Threshold instabilities in nonlinear self-excited oscillators*

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We present the results of a systematic study of threshold instabilities in a Wien bridge oscillator. The dynamical circuit equations can be cast into a time-dependent Ginzburg-Landau form where the order parameter is the complex zero-dimensional amplitude of the oscillatory voltage. In this context we have been able to simulate behavior which in classical theory is characteristic of first-order, second-order, and tricritical phase transitions, and have measured the associated threshold properties.

In recent years there has been considerable interest in the analogy between instabilities in dissipative nonlinear systems and critical points of equilibrium phase transitions.[1] For example, Graham and Haken[1] and DeGiorgio, Scully, Goldstein, and Lee[2] have related the behavior of a laser near threshold with the critical point of a second-order phase transition. Similar analogies have found their way into the theory of instabilities in chemical rate equations,[3] thermal instabilities in the Bernard problem,[4] and instabilities in simple bistable electrical circuits including the parametric oscillator[5] and the tunnel diode circuit.[6] Recently, Kawakubo *et al.*[7] have noted that the mean-square voltage fluctuations of a Wien bridge oscillator diverges near threshold in a manner analogous to mean-field theory for a second-order phase transition. However, there exists little experimental evidence regarding the extent to which this analogy holds.

With this motivation, we have examined in detail the behavior of a Wien bridge oscillator near threshold. We find that the behavior is well described by classical Landau (or mean-field) theory except in the immediate vicinity of the mean-field value of the critical feedback. In this region we argue that the system becomes analogous to a zero-dimensional superconductor (i.e., it has no spatial extent and a two-component order parameter). The details of these arguments will be described after the presentation of the experimental data.

I. EXPERIMENTAL

The circuit diagram for a Wien bridge oscillator is schematically shown in Fig. 1. The magnitude of the voltage fed back into the resonant cavity is determined by the dimensionless feedback parameter $\frac{1}{3}-\alpha$ where $\alpha = R_2/(R_1+R_2)$. The circuit oscillates when the feedback is greater than a critical value $\frac{1}{3}-\alpha_c$ (i.e., when $\alpha \leq \alpha_c$, where $\alpha_c = \frac{1}{3} - 1/A$). The circuit equation for Fig. 1 is given by

$$\frac{d^2V}{dt^2} + \frac{9\omega_0(\alpha-\alpha_c)}{1+3(\alpha-\alpha_c)}\frac{dV}{dt} + \omega_0^2 V = f(t),$$

which can be approximated near threshold as

$$\frac{d^2V}{dt^2} + 9\omega_0(\alpha-\alpha_c)\frac{dV}{dt} + \omega_0^2 V = f(t), \qquad (1)$$

where $\omega_0 = 1/RC$ is the resonance frequency at threshold and $f(t)$ is a statistically defined Langevin white-noise source. Experimentally $f(t)$ is either generated by Johnson noise in the circuit components, or externally introduced by the addition of a white-noise generator to the circuit in Fig. 1.

When $f(t)=0$, oscillation at fixed amplitude obtains when $\alpha - \alpha_c = 0$. For $\alpha < \alpha_c$ the amplitude of oscillation grows until nonlinearities in the components or the amplifier restore the condition $\alpha - \alpha_c = 0$. For simplicity we add to the circuit a specific well-defined power-dependent nonlinear element such as a light bulb or a thermistor. α is then determined by the temperature and hence the power dissipated in the nonlinear element. Thus for a light bulb with relaxation time τ, α trivially obeys the equation

$$\frac{d\alpha}{dt} + \frac{\alpha-\alpha_0}{\tau} = a|\psi|^2 + b|\psi|^4 + \cdots . \qquad (2)$$

α_0 is the linear part of α, a and b are positive constant, and ψ is the complex amplitude of the oscillatory voltage defined by $V(t) = \mathrm{Re}\,\psi(t)e^{i\omega_0 t}$. The condition for oscillation with fixed amplitude

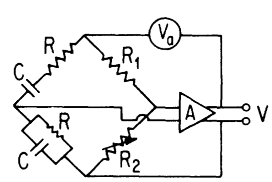

FIG. 1. Circuit schematic for the Wien bridge oscillator.

becomes

$$\alpha_0 - \alpha_c + \tau a\langle|\psi|^2\rangle + \tau b\langle|\psi|^4\rangle + \cdots = 0, \tag{3}$$

where $\langle|\psi|^2\rangle$ is the time average of $|\psi|^2$. Note that in the absence of noise [i.e., $f(t)=0$] the stationary or steady-state value of $|\psi|^2$ obtained by setting $d\alpha/dt = d\psi/dt = 0$ is equal to the time average, since after initial transients there are no fluctuations of ψ about $\langle\psi\rangle$. Furthermore, in the approximation $f(t) = 0$, $(\langle\psi\rangle)^2 = \langle|\psi|\rangle^2 = \langle|\psi|^2\rangle$. With the identification that $\alpha_0 \to T$ and $\alpha_c \to T_c$ Eq. (3) becomes equivalent to the equilibrium condition in Landau theory for $T<T_c$. Thus neglecting the $|\psi|^4$ term in Eq. (3) we obtain the mean-field expression for the onset of oscillations $\langle\psi\rangle = [(\alpha_c - \alpha_0)/\tau a]^{1/2}$. Similarly, for $\alpha_0 > \alpha_c$, the steady-state solutions are $\langle\alpha\rangle = \alpha_0$ and $\langle\psi\rangle = 0$.

In a similar manner, one can show that the amplitude E of an applied voltage V_a oscillating with frequency ω_0 (i.e., $V_a = \text{Re}\, iEe^{i\omega_0 t}$) takes the place in the Landau analogy of the field conjugate to ψ. The condition for steady-state oscillations becomes

$$(\alpha_0 - \alpha_c)\langle|\psi|\rangle + \tau a\langle|\psi|^3\rangle + \tau b\langle|\psi|^5\rangle + \cdots - E/9\omega_0^2 = 0, \tag{4}$$

which is again equivalent to classical Landau theory when E is the field conjugate to ψ. We see therefore that in the absence of noise [i.e., $f(t)=0$] the steady-state or time-averaged behavior of the circuit is reminiscent of the mean-field approximation for the equilibrium behavior of a system which has a phase transition. It is tempting therefore to assume that the analogy between the time-averaged circuit response and equilibrium phase transitions holds in the presence of fluctuations and hence define the usual critical exponents β, δ, and γ as[8]

$$\begin{aligned}
&\langle\psi\rangle \sim (\alpha_c - \alpha_0)^\beta, \quad E=0, \quad \alpha_0 \leq \alpha_c,\\
&\langle\psi\rangle \sim E^{1/\delta}, \quad \alpha_0 = \alpha_c \qquad\qquad (5)\\
&x \equiv \frac{\partial\langle\psi\rangle}{\partial E} \sim (\alpha_0 - \alpha_c)^{-\gamma}, \quad E=0, \quad \alpha_0 \geq \alpha_c,
\end{aligned}$$

where in the above simple classical or mean-field-like model $\langle|\psi|\rangle = \langle\psi\rangle$ and $\beta = \frac{1}{2}$, $\delta = 3$, and $\gamma = 1$. In Sec. III we will confirm the general structure of the phase-transition analogy but will show that the lack of spatial extent of ψ leads to the nontransitional behavior characteristic of, for example, a zero-dimensional superconductor.

Figures 2(a)–2(d) show experimental results obtained on the analog second-order phase transition as described above. We show the measured values of $\langle|\psi|\rangle$ and $\partial\langle|\psi|\rangle/\partial E$ as a function of the feedback parameter α_0 and the applied field E. The fact that the measured value of $x = \partial\langle|\psi|\rangle/\partial E$ appears not to diverge as $\alpha_0 \to \alpha_c$ can at least be partially

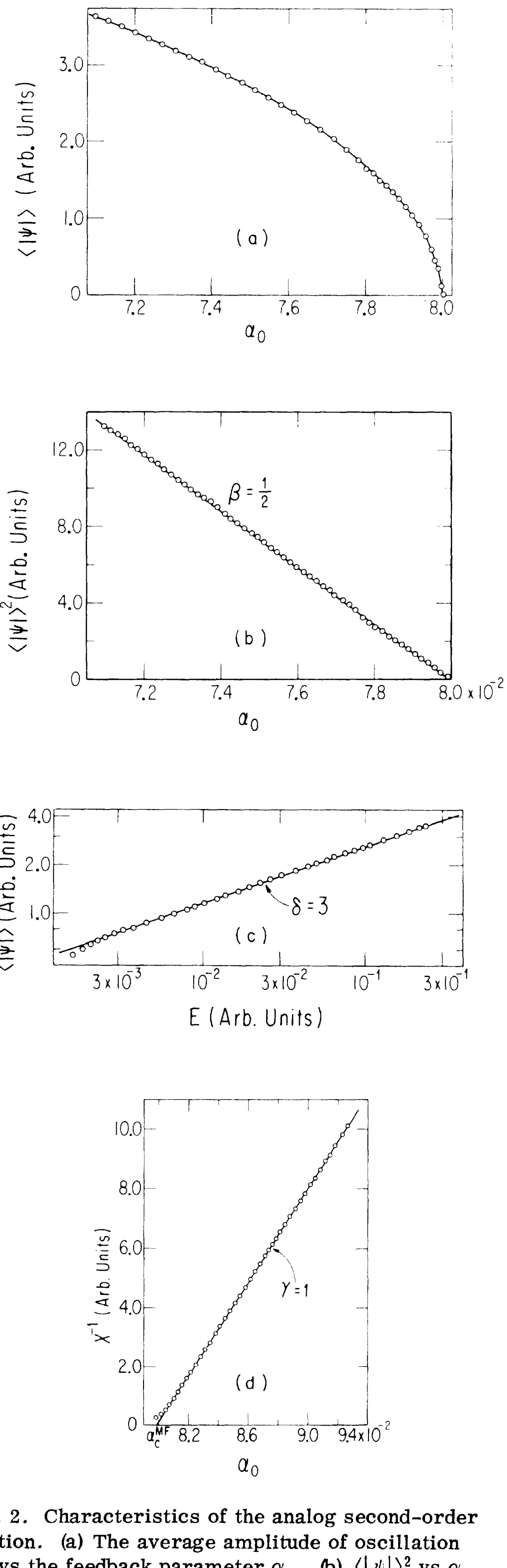

FIG. 2. Characteristics of the analog second-order transition. (a) The average amplitude of oscillation $\langle|\psi|\rangle$ vs the feedback parameter α_0. (b) $\langle|\psi|\rangle^2$ vs α_0. (c) $\langle|\psi|\rangle$ vs the amplitude E of the driving voltage. (d) The inverse generalized susceptibility $\chi^{-1} = (\partial\langle|\psi|\rangle/\partial E)^{-1}$ vs α_0.

attributed to the fact that the measurements were made at finite E and not in the limit $E \to 0$. We find therefore the onset of oscillations are well described by the classical theory, with at most small deviations in the vicinity of threshold.

To the present time we have considered only a particular type of nonlinear element as described by Eq. (2). It is also possible to simulate a first-order phase transition by incorporating nonlinear elements such that Eq. (2) has the form

$$\frac{d\alpha}{dt} + \frac{\alpha - \alpha_0}{\tau} = -a|\psi|^2 + b|\psi|^4 + \cdots . \quad (6)$$

This is achieved experimentally by replacing R_1 and R_2 in Fig. 1 with appropriately selected nonlinear elements into both the R_1 and the R_2 arms of the bridge. The condition for steady-state oscillation becomes

$$(\alpha_0 - \alpha_c)\langle|\psi|\rangle - \tau a\langle|\psi|^3\rangle + \tau b\langle|\psi|^5\rangle - (E/9\omega_0^2) = 0 , \quad (7)$$

where we have neglected terms of order higher than $|\psi|^4$ in Eq. (6). The experimental results for the analog first-order phase transition are shown in Figs. 3(a)–3(c). In Fig. 3(a) the data for various values of E are shown in comparison with predictions from Eq. (7). The three adjustable parameters which must be chosen when comparing Eq. (7) with the data are α_c, τa, and τb, or equivalently α_c, α^*, and $|\psi(\alpha^*)|$, where α^* is the feedback value at the first-order instability and $|\psi(\alpha^*)|$ is the magnitude of the discontinuity in $|\psi|$ at $\alpha = \alpha^*$. These parameters are fixed from the $|\psi|$ vs α curve at $E = 0$. The remaining $E \neq 0$ curves are fitted without further adjustment of the parameters. The fit is quite good. For $E \cong 30$ [in the arbitrary units used in Fig. 3(a)] there is a second-order transition at $\alpha_0 = \alpha_0^+$. The critical exponents near this transition are identical to those observed in Fig. 2, as exemplified by Fig. 3(c). For $E > 30$ the transition for oscillatory to nonoscillatory behavior becomes continuous.

It is also possible to simulate a tricritical point by choosing a nonlinear element such that a in Eq. (7) can be varied continuously. The characteristic $|\psi|^4$ behavior expected when $a = 0$ is seen in Figs. 4(a) and 4(b). The deviation from $\langle|\psi|\rangle \sim (\alpha_c - \alpha_0)^{1/4}$ as $\alpha_0 \to \alpha_c$ is considerably larger than the deviation from mean-field theory observed at the analog second-order phase transition. This deviation will be discussed in Sec. III.

In summary, the experimental analysis shows that the onset of oscillations can be adequately described by a classical Landau-like analysis except possibly in the immediate vicinity of $\alpha_0 = \alpha_c$.

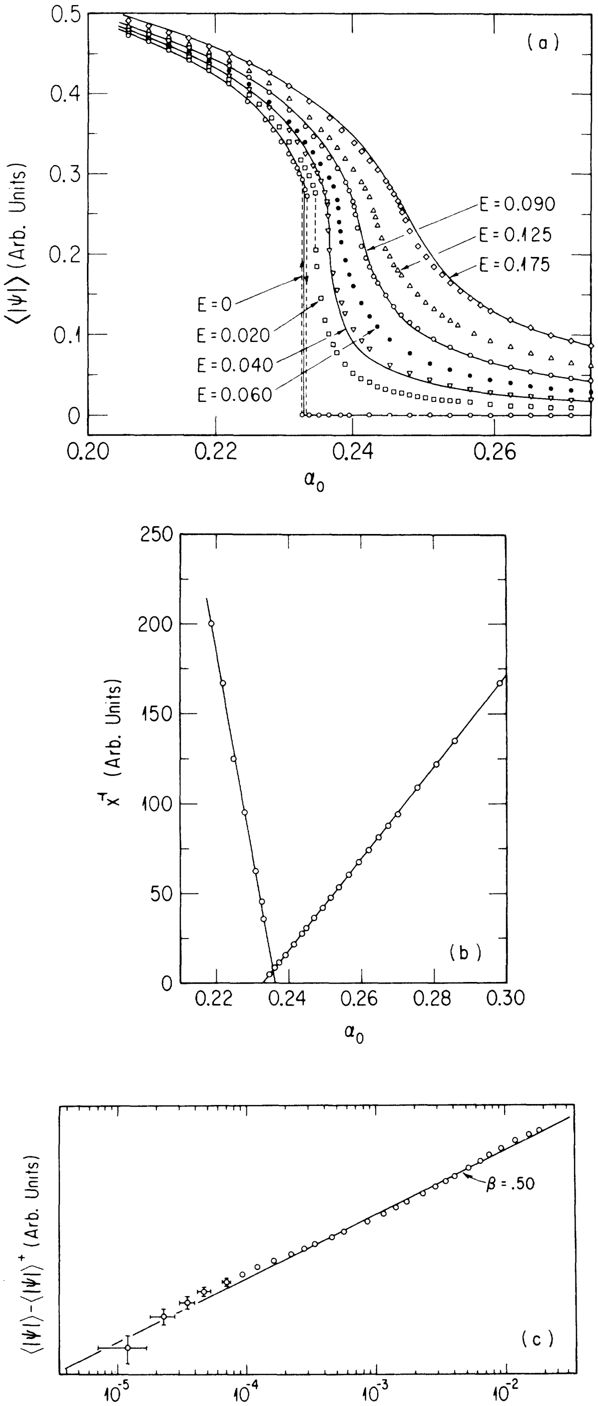

FIG. 3. Characteristics of the analog first-order phase transition. (a) $\langle|\psi|\rangle$ vs α_0 for various values of E. Dashed lines: experimentally observed limits of hysteresis loop. Solid lines: Theoretical fit to Eq. (7) for $E = 0$, 0.04, 0.09, and 0.175. (b) Inverse generalized susceptibility $\chi^{-1} = (\partial\langle|\psi|\rangle/\partial E)^{-1}$ vs α_0. (c) Behavior near the critical point at $\alpha_0 = \alpha_0^+$ and $\langle|\psi|\rangle = \langle|\psi|\rangle^+$; order parameter $\langle|\psi|\rangle - \langle|\psi|\rangle^+$ vs feedback $\alpha_0 - \alpha_0^+$.

II. DEVIATION FROM CLASSICAL THEORY

When $f(t)$ is nonzero, fluctuations of $|\psi|$ from $\langle|\psi|\rangle$ may become important and the validity of the simplified mean-field-like analysis given in Sec. I is called into question. Intuitively one might expect that for $\alpha_0 \gg \alpha_c$ or $\alpha_0 \ll \alpha_c$ some aspects of the mean-field analysis would be approximately correct. The problem is in determining how and when the classical theory fails as $\alpha_0 \to \alpha_c$. There are at least two equivalent approaches which one can take to answer this question. Since ψ has no spatial extent, the problem can be transformed into a one-dimensional statics problem with the transformation $it \to X$. The circuit response can then be determined exactly using the techniques developed by Scalapino, Sears, and Ferrell.[9] Alternatively, the problem can be treated as a dynamics problem in zero dimensions via application of the time-dependent Ginzburg-Landau formalism.

We choose the latter approach. Let $\psi(t) = |\psi| e^{i\varphi}$. Under the assumption that ψ is slowly varying compared to V, Eq. (1) can be rewritten by averaging over a few oscillatory cycles.[10] We obtain

$$\begin{aligned} &(d|\psi|/dt) + (9\omega_0/2)(\alpha - \alpha_c)|\psi| = \eta_{|\psi|}(t), \\ &|\psi|(d\varphi/dt) = \eta_\varphi(t), \end{aligned} \tag{8}$$

where

$$\eta_{|\psi|}(t) = \mathrm{Re}\{[f(t)/2i\omega_0]e^{-i(\omega_0 t+\varphi)}\},$$

$$\eta_\varphi(t) = -\mathrm{Re}\{[f(t)/2\omega_0]e^{-i(\omega_0 t+\varphi)}\},$$

and we have neglected terms small compared to $d\psi/dt$. In the limit $\alpha_0 \to \alpha_c$,

$$\frac{d^2\psi}{dt^2} \ll \frac{1}{\tau}\frac{d\psi}{dt}$$

and α can be replaced by its steady-state value in the absence of noise (i.e., $\alpha = \langle\alpha\rangle + \tau a|\psi|^2$). Equation (8) becomes

$$\begin{aligned} &\frac{d|\psi|}{dt} + \frac{9\omega_0}{2}(\alpha_0 - \alpha_c + \tau a|\psi|^2)|\psi| = \eta_{|\psi|}(t), \\ &|\psi|\frac{d\varphi}{dt} = \eta_\varphi(t). \end{aligned} \tag{9}$$

Equation (9) is in the form of a time-dependent Ginzburg-Landau (TDGL) equation,[11]

$$\frac{d\psi_i}{dt} = -\frac{\Gamma}{T}\frac{\delta}{\delta\psi_i}F\{\psi_i\} + \eta_{\psi_i}(t), \tag{10}$$

where $\psi_i = \psi_1$ or ψ_2, $F\{\psi_i\}$ is the Landau functional, and Γ is the inverse relaxation time associated with $\eta_{\psi_i}(t)$. $\eta_{\psi_i}(t)$ is a statistically defined white-noise source with zero mean and correlation function given by

$$\langle \eta_{\psi_i}(t)\eta_{\psi_j}(t')\rangle_{\text{eq. av.}} = 2\Gamma\delta_{ij}\delta(t-t'), \tag{11}$$

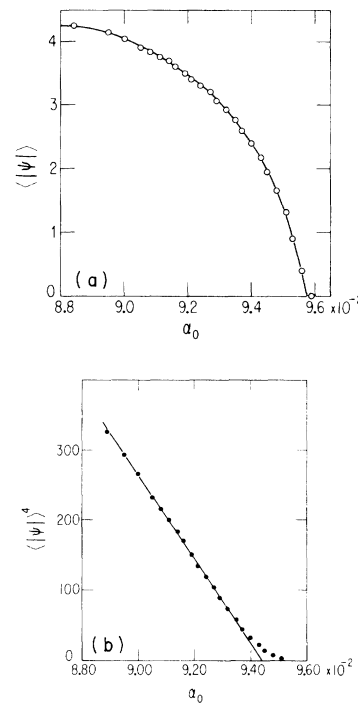

FIG. 4. Characteristics of the analog tricritical point. (a) $\langle|\psi|\rangle$ vs α_0. (b) $\langle|\psi|\rangle^4$ vs α_0.

where $\langle\cdots\rangle_{\text{eq. av.}}$ represents the equilibrium statistical average. To show the analogy between Eq. (10) and Eq. (9) we transform Eq. (10) to radial coordinates. Let $\psi = \psi_1 + i\psi_2 = |\psi|e^{i\varphi}$. Then if F is independent of φ Eq. (10) becomes

$$\frac{d|\psi|}{dt} = -\frac{\Gamma}{T}\frac{\delta F}{\delta|\psi|} + \eta_{|\psi|}(t), \quad |\psi|\frac{d\varphi}{dt} = \eta_\varphi(t), \tag{12}$$

where

$$\eta_{|\psi|}(t) = \eta_{\psi_1}(t)\cos\varphi + \eta_{\psi_2}(t)\sin\varphi,$$

$$\eta_\varphi(t) = \eta_{\psi_2}(t)\cos\varphi - \eta_{\psi_1}(t)\sin\varphi.$$

Note that $\eta_{|\psi|}(t)$ and $\eta_\varphi(t)$ do not represent white-noise sources because of the time dependence of $\varphi(t)$. Comparing Eq. (12) with Eq. (9) we obtain for F

$$F\{|\psi|\} = (9\omega_0 T/4\Gamma)(\alpha_0 - \alpha_c)|\psi|^2 + (9\omega_0\tau a T/8\Gamma)|\psi|^4.$$

Decoupling the statics from the dynamics forces, $T = 4\Gamma\omega_0$. Thus the parameter T, which is a measure of the fluctuations in $|\psi|$, is defined in terms of the circuit parameters Γ and ω_0. The Landau function then becomes

$$F\{|\psi|\} = \tfrac{9}{2}\omega_0^2(\alpha_0 - \alpha_c)|\psi|^2 + \tfrac{9}{4}\omega_0^2\tau a|\psi|^4. \tag{13}$$

$F\{|\psi|\}$ in Eq. (13) represents a fictitious Landau free-energy function for the circuit chosen such that the TDGL equation reproduces the circuit equation. All time-averaged moments of ψ can now be obtained by computing equilibrium averages over the Landau function given in Eq. (11). Since ψ is a zero-dimensional field these averages become simple integrals, as illustrated below:

$$\langle g(\psi)\rangle = \langle g(\psi)\rangle_{\text{eq. av.}} = \frac{1}{Z}\int d\psi\, g(\psi)e^{-F\{|\psi|\}/T}, \qquad (14)$$

where the partition function Z is given by

$$Z = \int d\psi\, e^{-F\{|\psi|\}/T}, \qquad (15)$$

and where $g(\psi)$ is an arbitrary function of ψ and $d\psi = |\psi|\, d|\psi|\, d\varphi$. The equality of the time average $\langle g(\psi)\rangle$ and the equilibrium average $\langle g(\psi)\rangle_{\text{eq. av.}}$ is just the usual requirement of ergodicity built into the TDGL formalism, and in the remainder of the text we use these symbols interchangeably.

Therefore we have shown that the circuit response can be represented by an equilibrium average over a properly chosen free-energy function. Furthermore, as anticipated in Sec. II, the form of $F\{|\psi|\}$ thus obtained has, in the classical or mean-field approximation, a phase transition at $\alpha_0 = \alpha_c$.

At this stage the approximation that $\alpha = \alpha_0 + \tau a|\psi|^2$ must be scrutinized. Examination of Eqs. (8) and (2) shows that approach of $|\psi|$ to $\langle|\psi|\rangle$ is not monotonic as suggested by Eq. (9), but rather is oscillatory. In the limit $\alpha_0 \to \alpha_c$, $\ddot{\psi} \ll (1/\tau)\dot{\psi}$ [or equivalently $18\omega_0(\alpha_0 - \alpha_c) \ll 1/\tau$] and Eq. (13) is approximately valid. However, if $\ddot{\psi} \gtrsim (1/\tau)\dot{\psi}$, one cannot let $\alpha = \langle\alpha\rangle$ but rather must appropriately average over the fluctuations in α. Since the approach to $|\psi|$ to $\langle|\psi|\rangle$ and α to $\langle\alpha\rangle$ is oscillatory, the usual TDGL formalism must be modified. It is found however that even in this region Eq. (13) is still valid.[12]

III. DISCUSSION

The previous two sections have demonstrated that the Wien bridge oscillator is well described by a zero-dimensional Ginzburg-Landau (GL) field theory. A zero-dimensional field displays no long-range order in time. For example,

$$\langle\psi\rangle_{\text{eq. av.}} = \frac{1}{Z}\int d|\psi|\, d\varphi\, |\psi|^2 e^{i\varphi} e^{-F\{|\psi|\}/T} = 0. \qquad (16)$$

Since $\langle|\psi|\rangle_{\text{eq. av.}}$ is nonzero, the absence of long-range order in time is a manifestation of the independence of the system's free energy of the phase φ; φ is allowed to fluctuate over all possible values. Therefore while we find experimentally that the classical theory describes $\langle|\psi|\rangle_{\text{eq. av.}}$ quite well except when $\alpha_0 \cong \alpha_c$, it fails badly in describing $\langle\psi\rangle_{\text{eq. av.}}$, since it predicts $\langle\psi\rangle_{\text{eq. av.}} \neq 0$ for $\alpha_0 < \alpha_c$. This dichotomy of behavior is not new to the literature, but in fact happens in many systems of limited dimensionality.[9]

The measured moments for the second-order phase transition can be calculated from the appropriate integrals over $F\{|\psi|\}$. The susceptibility, for example, is given by

$$x = \frac{1}{ZT}\int |\psi|^2 e^{-F\{|\psi|\}/T} d\psi. \qquad (17)$$

For $\alpha_0 \gg \alpha_c$ the dominant contribution to $F\{|\psi|\}$ comes from the $|\psi|^2$ term in Eq. (13) and the susceptibility is mean-field-like. In this region we obtain

$$x = \frac{1}{9\omega_0^2}\,\frac{1}{\alpha_0 - \alpha_c}. \qquad (18)$$

As α_0 approaches α_c the $|\psi|^4$ term in $F\{|\psi|\}$ becomes important and classical theory breaks down. At $\alpha_0 = \alpha_c$, $x = 2/3\omega_0(\pi\tau a T)^{1/2}$ is finite, but large if the noise is small. The condition for the validity of the mean-field-like theory is

$$\alpha_0 - \alpha_c \gg (\tau a T/9\omega_0^2)^{1/2}. \qquad (19)$$

Equation (19) is analogous to the Ginzburg criterion for the validity of mean-field theory. T, which is a measure of the magnitude of the fluctuations in ψ (i.e., $T = 4\Gamma\omega_0$), may be varied at will by introducing an external noise source into the circuit. The role of T in the circuit response is illustrated in Fig. 5, where we plot the probability amplitude of $|\psi|$ [$\equiv P(|\psi|)$] vs $|\psi|$ for various values of the applied noise at fixed $\alpha_0 > \alpha_c$. $P(|\psi|)$, which is ex-

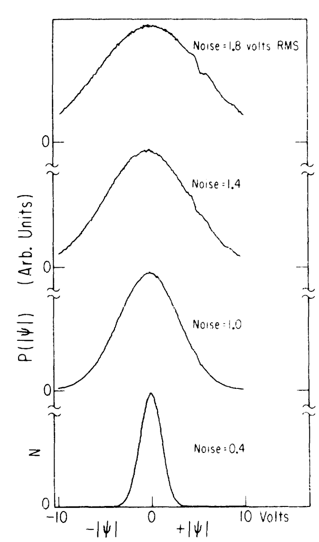

FIG. 5. Probability of obtaining a particular amplitude of oscillation $P(|\psi|)$ vs $|\psi|$ for the second-order transition with $\alpha_0 = 0.0950$, at various values of the applied white noise.

perimentally measured with a high-frequency gate and a pulse-height analyzer, is given by $P(|\psi|) = e^{-F(|\psi|)/T}$. For $\alpha_0 - \alpha_c \gg (\tau a T/9\omega_0^2)^{1/2}$, $P(|\psi|)$ should be Gaussian with a half-width (at fixed α_0) proportional to $T^{1/2}$. Since $T \sim \Gamma \sim$ (noise power), we expect the Gaussian half-width to be proportional to the applied-noise voltage. Our experimental results, as illustrated in Fig. 5, are consistent with this picture.

The region over which the classical theory does not apply can also be adjusted by varying the applied noise. For the second-order transition, ordinary thermal and amplifier noise is so small that the deviations from classical behavior are difficult to measure. The large apparent deviation from classical theory for the tricritical-point circuit can be attributed to one of two possible sources: (1) crossover from a tricritical region to a critical region as α_0 increases owing to the inability to experimentally set the coefficient of $|\psi|^2$ (in Eq. 2) exactly equal to zero, or (2) large ambient electrical noise in the circuit.

To examine the role of noise in the validity of the classical theory, we have measured the frequency-dependent voltage-voltage correlation function $G(\omega)$ as a function of α_0 in the presence of a large amount of externally induced noise.[13] $G(\omega)$ is experimentally measured using a Saicor model 52-B frequency spectrum analyzer. The theory above predicts that $G(\omega)$ should be a Lorentzian peaked at ω_0. As $\alpha_0 \to \alpha_c$ the Lorentzian half-width γ should narrow, corresponding to critical slowing down. For the nonconserved order-parameter field ψ defined above, this slowing down should be proportional to the inverse of the generalized susceptibility, χ^{-1}. Figures 6(a) and 6(b) demonstrate that γ does indeed vary as χ^{-1}, except that the absence of long-range order is clearly visible when the noise is large. There are two specific features of Fig. 6(b) which should be noted: The critical value of $\alpha_0 - \alpha_c$ for which classical theory first fails is increasing with increasing noise, as suggested by Eq. (19), and the mean-field value of the critical feedback, α_c, decreases as the noise increases. This second effect can be understood as a destabilization of the oscillatory state as the noise increases. One can estimate the magnitude of this by arguing that the oscillatory phase will be destabilized when the depth of the free-energy minimum (for $|\psi| \neq 0$) is small compared to T. Formalizing this idea we obtain

$$\alpha_c(T) = \alpha_c^0 - (\tau a T/9\omega_0^2)^{1/2},$$

where α_c is the critical point as a function of noise and α_c^0 is the unrenormalized mean-field value of the critical feedback ($= \frac{1}{3} - 1/A$).

In conclusion, we have demonstrated that the

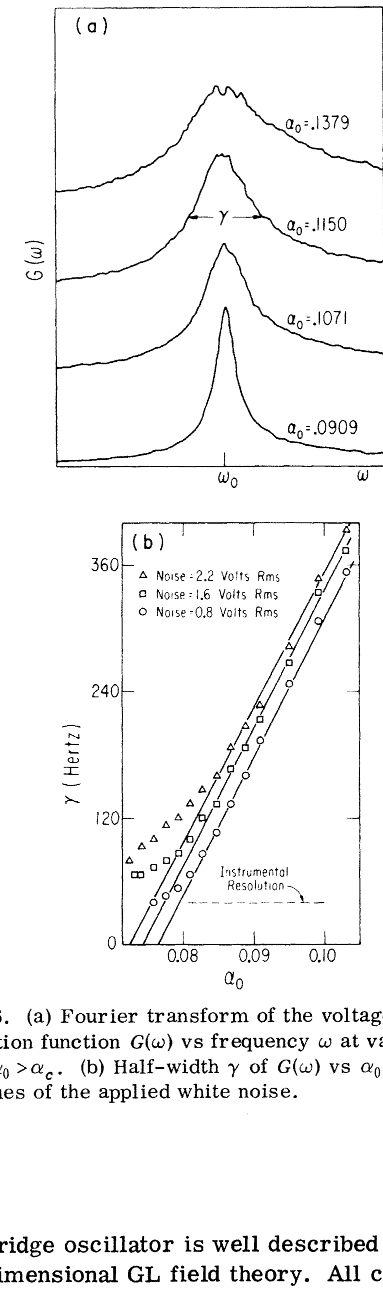

FIG. 6. (a) Fourier transform of the voltage-voltage correlation function $G(\omega)$ vs frequency ω at various values of $\alpha_0 > \alpha_c$. (b) Half-width γ of $G(\omega)$ vs α_0 for various values of the applied white noise.

Wien bridge oscillator is well described by a zero-dimensional GL field theory. All circuit characteristics can therefore be obtained by performing averages within the framework of equilibrium statistical mechanics. In the absence of externally induced noise, a mean-field-like theory is adequate to describe the circuit response except in the immediate vicinity of α_c. We also observe that the oscillator has no long-time phase coherence.

Viewed from a different perspective, these devices become unique tools with which one can simulate mean-field theory at rather complex multicritical points, of which the tricritical point is just one example. In this context, it is possible to use coupled Wien bridge oscillators to simulate either bicritical- or tetracritical-point phenomena and examine not only the mean-field values of the critical indices but also the regions of relative stability of the various ordered phases.

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[1](a) For a review, see H. Haken, Rev. Mod. Phys. 47, 67 (1975); see also R. L. Strantonovich, *Topics in the Theory of Random Noise* (Gordon and Breach, New York) Vols. I and II. (b) R. Graham and H. Haken, Z. Phys. 237, 31 (1970).

[2]V. DeGiorgio and M. O. Scully, Phys. Rev. A 2, 1170 (1970); J. C. Goldstein, M. O. Scully, and P. A. Lee, Phys. Lett. 35A, 317 (1971).

[3]A. Nitzan, P. Ortoleva, J. Deutch, and J. Ross, J. Chem. Phys. 61, 1059 (1974).

[4]V. M. Zaitsev and M. I. Shliomis, Zh. Eksp. Teor. Fiz. 59, 1853 (1970) [Sov. Phys.-JETP 32, 866 (1971)]; J. P. Boon, J. Phys. Chem. Liquids 3, 157 (1972).

[5]See, for example, R. Landauer and J. W. F. Woo, in *Proceedings of the Sixth Conference of the International Union of Pure and Applied Physics, Chicago, 1971*, edited by S. A. Rice, K. F. Freed, and J. C. Light (University of Chicago Press, Chicago, 1972), and references therein.

[6]See, for example, R. Landauer, J. Appl. Plys. 33, 2209 (1962); R. Landauer, Ferroelectrics 2, 47 (1971); T. Kawakubo, S. Kabashima, and K. Nishimura, J. Phys. Soc. Jpn. 34, 1460 (1973); and E. Pytle and H. Thomas, Phys. Rev. 179, 431 (1969).

[7]T. Kawakubo, S. Kabashima, and M. Ogishima, J. Phys. Soc. Jpn. 34, 1149 (1973).

[8]See, for example, L. P. Kadanoff *et al*., Rev. Mod. Phys. 39, 395 (1967); H. E. Stanley, *Introduction to Phase Transitions and Critical Phenomena* (Oxford U.P., Oxford, England, 1971).

[9]See, for example, D. J. Scalapino, M. Sears, and R. A. Ferrell, Phys. Rev. B 6, 3409 (1972).

[10]The analysis in this section of the paper is somewhat similar to the Fokker-Planck analysis of a Wien bridge oscillator done independently by T. Kawakubo and S. Kabashima, J. Phys. Soc. Jpn. 37, 1199 (1974). This work does differ considerably both in detail and point of view.

[11]See, for example, B. I. Halperin, P. C. Hohenberg, and S. K. Ma, Phys. Rev. B 10, 139 (1974).

[12]P. M. Horn, T. Carruthers, and M. T. Long (unpublished).

[13]It is difficult to measure $\langle|\psi|\rangle$ or $\chi = \partial\langle|\psi|\rangle/\partial E$ where the noise is large because of the need for large experimental averaging times.