eq. (25), $I_0(\lambda, q)$ is given in Eq. (7), whereas $I_z(\lambda, q)$ is defined by

$$I_z(\lambda,q)=\frac{1}{(2\pi)^2}\int_0^\infty b\,db\int_0^\infty s\,ds\int_{-\infty}^\infty z^2\frac{e^{-\lambda(s^2+z^2)^{1/2}}}{(s^2+z^2)^{1/2}}dz\int_0^{2\pi}d\varphi_b\,e^{iqb\cos\varphi_b}\int_0^{2\pi}d\varphi_s\left[1-\left(\frac{b^2+s^2-2bs\cos\varphi_s}{b^2}\right)^{i\eta}\right]. \tag{26}$$

The detailed reduction of Eq. (26) is given in Appendix A1, where $I_z(\lambda, q)$ is given by Eq. (A10),

$$I_z(\lambda,q) = -8i\eta\lambda^{-4-2i\eta}q^{-2+2i\eta}\Gamma(1+i\eta)\Gamma(1-i\eta)$$
$$\times[(1+i\eta)\,{}_2F_1(1-i\eta,1-i\eta;1;-\lambda^2/q^2)+(\lambda^2/q^2)(1-i\eta)^2\,{}_2F_1(2-i\eta,2-i\eta;2;-\lambda^2/q^2)]. \tag{27}$$

For excitation to $m_l = \pm 1$ states, one sees that by introducing the cylindrical coordinates for $\vec{r}$, $F^{(\zeta)}_{3d,1s}(q, m_l = \pm 1)$ vanishes from Eq. (1) since the integrand under the integral is an odd function of z.

Similarly for $1s$-$3d_{\pm 2}$ excitation we may write

$$F^{(\zeta)}_{3d,1s}(q,m_l=\pm 2) = -i\frac{K_i}{81}e^{\mp i2\phi_q}\left(\frac{\partial I_s(\lambda,q)}{\partial\lambda}\right)\bigg|_{\lambda=4/3} = e^{\mp i2\phi_q}h_2(q). \tag{28}$$

In Eq. (28), $I_s(\lambda, q)$ is defined via

$$I_s(\lambda,q) = \frac{1}{(2\pi)^2}\int_0^\infty b\,db\int_0^\infty s^3ds\int_{-\infty}^{+\infty}\frac{e^{-\lambda(s^2+z^2)^{1/2}}}{(s^2+z^2)^{1/2}}dz\int_0^{2\pi}d\varphi_b\,e^{\mp i2(\varphi_b-\phi_q)+i\vec{q}\cdot\vec{b}}$$
$$\times\int_0^{2\pi}d\varphi_s\,e^{\mp i2(\varphi_s-\varphi_b)}\left[1-\left(\frac{b^2+s^2-2bs\cos(\varphi_s-\varphi_b)}{b^2}\right)^{i\eta}\right], \tag{29}$$

where $I_s(\lambda, q)$ is given by Eq. (A16) in Appendix A 2,

$$I_s(\lambda,q) = 8i\eta(1-i\eta)(2-i\eta)\lambda^{-2-2i\eta}q^{-4+2i\eta}\Gamma(1+i\eta)\Gamma(1-i\eta)$$
$$\times[2\,{}_2F_1(3-i\eta,1-i\eta;1;-\lambda^2/q^2)-4(1+i\eta)\,{}_2F_1(3-i\eta,1-i\eta;2;-\lambda^2/q^2)$$
$$+(1+i\eta)(2+i\eta)\,{}_2F_1(3-i\eta,1-i\eta;3;-\lambda^2/q^2)]. \tag{30}$$

The differential cross section for $1s$-$3d$ excitation is constructed from Eqs. (25) and (28) in the usual way; that is

$$\left(\frac{d\sigma}{d\Omega}\right)^{(\zeta)}_{1s-3d} = \frac{K_f}{K_i}[|F^{(\zeta)}_{3d,1s}(q,m_l=0)|^2 + 2|F^{(\zeta)}_{3d,1s}(q,m_l=2)|^2]. \tag{31}$$

Substituting $d^{(2)}_{mm'}(\beta)$,[14] Eqs. (5), (25), and (28) into Eq. (4), we find that the $1s$-$3dm_l$ Glauber amplitudes, quantized along $\vec{K}_i$, are

$$F^{(i)}_{3d,1s}(q,m_l=0) = \tfrac{1}{2}(3\sin^2\theta_q-1)h_0(q) + \sqrt{\tfrac{3}{2}}\cos^2\theta_q\,h_2(q), \tag{32}$$

$$F^{(i)}_{3d,1s}(q,m_l=\pm 1) = e^{\mp i\phi_q}\sin\theta_q\cos\theta_q \times[-\sqrt{\tfrac{3}{2}}\,h_0(q)+h_2(q)], \tag{33}$$

$$F^{(i)}_{3d,1s}(q,m_l=\pm 2) = e^{\mp i2\phi_q}[\tfrac{1}{2}\sqrt{\tfrac{3}{2}}\cos^2\theta_q\,h_0(q) + \tfrac{1}{2}(1+\sin^2\theta_q)h_2(q)], \tag{34}$$

where $h_0(q)$ and $h_2(q)$ are defined in Eqs. (25) and (28). Since $h_0(q)$ and $h_2(q)$ appear in all $F^{(i)}_{3d,1s}(q,m_l)$ ($m_l = 0, \pm 1, \pm 2$), one notes that all the $3d$ magnetic sublevels are excited coherently and (being degenerate) radiate coherently.[6]

The Glauber cross sections again are independent of whether using $C^g(\hat{\xi})$ or $C(\vec{K}_i)$, namely,

$$\left(\frac{d\sigma}{d\Omega}\right)^{(i)}_{1s-3d} = \frac{K_f}{K_i}[|F^{(i)}_{3d,1s}(q,m_l=0)|^2 + 2|F^{(i)}_{3d,1s}(q,m_l=1)|^2 + 2|F^{(i)}_{3d,1s}(q,m_l=2)|^2]$$
$$= \frac{K_f}{K_i}[|h_0(q)|^2+2|h_2(q)|^2] = \left(\frac{d\sigma}{d\Omega}\right)^{(\zeta)}_{1s-3d} \tag{35}$$

and

$$\sigma^{(i)}_{1s-3d}(E_i) = \frac{K_f}{K_i}\int[|h_0(q)|^2+2|h_2(q)|^2]d\Omega = \sigma^{(\zeta)}_{1s-3d}(E_i), \tag{36}$$

respectively.

The parameters $\lambda_0(3d)$ and $\lambda_1(3d)$ are evaluated via Eqs. (32)–(34):

$$\lambda_0(3d) = \frac{|F^{(i)}_{3d,1s}(q,m_l=0)|^2}{|F^{(i)}_{3d,1s}(q,m_l=0)|^2+2|F^{(i)}_{3d,1s}(q,m_l=1)|^2+2|F^{(i)}_{3d,1s}(q,m_l=2)|^2}, \tag{37}$$

$$\lambda_1(3d) = \frac{2|F^{(i)}_{3d,1s}(q,m_l=1)|^2}{|F^{(i)}_{3d,1s}(q,m_l=0)|^2+2|F^{(i)}_{3d,1s}(q,m_l=1)|^2+2|F^{(i)}_{3d,1s}(q,m_l=2)|^2}. \tag{38}$$

The phase differences χ between $F^{(i)}_{3d,1s}(q,m_l)$ vanish again in Glauber theory.

Mahan[3] has shown that the polarization fraction of the resulting radiation from $3d$ to $2p$, using the general theory of Percival and Seaton,[12] is given by

$$P_d(E_i)=\frac{Q_{d_0}+Q_{d_1}-2Q_{d_2}}{2.07Q_{d_0}+3.8Q_{d_1}+2.8Q_{d_2}}\,. \qquad (39)$$

In Eq. (39), Q_{dm_l} ($m_l=0$, 1, and 2), total excitation cross sections to the $3dm_l$ sublevels, are evaluated via Eqs. (32)–(34); thus

$$Q_{dm_l}=\frac{K_f}{K_i}\int|F^{(i)}_{3d,1s}(q,m_l)|^2\,d\Omega\,. \qquad (40)$$

D. Excitation and polarization of Balmer-α radiation

Kleinpoppen and Kraiss[1] define the cross section of the Balmer-α line, excited from the ground state, as follows:

$$\sigma_\alpha(E_i)=\sigma_{1s-3s}(E_i)+0.12\sigma_{1s-3p}(E_i)+\sigma_{1s-3d}(E_i)\,. \qquad (41)$$

This is the sum of the cross sections for the excitation of the $3s$ state, of the $3p$ state multiplied by the branching ratio of the $3p$-$2s$ transition, and of the $3d$ state.

We also define the polarization fraction of the Balmer-α line emitted by hydrogen atoms following excitation to $n=3$ states as follows:

$$P_\alpha(E_i)=\frac{\sigma_{1s-3p}(E_i)P_p(E_i)+\sigma_{1s-3d}(E_i)P_d(E_i)}{\sigma_{1s-3s}(E_i)+\sigma_{1s-3p}(E_i)+\sigma_{1s-3d}(E_i)}\,. \qquad (42)$$

III. RESULTS AND DISCUSSION

The theoretical results as obtained using Glauber approximation (GA) are divided into the three following categories.

(i) The individual $n=3$ cross sections and the Balmer-α cross section of hydrogen atoms by electron impact with incident energies from 18 to 500 eV are calculated in GA. We find that our analytic results of the individual $n=3$ cross sections are in good agreement with those obtained previously from a one-dimensional representation for energies from 18 to 140 eV for $3s$,[7] $3p$,[7] and to 400 eV for[8] $3d$ excitations. At higher energies, the experimental data are right on the Glauber curves. We do not present the GA calculations, experimental measurements,[1-3] and other theoretical results[17-22] since they have already been presented by Mahan *et al.* in Fig. 8 of Ref. 3. We

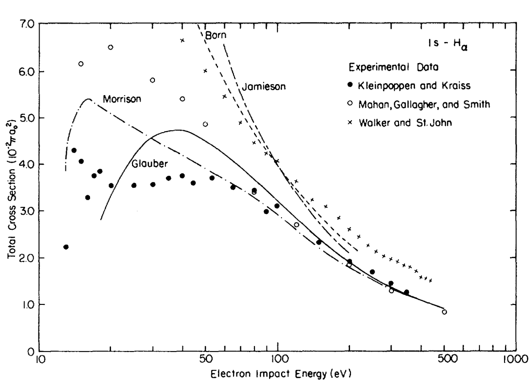

FIG. 1. Total H_α excitation cross section.

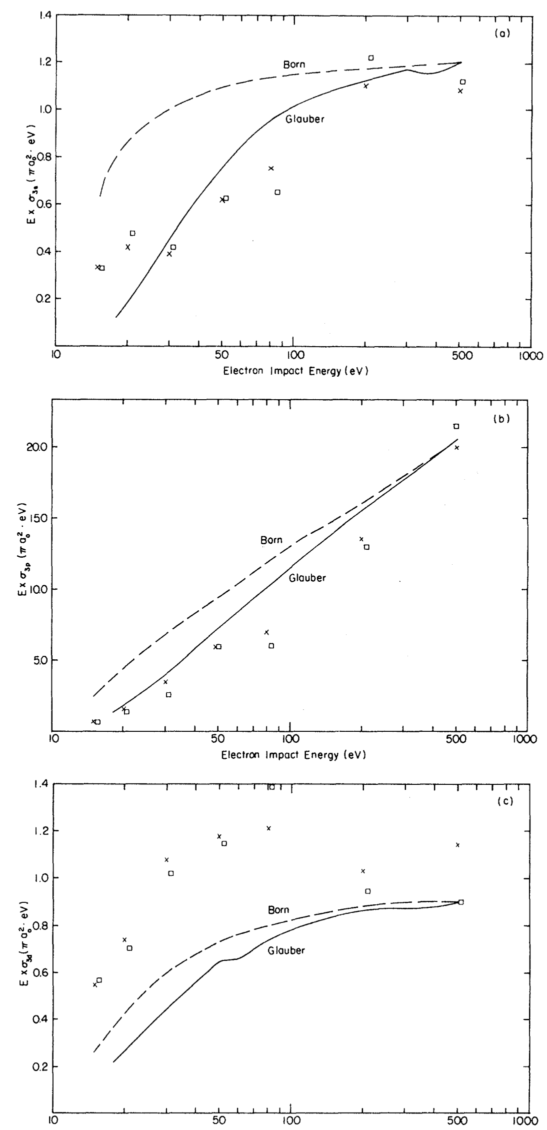

FIG. 2. Cross section times energy for the direct (a) $3s$, (b) $3p$, and (c) $3d$ excitations. The theoretical results are GA and Born approximation, Mahan *et al.*, Ref. 3. The experimental measurements of Mahan *et al.* are given as **×** and □ using the in-phase and in-phase-plus-out-of-phase fits.

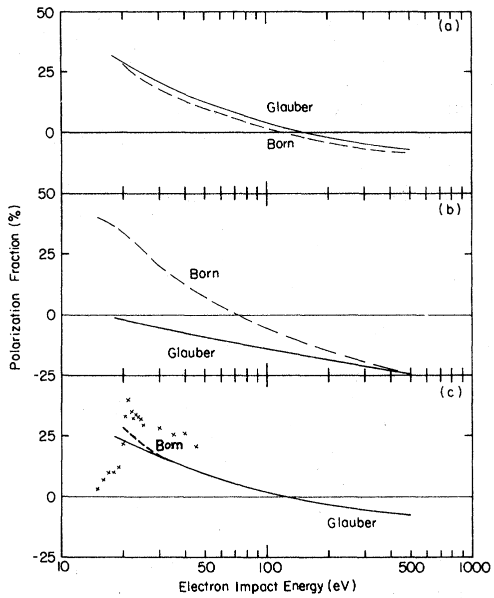

FIG. 3. Polarization fraction of (a) $3p$, (b) $3d$, and (c) H_α as functions of the electron energy. The theoretical curves are GA and Born approximation, Mahan *et al.*, Ref. 3. The experimental data of Kleinpoppen and Kraiss are given as ×.

note that the present $n=3$ experimental data would agree fairly well with the Glauber prediction, except for the $1s$-$3d$ excitation cross section. Similar patterns also appear in the $n=3$ excitation of helium by electron impact; for instance Glauber results[23] for the $1\,^1S$-$3\,^1P$ excitation agree well with experimental measurements whereas the Glauber values[13] lie below experimental data as well as the Born values for the $1\,^1S$-$3\,^1D$ excitation. The Balmer-α cross section of hydrogen atoms by electron impact is represented in Fig. 1. We have also calculated $E_i \cdot \sigma_{1s-3l}(E_i)$ and shown the Glauber results together with the experimental data and the Born calculations of Mahan *et al.*[3] in Fig. 2. We again note the Glauber predictions are in reasonable agreement with the experimental findings for $1s$-$3s$ and $1s$-$3p$ excitation, whereas the Born approximation seems to be better than GA in comparison with $1s$-$3d$ excitation data.

(ii) The polarization fractions of the resulting radiation from $3p$ to $2s$, $3d$ to $2p$, and of the Balmer-α line emitted by hydrogen atoms following excitation to $n=3$ states are evaluated in GA by means of Eqs. (21), (39), and (42). We note from Fig. 3(a) that the Glauber curve closely resembles those obtained from the Born calculations[3] for the polarization fraction of radiation from $3p$ to $2s$, the pattern follows that of the $2p$-$1s$ case.[9] The reason for the close agreement of the predicted polarization fractions—although the computed total and differential cross sections are not so close—is explained by Gerjuoy, Thomas, and Sheorey in Ref. 9. However, we see from Fig. 3(b) that for the polarization fraction of the $3d$-$2p$ line, the Glauber values show a large deviation from the Born calculations at low incident impact energies although they both predict almost the same value at 500 eV. A more quantitative picture is provided by Table I. Since the Glauber predicted Q_{d_2} are always larger than the average values of Q_{d_0} and Q_{d_1}, $P_d(E_i)$ is negative via Eq. (39). We also present the Glauber predictions, the Born calculations, and the experimental results for the polarization fraction for the Balmer-α line in Fig. 3(c). Since the main contributions to the polarization fraction for the Balmer-α line come from the p-s transition, it is understandable that results obtained using Glauber and Born approximations are close to each other. However, at energies <30 eV, the difference for the Balmer-α polarization fraction predicted by these two models is no longer small. Further experimental and theoretical (using other models) investigations are desirable.

TABLE I. Sublevel excitation cross sections calculated using GA in units πa_0^2 for e^--H collisions.

E_i (eV)	$3s$	$3p_0$	$3p_1$	$3d_0$	$3d_1$	$3d_2$
20	0.889−2	0.702−1	0.104−1	0.520−2	0.730−3	0.318−2
30	0.148−1	0.860−1	0.242−1	0.412−2	0.173−2	0.361−2
40	0.158−1	0.817−1	0.317−1	0.284−2	0.215−2	0.337−2
50	0.151−1	0.738−1	0.352−1	0.202−2	0.221−2	0.306−2
70	0.129−1	0.594−1	0.369−1	0.123−2	0.195−2	0.255−2
100	0.102−1	0.447−1	0.352−1	0.818−3	0.147−2	0.203−2
150	0.729−2	0.312−1	0.307−1	0.609−3	0.940−3	0.153−2
200	0.564−2	0.238−1	0.269−1	0.520−3	0.648−3	0.124−2
250	0.459−2	0.192−1	0.239−1	0.463−3	0.474−3	0.104−2
300	0.386−2	0.161−1	0.215−1	0.419−3	0.362−3	0.897−3
500	0.236−2	0.970−2	0.156−1	0.304−3	0.164−3	0.583−3

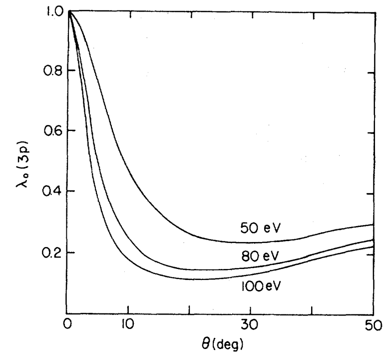

FIG. 4. Parameter $\lambda_0(3p)$ as a function of scattering angles for 50-, 80-, 100-eV incident electron energies.

(iii) The parameters $\lambda_0(3p)$, $\lambda_0(3d)$, and $\lambda_1(3d)$ are calculated in GA via Eqs. (20), (37), and (38) as functions of scattering angles for 50-, 80-, and 100-eV incident electron energies. The results are shown in Figs. 4 and 5. However, both Glauber and Born approximations predict the zero-phase differences between the excitation amplitudes for different magnetic sublevels of the substates. Experimental measurements of (λ, χ) using the coincidence techniques[5] would provide an important check for the Glauber approximation.

ACKNOWLEDGMENTS

We wish to thank Professor H. Kleinpoppen of the University of Stirling and Professor A. C. Gallagher of Joint Institute for Laboratory Astrophysics

for supplying their experimental information prior to publication.

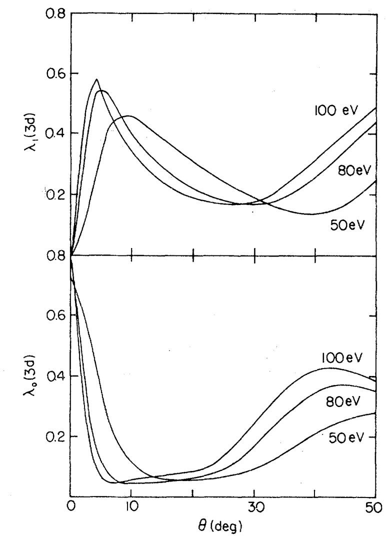

FIG. 5. Parameters $\lambda_0(3d)$ and $\lambda_1(3d)$ as functions of scattering angles for 50-, 80-, and 100-eV incident electron energies.

APPENDIX

A1. Reduction of the generating function $I_z(\lambda,q)$ using Thomas and Chan method (Ref. 10)

The generating function $I_z(\lambda, q)$ is defined by Eq. (26),

$$I_z(\lambda, q) = \frac{1}{(2\pi)^2}\int_0^\infty b\,db\int_0^\infty s\,ds\int_{-\infty}^\infty z^2\frac{e^{-\lambda(s^2+z^2)^{1/2}}}{(s^2+z^2)^{1/2}}dz\int_0^{2\pi}d\varphi_b\,e^{iqb\cos\varphi_b}\int_0^{2\pi}d\varphi_s\left[1-\left(\frac{b^2+s^2-2bs\cos\varphi_s}{b^2}\right)^{i\eta}\right]. \tag{A1}$$

By employing the standard formulas[24] for K_ν and J_ν,

$$\int_{-\infty}^\infty z^2\frac{e^{-\lambda(s^2+z^2)^{1/2}}}{(s^2+z^2)^{1/2}}\,dz = \frac{2s}{\lambda}K_1(\lambda s)\ , \tag{A2}$$

$$\int_0^{2\pi} d\varphi\, e^{iqb\cos\varphi} = 2\pi J_0(qb)\,, \tag{A3}$$

and then changing variable s to sb, we find that $I_z(\lambda, q)$ can be written as

$$I_z(\lambda, q) = 2\int_0^\infty b^5\,db\,J_0(qb)\left(\frac{1}{\lambda b}\int_0^\infty s^2K_1(\lambda bs)\,ds - \frac{1}{\lambda b}\int_0^\infty s^2\,ds\,K_1(\lambda bs)\,\frac{1}{2\pi}\int_0^{2\pi}d\varphi(1+s^2-2s\cos\varphi)^{i\eta}\right). \tag{A4}$$

We now utilize the result[25] that

$$\int_0^\infty s^2K_1(\lambda bs)\,ds = 2(\lambda b)^{-3}. \tag{A5}$$

We obtain

$$I_z(\lambda, q) = 2\int_0^\infty b^5\,db\,J_0(qb)\left(\frac{2}{(\lambda b)^4} - M_2(\lambda b)\right), \tag{A6}$$

where

$$M_2(x) \equiv \frac{1}{x}\int_0^\infty s^2\,ds\,K_1(xs)\times\frac{1}{2\pi}\int_0^{2\pi}d\varphi\,(1+s^2-2s\cos\varphi)^{i\eta}. \tag{A7}$$

Equation (A7) for $M_2(x)$ was derived by Chan and Chang[13] [Eq. (A17) of Ref. 13] and was given in terms of the modified Lommel functions[10]

$$M_2(x)=2x^{-4}\{1-(2i\eta)^2\,[(1+i\eta)(ix)^{-2i\eta}\mathfrak{L}_{2i\eta-1,0}(ix)+(1-i\eta)(ix)^{-2i\eta+1}\mathfrak{L}_{2i\eta-2,1}(ix)]\}. \tag{A8}$$

Substituting Eq. (A8) into Eq. (A6), we have explicitly removed the $\delta(\vec{q})$ by noting the exact cancellation between the term $[2(\lambda b)^{-4}]$ and the same factor stemming from $M_2(\lambda b)$. Thus we have

$$I_z(\lambda,q)=\frac{4}{\lambda^4}\,(2i\eta)^2\left((1+i\eta)\int_0^\infty b\,db\,J_0(qb)\,(i\lambda b)^{-2i\eta}\mathfrak{L}_{2i\eta-1,0}(i\lambda b)\right.$$
$$\left.+(1-i\eta)\int_0^\infty b\,db\,J_0(qb)(i\lambda b)^{-2i\eta+1}\,\mathfrak{L}_{2i\eta-2,1}(i\lambda b)\right). \tag{A9}$$

With help from Eqs. (18) and (A7) of Ref. 10 and Eqs. (B3) and (B5) of Ref. 11, the integrals in Eq. (A9) which involve only modified Lommel function $\mathfrak{L}_{\mu,\nu}$ may be evaluated in close form. Therefore, we obtain

$$I_z(\lambda,q)=-8i\eta\lambda^{-4-2i\eta}q^{-2+2i\eta}\Gamma(1+i\eta)\Gamma(1-i\eta)$$
$$\times[(1+i\eta)\,{}_2F_1(1-i\eta,1-i\eta;1;-\lambda^2/q^2)+(\lambda^2/q^2)(1-i\eta)^2\,{}_2F_1(2-i\eta,2-i\eta;2;-\lambda^2/q^2)]. \tag{A10}$$

A2. Reduction of the generating function $I_s(\lambda,q)$ using Thomas and Chan method (Ref. 10)

The generating function $I_s(\lambda,q)$ is defined by Eq. (29),

$$I_s(\lambda,q)=\frac{1}{(2\pi)^2}\int_0^\infty b\,db\int_0^\infty s^3\,ds\int_{-\infty}^\infty dz\,\frac{e^{-\lambda(s^2+z^2)^{1/2}}}{(s^2+z^2)^{1/2}}\int_0^{2\pi}d\varphi_b\,e^{\mp i2(\varphi_b-\phi_q)+i\vec{q}\cdot\vec{b}}$$
$$\times\int_0^{2\pi}d\varphi_s\,e^{\mp i2(\varphi_s-\varphi_b)}\left[1-\left(\frac{b^2+s^2-2bs\cos(\varphi_s-\varphi_b)}{b^2}\right)^{i\eta}\right]. \tag{A11}$$

Following the procedure in Appendix A1 and employing[24]

$$\int_{-\infty}^\infty dz\,\frac{e^{-\lambda(s^2+z^2)^{1/2}}}{(s^2+z^2)^{1/2}}=K_0(\lambda s) \tag{A12}$$

and

$$\int_0^{2\pi}d\varphi\,e^{\mp i2\varphi+iqb\cos\varphi}=-2\pi J_2(qb), \tag{A13}$$

we find that $I_s(\lambda,q)$ can be written

$$I_s(\lambda,q)=2\int_0^\infty b^5\,db\,J_2(qb)\int_0^\infty s^3\,ds\,K_0(\lambda bs)\,\frac{1}{2\pi}\int_0^{2\pi}d\varphi\,e^{\mp i2\varphi}(1+s^2-2s\cos\varphi)^{i\eta}=2\int_0^\infty b^5\,db\,J_2(qb)M_3(\lambda b), \tag{A14}$$

where

$$M_3(x)=\int_0^\infty s^3\,ds\,K_0(xs)\times\frac{1}{2\pi}\int_0^{2\pi}d\varphi\,e^{\mp i2\varphi}(1+s^2-2s\cos\varphi)^{i\eta}.$$

By applying the result of $M_3(x)$ derived by Chan and Chang[13] [Eq. (B10) of Ref. 13]

$$M_3(x)=(2i\eta)^2(ix)^{-2i\eta-2}\mathfrak{L}_{2i\eta-1,0}(ix)-8i\eta(1+i\eta)(ix)^{-2i\eta-3}\mathfrak{L}_{2i\eta,1}(ix)+4i\eta(2+i\eta)(ix)^{-2i\eta-4}\mathfrak{L}_{2i\eta+1,2}(ix),$$

we have

$$I_s(\lambda,q)=2\int_0^\infty b^5\,db\,J_2(qb)\times[(2i\eta)^2(i\lambda b)^{-2i\eta-2}\mathfrak{L}_{2i\eta-1,0}(i\lambda b)-8i\eta(1+i\eta)(i\lambda b)^{-2i\eta-3}\mathfrak{L}_{2i\eta,1}(i\lambda b)+4i\eta(2+i\eta)(i\lambda b)^{-2i\eta-4}\mathfrak{L}_{2i\eta+1,2}(i\lambda b)]. \tag{A15}$$

Carrying out the integrals by using Eq. (A7) of Ref. 10 and Eqs. (B3) and (B5) of Ref. 11, we finally obtain

$$I_s(\lambda,q)=8i\eta(1-i\eta)(2-i\eta)\lambda^{-2-2i\eta}q^{-4+2i\eta}\Gamma(1+i\eta)\Gamma(1-i\eta)\times[2\,{}_2F_1(3-i\eta,1-i\eta;1;-\lambda^2/q^2)-4(1+i\eta)\,{}_2F_1(3-i\eta,1-i\eta;2;-\lambda^2/q^2)+(1+i\eta)(2+i\eta)\,{}_2F_1(3-i\eta,1-i\eta;3;-\lambda^2/q^2)]. \quad \text{(A16)}$$

B1. Reduction of the generating function $I_z(\lambda,q)$ using Thomas and Gerjuoy method (Ref. 11)

The generating function $I_z(\lambda,q)$ is defined by Eq.(26), again. By employing Eqs. (A2) and (A3) and changing variable s to sb, we find that $I_z(\lambda,q)$ can be written

$$I_z(\lambda,q)=\frac{1}{\pi}\,\frac{1}{\lambda}\int_0^\infty b^4\,db\,J_0(qb)\int_0^\infty s^2\,ds\,K_1(\lambda bs)\int_0^{2\pi}d\varphi[1-(1+s^2-2s\cos\varphi)^{i\eta}]. \quad \text{(B1)}$$

We now utilize the result[26] that

$$\int_0^\infty b^4\,db\,J_0(qb)K_1(\lambda sb)=\frac{16}{(\lambda s)^5}\,{}_2F_1(3,2;1;-q^2/\lambda^2s^2). \quad \text{(B2)}$$

We obtain

$$I_z(\lambda,q)=\frac{32}{\lambda}\int_0^\infty s^2\,ds\,(\lambda s)^{-5}\,{}_2F_1(3,2;1;-q^2/\lambda^2s^2)\times\left(1-\frac{1}{2\pi}\int_0^{2\pi}(1+s^2-2s\cos\varphi)^{i\eta}\,d\varphi\right). \quad \text{(B3)}$$

Since $q\neq0$ for excitation, the first term (independent of η) under the integral in Eq. (B3) gives zero contribution to the integral.[11] By introducing integral representation of TG equation (14) to replace the integral over φ in Eq. (B3) by an equivalent integral involving Bessel functions, we have

$$I_z(\lambda,q)=\frac{32}{\lambda}\int_0^\infty s^2\,ds\,(\lambda s)^{-5}\,{}_2F_1(3,2;1;-q^2/\lambda^2s^2)\times2^{2i\eta}\frac{\Gamma(1+i\eta)}{\Gamma(1-i\eta)}\times\int_0^\infty dt\,t^{-2i\eta}\frac{d}{dt}\,(J_0(st)J_0(t)). \quad \text{(B4)}$$

The hypergeometric function in Eq. (B4) is simply[27]

$${}_2F_1(3,2;1;-q^2/\lambda^2s^2)=s^4\left(s^2+\frac{q^2}{\lambda^2}\right)^{-2}\left[1-\frac{4q^2/\lambda^2}{s^2+q^2/\lambda^2}+3\left(\frac{q^2/\lambda^2}{s^2+q^2/\lambda^2}\right)^2\right]. \quad \text{(B5)}$$

Substituting Eq. (B5) into Eq. (B4), one finds that the three terms in Eq. (B4) are multiple integrals $\mathcal{K}_{0,0}$, $\mathcal{K}_{0,1}$, and $\mathcal{K}_{0,2}$ of the class $\mathcal{K}_{m,r}$ discussed in TG Appendix B. Therefore,

$$I_z(\lambda,q)=2^{5+2i\eta}\frac{\Gamma(1+i\eta)}{\Gamma(1-i\eta)}\,\frac{1}{\lambda^6}\times\left[\mathcal{K}_{0,0}-\frac{4q^2}{\lambda^2}\mathcal{K}_{0,1}+3\left(\frac{q^2}{\lambda^2}\right)^2\mathcal{K}_{0,2}\right], \quad \text{(B6)}$$

which reduces, via TG equation (B7), to

$$I_z(\lambda,q)=-8\lambda^{-4-2i\eta}q^{-2+2i\eta}\Gamma(1-i\eta)\Gamma(1+i\eta)\,[2\,{}_2F_1(1-i\eta,-i\eta;1;-\lambda^2/q^2)-4(1-i\eta)\,{}_2F_1(2-i\eta,-i\eta;1;-\lambda^2/q^2)+(1-i\eta)(2-i\eta)\,{}_2F_1(3-i\eta,-i\eta;1;-\lambda^2/q^2)]. \quad \text{(B7)}$$

The sum of three hypergeometric functions in Eq.(B7) can be reduced further, via the Gaussian recursion relations,[28] to a sum of two hypergeometric functions which is exactly Eq. (A10).

B2. Reduction of the generating function $I_s(\lambda,q)$ using Thomas and Gerjuoy method (Ref. 11)

Again, the generating function $I_s(\lambda,q)$ is defined by Eq. (29). Following the procedure in Appendix B 1 and employing Eqs. (A12) and (A13), we find that $I_s(\lambda,q)$ can be written

$$I_s(\lambda,q)=\frac{1}{\pi}\int_0^\infty b^5\,db\,J_2(qb)\int_0^\infty s^3\,ds\,K_0(\lambda sb)\int_0^{2\pi}d\varphi\,e^{\mp i2\varphi}(1+s^2-2s\cos\varphi)^{i\eta}. \quad \text{(B8)}$$

The integral over b may be done immediately via[26,27]

$$\int_0^\infty b^5\,db\,J_2(qb)K_0(\lambda sb)=2^5\times 3^2\frac{q^2}{(\lambda s)^8}\,{}_2F_1(4,4;3;-q^2/\lambda^2 s^2)$$
$$=2^5\times 3^2\frac{q^2}{(\lambda s)^8}\,s^8\left(s^2+\frac{q^2}{\lambda^2}\right)^{-4}\left(1-\frac{4}{3}\,\frac{q^2/\lambda^2}{s^2+q^2/\lambda^2}\right). \tag{B9}$$

Again, by introducing the TG integral representation and by substituting Eq. (B9) into Eq. (B8), we have

$$I_s(\lambda,q)=-2^6\times 3^2\frac{q^2}{\lambda^8}\,2^{2i\eta}\frac{\Gamma(1+i\eta)}{\Gamma(1-i\eta)}\left(\mathcal{K}_{2,0}-\frac{4}{3}\,\frac{q^2}{\lambda^2}\mathcal{K}_{2,1}\right), \tag{B10}$$

which reduces, via TG equation (B7), to

$$I_s(\lambda,q)=-8i\eta(1-i\eta)\Gamma(1+i\eta)\Gamma(3-i\eta)\lambda^{-2-2i\eta}q^{-4+2i\eta}$$
$$\times[3\,{}_2F_1(3-i\eta,2-i\eta;3;-\lambda^2/q^2)-(3-i\eta)\,{}_2F_1(4-i\eta,2-i\eta;3;-\lambda^2/q^2)]. \tag{B11}$$

Using the Guassian recursion relations,[28] one can prove that Eq. (B11) is the same as Eq. (A16).

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