Mapping of operator equations into C-number differential form and the atomic coherent-state representation

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(Received 16 August 1976)

The mapping of operator equations into C-number differential form by means of the atomic coherent-state representation leads to mathematical difficulties that have no counterpart in the case of the Glauber-Sudarshan representation for Bose systems. We give sufficient conditions for the solutions of C-number equations in the atomic coherent-state representation to be physically meaningful.

I. INTRODUCTION

The mapping of Bose operator equations into C-number differential form using the Glauber-Sudarshan representation[1] has been used extensively in the past.[2,3] Recently, similar mapping techniques based on the atomic coherent state representation[4] have been developed and applied to the Markoffian superradiance master equation[5] and to the non-Markoffian superfluorescence equation.[6] The resulting C-number differential equations describe the evolution of quasi-probability functions corresponding to collective atomic operators.

There are special mathematical difficulties that arise with the atomic coherent state representation which have no counterpart in the case of the Glauber-Sudarshan representation. This difference stems from the lack of uniqueness of the C-number functions associated to given operators in the atomic coherent state representation.

Before stating the problem, we need to discuss the procedure for constructing C-number differential equations from given operator equations. For definiteness, we consider a linear equation

$$\frac{dW}{dt} = \mathcal{F}(W) \tag{1.1}$$

for the density operator W, and assume that W is defined over the Hilbert space of the angular momentum spanned by the basis vectors $|j, m\rangle$ with a fixed value of j. Following Arecchi *et al.*,[4] we define a real valued C-number function $P(\Omega, t)$ from the integral representation

$$W(t) = \int d\Omega\, |\Omega\rangle\langle\Omega| P(\Omega, t) \;, \qquad d\Omega = \sin\theta\, d\theta\, d\varphi \;, \tag{1.2}$$

where $|\Omega\rangle$ is the coherent atomic state of fixed angular momentum j. As shown in Ref. 4, the function $P(\Omega, t)$ is not uniquely defined by Eq. (1.2). In fact, if $P_1(\Omega, t)$ satisfies Eq. (1.2), the addition to $P_1(\Omega, t)$ of any convergent linear combination of spherical harmonics $Y_l^m(\Omega)$ with $l > 2j$ leads to a new distribution function $P_2(\Omega, t)$ which satisfies Eq. (1.2) as well. In order to define $P(\Omega, t)$ uniquely, it is common to prescribe that it be given by a linear combination of only the first $(2j+1)^2$ spherical harmonics

$$P(\Omega, t) = \sum_{l=0}^{2j} \sum_{m=-l}^{l} C_{lm}(t) Y_l^m(\theta, \varphi) \;. \tag{1.3}$$

If we now substitute Eq. (1.2) into Eq. (1.1), we obtain

$$\int d\Omega\, \Lambda(\Omega)(\partial/\partial t) P(\Omega, t) = \int d\Omega\, P(\Omega, t) \mathcal{F}(\Lambda(\Omega)) \;, \qquad \Lambda(\Omega) \equiv |\Omega\rangle\langle\Omega| \;. \tag{1.4}$$

The operator functional $\mathcal{F}(\Lambda(\Omega))$ can be mapped into a differential form acting on the angular variables of the projector $\Lambda(\Omega)$.[7] After the appropriate integration by parts, Eq. (1.4) can be cast into the form

$$\int d\Omega\, \Lambda(\Omega)(\partial P/\partial t - \mathcal{L}P) + \mathcal{S} = 0 \;, \tag{1.5}$$

where $\mathcal{L}$ is a differential operator on the variables θ and φ of the Bloch sphere and $\mathcal{S}$ stands for the surface terms.

Narducci *et al.*[5] have demonstrated that under reasonable conditions the surface terms originating from the Markoffian superradiance master equation vanish identically. On the other hand, Gronchi and Lugiato[6] resorted to physical arguments in order to discard the surface terms from their equations.

It is interesting, therefore, to establish the mathematical conditions for the vanishing of the surface terms. The first aim of our paper is to prove that the condition (1.3) guarantees that the surface terms vanish identically. The operator equation (1.1) is now equivalent to

$$\int d\Omega\, \Lambda(\Omega)(\partial P/\partial t - \mathcal{L}P) = 0 \ . \tag{1.6}$$

Until now, it has been tacitly assumed[5,6] that one could obtain $P(\Omega, t)$ by solving the differential equation

$$\frac{\partial P}{\partial t} - \mathcal{L}P = 0 \ . \tag{1.7}$$

In reality, Eq. (1.6) does not imply that the function $\partial P/\partial t - \mathcal{L}P$ is equal to zero, but rather that it is a linear combination of spherical harmonic functions $Y_l^m(\Omega)$ with $l > 2j$. This is a consequence of the results discussed in Appendix D of Ref. 4. [See, in particular, Eq. (D.22).] Indeed, an example of P function that satisfies Eqs. (1.3) and (1.6), but not Eq. (1.7) is not hard to find, as we show in the main text.

Still, equations of the type (1.6) have been solved until now by looking for a solution $\overline{P}(\Omega, t)$ such that

$$\frac{\partial \overline{P}}{\partial t} - \mathcal{L}\overline{P} = 0 \ . \tag{1.8}$$

Since, in general, $\overline{P}$ is not equal to P, we raise the question: Is it possible to claim that $\overline{P}(\Omega, t)$, solution of Eq. (1.8), is physically equivalent to $P(\Omega, t)$? Or, more precisely, can we prove under reasonable assumptions that

$$\int d\Omega\, \Lambda(\Omega)\overline{P}(\Omega, t) = \int d\Omega\, \Lambda(\Omega) P(\Omega, t) \ ? \tag{1.9}$$

Our paper is addressed to the two related problems: (1) the vanishing of the surface terms in Eq. (1.5), and (2) the proof of Eq. (1.9) under appropriate conditions.

In Sec. II we discuss a sufficient condition for the vanishing of the surface terms. In Sec. III we present a simple example in which the density functions $P(\Omega, t)$ and $\overline{P}(\Omega, t)$ are manifestly different from one another. The proof of Eq. (1.9) and the conditions for its validity are given in Sec. IV.

II. QUESTION OF THE SURFACE TERMS

The mapping of operator equations such as Eq. (1.1) into the general integral form of Eq. (1.5) is best accomplished by the simple rules developed in Ref. (7). For definiteness, we consider a class of equations of the form

$$\dot{W}(t) = \sum_{n,m} C_{nm} A_n W(t) B_m \ , \tag{2.1}$$

where A_n and B_m are arbitrary products of angular momentum operators. From the integral representation (1.2) one easily arrives at

$$\int d\Omega\, \Lambda(\Omega)\frac{\partial P}{\partial t} = \sum_{n,m} C_{nm} \int d\Omega\, P(\Omega, t) \times \mathfrak{D}^L(A_n)\mathfrak{D}^R(B_m)\Lambda(\Omega) \ , \tag{2.2}$$

where $\mathfrak{D}^L$ and $\mathfrak{D}^R$ are the appropriate differential operators acting on the angular variables of the atomic coherent-state projector $\Lambda(\Omega)$. In the physically interesting cases, equations of the type (2.1) involve, at most, bilinear combinations of angular momentum operators. We direct our attention to this case in order to limit the amount of algebraic manipulations. Our immediate goal is to integrate the right-hand side of Eq. (2.2) by parts and to study the conditions under which the surface terms vanish identically.

To this purpose, consider first the typical operator combination $AW(t)$, where A stands for J_z, J^+, or J^-. Following the rules developed in Ref. (7) we can write

$$AW(t) = \int d\Omega\, P(\Omega, t)\mathfrak{D}^L(A)\Lambda(\Omega) = \int d\Omega\, \Lambda(\Omega)\tilde{\mathfrak{D}}^L(A)[P(\Omega, t)] + \mathcal{S} \ . \tag{2.3}$$

Explicit expressions for the $\tilde{\mathfrak{D}}^L$ operators are given in Sec. VI of Ref. (7). An elementary calculation shows that the surface terms vanish if the function $P(\Omega, t)$ is single valued, and if the condition

$$\lim_{\theta \to 0, \pi} \sin\theta\, P(\Omega, t) = 0 \tag{2.4}$$

is satisfied. Consider now the bilinear combination $ABW(t)$. Proceeding as indicated above we find

$$ABW(t) = A \int d\Omega\, P(\Omega, t)\mathfrak{D}^L(B)\Lambda(\Omega) = A \int d\Omega\, \Lambda(\Omega)\tilde{\mathfrak{D}}^L(B)[P(\Omega, t)] \ . \tag{2.5}$$

In Eq. (2.5) we have integrated by parts, and assumed that $P(\Omega, t)$ is single valued in φ and that it satisfies Eq. (2.4). Proceeding further, we find

$$ABW(t) = \int d\Omega\, \mathfrak{D}^L(A)[\Lambda(\Omega)]P'(\Omega, t) = \int d\Omega\, \Lambda(\Omega)\tilde{\mathfrak{D}}^L(A)[P'(\Omega, t)] + \mathcal{S} \ , \tag{2.6}$$

where $P'(\Omega, t) = \tilde{\mathfrak{D}}^L(B)P(\Omega, t)$. A detailed analysis of the surface terms that originate from every bilinear combination of angular momentum operators reveals that single valuedness in the variable φ and the condition (2.4) are no longer sufficient to insure that the surface terms will vanish identically. This situation is a consequence of the fact that $\tilde{\mathfrak{D}}^L(B)$ may involve trigonometric functions such as $\tan\frac{1}{2}\theta$ or $\cot\frac{1}{2}\theta$ which diverge at π and 0, respectively. Appendix A contains a detailed sample calculation that illustrates the point. Here we state that the surface terms will vanish if $P(\Omega,t)$ is single valued and if

$$\lim_{\theta\to 0,\pi}\left(\sin\theta\,\frac{\partial P(\Omega,t)}{\partial\theta}\pm i\,\frac{\partial P(\Omega,t)}{\partial\varphi}\right)=0\ . \tag{2.7}$$

It is shown below that as long as Eq. (1.3) holds, the surface terms associated to Eq. (2.6) and all other similar expressions derived from bilinear combinations of angular momentum operators will vanish. From Eq. (1.3) it follows that

$$\frac{\partial P(\Omega,t)}{\partial\varphi}=i\sum_{l=0}^{2j}\sum_{m=-l}^{l}C_{lm}(t)mP_l^m(\theta)e^{im\varphi} \tag{2.8}$$

and

$$\frac{\partial P}{\partial\theta}<\infty\ . \tag{2.9}$$

Since the associated Legendre functions P_l^m, $m\neq 0$, contain a factor $(\sin\theta)^{|m|}$, it is obvious that Eq. (2.7) is satisfied. It may be remarked that, under the assumption (1.3), all the surface terms discussed in Ref. 6 are identically zero.

In conclusion, we can state that all the physically interesting operator equations that have been studied thus far can be mapped into the integral form

$$\int d\Omega\,\Lambda(\Omega)(\partial P/\partial t-\mathcal{L}P)=0$$

if the density function $P(\Omega,t)$ satisfies the condition (1.3), or more generally if $P(\Omega,t)$ is a *finite* linear combination of spherical harmonics.

III. PHYSICALLY EQUIVALENT *P* FUNCTIONS

It was remarked in the Introduction and shown in the previous section that if $P(\Omega,t)$ is a finite linear combination of spherical harmonics, the integral equation

$$\int d\Omega\,\Lambda(\Omega)(\partial P/\partial t-\mathcal{L}P)=0 \tag{3.1}$$

is equivalent to the operator equation $\dot{W}=\mathfrak{F}(W)$. The operator W and the corresponding P function are related to one another by the representation

$$W(t)=\int d\Omega\,\Lambda(\Omega)P(\Omega,t)\ . \tag{3.2}$$

We have also indicated that the function $\overline{P}(\Omega,t)$, solution of the differential equation

$$\frac{\partial\overline{P}(\Omega,t)}{\partial t}-\mathcal{L}\overline{P}(\Omega,t)=0 \tag{3.3}$$

is not, in general, a finite linear combination of spherical harmonics. [The new symbol $\overline{P}$ emphasizes that usually $P(\Omega,t)$ is not equal to $\overline{P}(\Omega,t)$.]

This observation will be supported by a simple example. The more general issue of the physical relation between $P(\Omega,t)$ and $\overline{P}(\Omega,t)$ will be discussed in detail in the next section.

Consider the model Hamiltonian

$$H=-\epsilon J_z \tag{3.4}$$

for a two-level system ($j=\frac{1}{2}$), and assume thermal equilibrium between the system and a reservoir at a temperature T. The (unnormalized) canonical density operator $W(\beta)$ is given by

$$W(\beta)=\exp(\beta\epsilon J_z),\quad \beta=1/kT\ . \tag{3.5}$$

For the present calculation it is convenient to regard $W(\beta)$ as the solution of the Bloch equation

$$\frac{\partial W}{\partial\beta}=-\tfrac{1}{2}\{H,W\}_+=\tfrac{1}{2}\epsilon(J_zW+WJ_z)\ . \tag{3.6}$$

We define the density function $P(\Omega,\beta)$, satisfying Eq. (1.3), by the integral representation

$$W(\beta)=\int d\Omega\,\Lambda(\Omega)P(\Omega,\beta)\ ,$$

and construct an integral equation of the type (3.1) following the procedure described in Ref. 7, Sec. VI. The result is

$$\int d\Omega\,\Lambda(\Omega)\left(\frac{\partial P}{\partial\beta'}+\frac{1}{2}\,\sin\theta\,\frac{\partial P}{\partial\theta}+\frac{3}{2}\,\cos\theta P\right)=0\ ,\qquad \beta'=\beta\epsilon\ . \tag{3.7}$$

As the next step, we consider the associated partial differential equation

$$\frac{\partial\overline{P}}{\partial\beta'}+\frac{1}{2}\,\sin\theta\,\frac{\partial\overline{P}}{\partial\theta}=-\frac{3}{2}\,\cos\theta\overline{P}\ , \tag{3.8}$$

subject to the initial conditon

$$\overline{P}(\Omega,\beta'=0)=1/2\pi\ . \tag{3.9}$$

The solution of Eq. (3.8) can be obtained by application of the method of characteristics. The result is

$$\overline{P}(\Omega,\beta')=(1/2\pi)[\cosh(\tfrac{1}{2}\beta')+\sinh(\tfrac{1}{2}\beta')\cos\theta]^{-3}\ . \tag{3.10}$$

It is obvious by inspection that even for $j=\frac{1}{2}$, $\overline{P}(\Omega,\beta')$ is not a finite linear combination of spherical harmonics. Still, we claim that $\overline{P}(\Omega,\beta')$ provides a representation of the density operator

(3.5) in the sense that

$$W(\beta') = \int d\Omega\, \Lambda(\Omega) \bar{P}(\Omega, \beta') \quad . \tag{3.11}$$

This can be checked by direct comparison of the matrix elements of the left- and right-hand sides of Eq. (3.11) in the angular momentum representation.

The simple structure of the density operator (3.5) enables us to evaluate the density function $P(\Omega, \beta')$ directly using the inversion formula developed in Sec. VII of Ref. (7). The inversion process requires the preliminary calculation of the function $Q(\Omega) = \langle \Omega | W(\beta') | \Omega \rangle$ and its expansion in terms of spherical harmonics

$$Q(\Omega, \beta') = \sum_{l=0}^{2j} \sum_{m=-l}^{l} q_{lm}(\beta') Y_l^m(\Omega) \quad . \tag{3.12}$$

In our case the task is especially simple since

$$Q(\Omega, \beta') = \langle \Omega | e^{\beta' J_z} | \Omega \rangle = \cosh(\tfrac{1}{2}\beta') - \sinh(\tfrac{1}{2}\beta') \cos\theta \quad . \tag{3.13}$$

The expansion coefficients are given by

$$q_{0,0} = (4\pi)^{1/2} \cosh(\tfrac{1}{2}\beta') \; , \quad q_{1,0} = -(4\pi/3)^{1/2} \sinh(\tfrac{1}{2}\beta') \; , \quad q_{1,1} = q_{1,-1} = 0 \; . \tag{3.14}$$

The P function is given by the finite sum

$$P(\Omega, \beta') = \sum_{l=0}^{2j} \sum_{m=-l}^{l} q_{lm}(\beta') (R_{00}^l)^{-1} Y_l^m(\Omega) \tag{3.15}$$

with

$$R_{00}^l = 4\pi \frac{(2j)!^2}{(2j+l+1)!\,(2j-l)!} \quad . \tag{3.16}$$

After setting $j = \frac{1}{2}$ and using the coefficients q_{lm} given by Eq. (3.14), the result follows

$$P(\Omega, \beta') = (1/2\pi)[\cosh(\tfrac{1}{2}\beta') - 3\sinh(\tfrac{1}{2}\beta')\cos\theta] \quad . \tag{3.17}$$

Clearly $P(\Omega, \beta')$ has quite a different functional form from $\bar{P}(\Omega, \beta')$. It is elementary to verify the validity of the integral representation for $W(\beta')$, again by direct calculation of the operator matrix elements.

Thus, we have derived two physically equivalent C-number representations for the same density operator. It is interesting to point out that while

$$\frac{\partial P}{\partial \beta'} + \frac{1}{2} \sin\theta \frac{\partial P}{\partial \theta} + \frac{3}{2} \cos\theta P \neq 0 \tag{3.18}$$

for $P(\Omega, \beta')$ given by Eq. (3.17), the integral equation (3.7) is instead identically satisfied, as it must be.

A final remark of interest concerns the question of positive definiteness. It is readily seen that $\bar{P}(\Omega, \beta')$ [Eq. (3.10)] is positive definite for all values of β' and θ. By contrast, $P(\Omega, \beta')$ [Eq. (3.17)] is not positive definite at least for sufficiently low temperatures (large β') such that

$$3 \tanh(\tfrac{1}{2}\beta') \cos\theta > 1 \quad .$$

Hence, we have produced a simple physical example of the lack of positive definiteness that characterizes the quasi-probability distributions associated to the overcomplete set of coherent atomic states.

IV. EQUIVALENCE OF THE DISTRIBUTION FUNCTIONS $P(\Omega,t)$ AND $\bar{P}(\Omega,t)$: MATHEMATICAL CONDITIONS

The discussion of the previous section makes it plausible that the solution $\bar{P}(\Omega, t)$ of the C-number differential equation

$$(\partial/\partial t - \mathcal{L}) \bar{P}(\Omega, t) = 0 \tag{4.1}$$

should be physically equivalent to the density function $P(\Omega, t)$, at least if appropriate conditions are satisfied. It is also clear that, in general, $\bar{P}(\Omega, t)$ and $P(\Omega, t)$ have different functional forms. In this section we establish sufficient conditions under which the equality

$$\int d\Omega\, P(\Omega, t) \Lambda(\Omega) = \int d\Omega\, \bar{P}(\Omega, t) \Lambda(\Omega) \tag{4.2}$$

holds. For definiteness, we label Eq. (4.2) as the condition for physical equivalence of $P(\Omega, t)$ and $\bar{P}(\Omega, t)$.

It is convenient to modify our notations in order to stress the dependence of the various quantities on the parameter j. Thus, in the Hilbert space of total angular momentum j, let $W^{(j)}(t)$ be the density operator, solution of Eq. (1.1) for a given initial condition $W^{(j)}(0)$, and $\Lambda^{(j)}(\Omega)$ the diagonal coherent atomic-state projector. We define the density function $P^{(j)}(\Omega, t)$ from the integral representation

$$W^{(j)}(t) = \int d\Omega\, \Lambda^{(j)}(\Omega) P^{(j)}(\Omega, t) \tag{4.3}$$

and require that $P^{(j)}$ be expressed as linear superposition of the first $(2j+1)^2$ spherical harmonics

$$P^{(j)}(\Omega, t) = \sum_{l=0}^{2j} \sum_{m=-l}^{l} C_{lm}^{(j)}(t) Y_l^m(\Omega) \quad . \tag{4.4}$$

Finally, we define $\bar{P}^{(j)}(\Omega, t)$ as the solution of the C-number differential equation

$$(\partial/\partial t - \mathcal{L}^{(j)}) \bar{P}^{(j)}(\Omega, t) = 0 \tag{4.5}$$

subject to the initial condition $\bar{P}^{(j)}(\Omega, 0) = f(\Omega)$, such that

$$W^{(j)}(0) = \int d\Omega\, f(\Omega)\Lambda^{(j)}(\Omega) \ . \quad (4.6)$$

It is important that we stress the following feature of Eq. (4.5): The differential operator $\mathfrak{L}^{(j)}$ depends analytically on the parameter j. Consequently, the function $\overline{P}^{(j)}(\Omega, t)$ and the scalar products

$$(Y_l^m, \overline{P}^{(j)}) \equiv \int d\Omega\ Y_l^m(\Omega)\overline{P}^{(j)}(\Omega, t) \quad (l, m \text{ arbitrary}) \quad (4.7)$$

will also be analytic functions of j.

Our main objective is to identify the conditions under which Eq. (4.2) holds. We do this in a few steps.

(a) We observe that Eqs. (4.3) and (4.4) will insure the validity of the integral relation

$$\int d\Omega\, \Lambda^{(j)}(\Omega)(\partial/\partial t - \mathfrak{L}^{(j)})P^{(j)}(\Omega, t) = 0$$

since, as we have shown, the surface terms vanish if Eq. (4.4) is satisfied. Consider now an arbitrary function $P_s^{(j)}(\Omega, t)$ (with $s > 2j$) defined as

$$P_s^{(j)}(\Omega, t) = P^{(j)}(\Omega, t) + \sum_{l=2j+1}^{s}\ \sum_{m=-l}^{l} C_{lm}^{(j)}(t) Y_l^m(\Omega) \ . \quad (4.8)$$

Since $P_s^{(j)}$ and $P^{(j)}$ differ only by a finite linear combination of spherical harmonics with $l > 2j$, $P_s^{(j)}$ will also satisfy the integral relation

$$\int d\Omega\, \Lambda^{(j)}(\Omega)(\partial/\partial t - \mathfrak{L}^{(j)})P_s^{(j)}(\Omega, t) = 0 \ . \quad (4.9)$$

It follows from Eq. (4.9) that

$$(Y_l^m, (\partial/\partial t - \mathfrak{L}^{(j)})P_s^{(j)}) = 0, \quad \text{for} \quad l \le 2j \ , \quad (4.10)$$

since $(\partial/\partial t - \mathfrak{L}^{(j)})P_s^{(j)}(\Omega, t)$ is a linear combination of spherical harmonics Y_l^m with $l > 2j$.

(b) We fix the coefficients in Eq. (4.8) as follows. For a given value of l consider the set of numbers $C_{lm}^{(j')}(t)$, with $j' \ge \frac{1}{2}l$, defined by Eq. (4.4), i.e.,

$$C_{lm}^{(j')}(t) = (Y_l^m, P^{(j')}), \quad |m| \le l \ . \quad (4.11)$$

Let us *assume* that the collection of numbers $C_{lm}^{(j')}(t)$ with l, m, and t fixed can be interpolated by an analytic function of j' such that the isolated points $j' = \frac{1}{2}n$ (n integer ≥ 0) and $n < l$ belong to the domain of analyticity of the function. The value of this function for $j' = j$ defines the coefficients $C_{lm}^{(j)}(t)$ in Eq. (4.8). The analytic function interpolating the values (4.11) can be constructed explicitly following the inversion procedure discussed in Sec. VII of Ref. 7. More precisely, in Ref. 7 it is shown that

$$Q^{(j)}(\Omega, t) = \mathrm{Tr}[\Lambda^{(j)}(\Omega)W^{(j)}(t)]$$

obeys a suitable linear time-evolution equation. Since the generator of this equation depends analytically on the parameter j, the scalar products

$$(Y_l^m, Q^{(j)}) \quad (4.12)$$

are analytic functions of j for all l and m. (In particular, these scalar products are identically zero for $l > 2j$.) On the other hand, for $l \le 2j$ we have

$$C_{lm}^{(j)}(t) = (Y_l^m, Q^{(j)})(R_{00}^l)^{-1} \ , \quad (4.13)$$

where R_{00}^l is given by Eq. (3.16). The function $C_{lm}^{(j)}(t)$ given by Eq. (4.13) is the required analytic function of j.

(c) Next, we *assume* the existence of the limits

$$\lim_{s\to\infty} P_s^{(j)}(\Omega, t) \equiv \tilde{P}^{(j)}(\Omega, t) \quad (4.14)$$

and

$$\lim_{s\to\infty} (\partial/\partial t - \mathfrak{L}^{(j)})P_s^{(j)}(\Omega, t) = (\partial/\partial t - \mathfrak{L}^{(j)})\tilde{P}^{(j)}(\Omega, t) \ . \quad (4.15)$$

Equations (4.14) and (4.15), together with the reasonable requirement that $\tilde{P}^{(j)}(\Omega, t)$ depends analytically on j represent sufficient conditions for the validity of the identity

$$(\partial/\partial t - \mathfrak{L}^{(j)})\tilde{P}^{(j)}(\Omega, t) = 0 \ . \quad (4.16)$$

Equation (4.16) shows that the function $\tilde{P}^{(j)}(\Omega, t)$, constructed as indicated above, is solution of the C-number differential equation.

Proof. First, we show that for all l, m

$$(Y_l^m, (\partial/\partial t - \mathfrak{L}^{(j)})\tilde{P}^{(j)}(\Omega, t)) = 0 \ . \quad (4.17)$$

From Eq. (4.15) we have

$$(Y_l^m, (\partial/\partial t - \mathfrak{L}^{(j)})\tilde{P}^{(j)}(\Omega, t)) = \lim_{s\to\infty}(Y_l^m, (\partial/\partial t - \mathfrak{L}^{(j)})P_s^{(j)}(\Omega, t)) \ . \quad (4.18)$$

Furthermore, from Eq. (4.10) we have

$$(Y_l^m, (\partial/\partial t - \mathfrak{L}^{(j)})P_s^{(j)}(\Omega, t)) = 0, \quad l \le 2j \quad (4.19)$$

and consequently [Eq. (4.18)]

$$(Y_l^m, (\partial/\partial t - \mathfrak{L}^{(j)})\tilde{P}^{(j)}(\Omega, t)) = 0, \quad l \le 2j \ . \quad (4.20)$$

Consider, now, the case $l > 2j$. Given an arbitrary value j' with $2j' \ge l$, Eq. (4.20) insures that

$$(Y_l^m, (\partial/\partial t - \mathfrak{L}^{(j')})\tilde{P}^{(j')}(\Omega, t)) = 0 \ , \quad l > 2j, \quad j' \ge \tfrac{1}{2}l \ . \quad (4.21)$$

On the other hand, the expression $(Y_l^m, (\partial/\partial t - \mathfrak{L}^{(j)})\tilde{P}^{(j)}(\Omega, t))$ is given by the analytic continuation of the function of $j'(Y_l^m, (\partial/\partial t - \mathfrak{L}^{(j')})\tilde{P}^{(j')})$ evaluated at $j' = j$.

Therefore, aside from unusual situations which are certainly excluded, for example, if the analytic functions in question satisfy Carlson's theorem,[8] Eq. (4.21) implies

$$(Y_l^m, (\partial/\partial t - \mathcal{L}^{(j)})\tilde{P}^{(j)}(\Omega, t)) = 0, \quad l > 2j \ . \tag{4.22}$$

In summary, Eqs. (4.19) and (4.22) imply the validity of Eq. (4.17), which in turn implies Eq. (4.16).

(d) By construction, the function $\tilde{P}^{(j)}(\Omega, t)$ satisfies the equality

$$\int d\Omega\, \tilde{P}^{(j)}(\Omega, t)\Lambda^{(j)}(\Omega) = \int d\Omega\, P^{(j)}(\Omega, t)\Lambda^{(j)}(\Omega) \ . \tag{4.23}$$

On the other hand, since

$$\tilde{P}^{(j)}(\Omega, 0) = f(\Omega) \ , \tag{4.24}$$

we obtain from Eqs. (4.5), (4.6), and (4.16) that the following equality is satisfied

$$\tilde{P}^{(j)}(\Omega, t) = \overline{P}^{(j)}(\Omega, t) \ . \tag{4.25}$$

In conclusion, Eqs. (4.23) and (4.25) show that the conditions (4.14) and (4.15) are sufficient to guarantee the validity of Eq. (4.2).

A final remark is in order concerning the analytic function that interpolates the infinite sequence of numbers $C_{lm}^{(j')}(t)$ corresponding to all values of j'. Such analytic function is not unique, since the addition of a term such as, for example, $K\sin 2\pi j'$, where K is an arbitrary constant, leads to another interpolating function ($\sin 2\pi j'$ is zero for integral or half-integral values of j'). It is clear, however, that this fact does not introduce any ambiguity in the definition of $C_{lm}^{(j)}(t)$ for $2j \leq l$.

APPENDIX

We consider the mapping of the operator J^+J^+W. Following the procedure outlined in Sec. II, we find

$$J^+J^+W = J^+ \int d\Omega\, P(\Omega, t)\mathfrak{D}^L(J^+)[\Lambda(\Omega)] \ , \tag{A1}$$

where the square brackets indicate that the differential operator $\mathfrak{D}^L(J^+)$ acts on the angular variables of the projector $\Lambda(\Omega)$. After a first integration by parts and the replacement of the operator J^+ by the appropriate differential form, we find

$$J^+J^+W = \int d\Omega\, \mathfrak{D}^L(J^+)[\Lambda(\Omega)]P'(\Omega, t) \ , \tag{A2}$$

where $P'(\Omega, t) = \tilde{\mathfrak{D}}^L(J^+)[P(\Omega, t)]$. A second integration by parts leads to the final result

$$J^+J^+W = \int d\Omega\, \tilde{\mathfrak{D}}^L(J^+)\tilde{\mathfrak{D}}^L(J^+)[P(\Omega, t)]\,\Lambda(\Omega) + \mathcal{S} \ , \tag{A3}$$

where the surface terms ($\mathcal{S}$) are given by

$$\mathcal{S} = \int_0^{2\pi} d\varphi\, \sin\theta \cos^2(\tfrac{1}{2}\theta)e^{i\varphi} \times \Lambda(\Omega)\tilde{\mathfrak{D}}^L(J^+)[P(\Omega, t)]\Big|_{\theta=0}^{\theta=\pi} + \mathcal{S}' \ , \tag{A4}$$

where $\mathcal{S}'$ consists of terms that are in fact identically zero because of the single-valuedness of $P(\Omega, t)$. After some elementary calculations, Eq. (A4) can be reduced to the following expression:

$$\mathcal{S} = \int_0^{2\pi} d\varphi\, e^{2i\varphi}|j, -j\rangle\langle j, -j| \lim_{\theta\to 0}\left(\sin\theta \frac{\partial P}{\partial\theta} + i\frac{\partial P}{\partial\varphi}\right). \tag{A5}$$

As shown in Sec. II, Eq. (A5) is identically zero if $P(\Omega, t)$ is a finite linear combination of spherical harmonics.

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