Quantum effects in the single-mode laser

F. Casagrande and L. A. Lugiato
Istituto di Fisica dell'Università, Milano, Italia
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The single-mode laser model as formulated by Haken and co-workers is considered. Within the approximation which takes into account only the interaction (at all orders in the coupling constant) between the mode and the single atoms, this model leads to a linear time-evolution equation for the Glauber quasiprobability distribution $P(\beta,\beta^*,t)$ of the field mode, containing derivatives of all orders in β and β^*. This equation is solved exactly in the stationary situation. Thus the quantum effects due to the terms with derivatives of order higher than the second one are fully taken into account. It is shown that these terms have a shrinking effect on the stationary distribution above threshold, counteracting the broadening effect due to the second-derivative term. The photon and photoelectron distributions are given. Exact relations are deduced, which allow the calculation of all the moments of the steady-state distribution from the first one. This distribution is compared with the stationary solutions of Risken's equation, of P. Mandel's equation, and with the stationary distribution deduced by Weidlich, Risken, and Haken. The connection with the single-mode laser theory of Scully and Lamb is analyzed in the high-intensity region. The complete Scully-Lamb master equation is translated into the Glauber representation, yielding a linear equation for $P(\beta,\beta^*,t)$ containing derivatives of all orders in β and β^*. Suitably connecting the parameters A, B, and C of the Scully-Lamb equation with the parameters of the model of Haken and co-workers, it is shown that there is perfect agreement between the strong-signal Scully-Lamb steady-state distribution and that of Weidlich, Risken, and Haken for all values of the pump parameter.

I. INTRODUCTION

A great deal of the recent progress concerning the laser system has been obtained via the semiclassical approach.[1] On the other hand, a variety of fully quantum-mechanical treatments also have been elaborated on in order to deal with fluctuations.[2] Among them we consider the laser model as formulated in Ref. 1(b), which describes the dynamics of the coupled system atoms plus radiation field in terms of a suitable master equation. This model has been analyzed in the single-mode case[3,4] under the approximation of considering only the interaction between the mode and the single atoms (i.e., taking the atom-field interaction at all orders but neglecting multiatom correlations). This analysis shows that the Glauber quasiprobability distribution[5] $P(\beta,\beta^*,t)$ of the field mode obeys a suitable linear time-evolution equation containing derivatives of all orders in β and β^*. P. Mandel[3] has simplified this equation by neglecting all terms with derivatives of order higher than second, thus obtaining a Fokker-Planck equation which generalizes Risken's equation[6] to the case of high-intensity lasers. This neglect has been justified by a well-known scaling argument,[7] based on the fact that the number of photons at threshold is large. Essentially, this approximation amounts to assuming a quasiclassical behavior for the laser above threshold: in fact, an equation similar to the Fokker-Planck equation of Ref. 3 has been previously set up by Schmid[8] on the basis of semiclassical arguments.

In this paper we consider the full equation for $P(\beta,\beta^*,t)$, taking into account the quantum effects arising from the terms with derivatives of order higher than second, and solving exactly this equation in the stationary situation. In such a way we can also test the validity of the aforementioned scaling argument. The steady-state distribution that we get agrees very well with the stationary solution of Risken's equation in the threshold region. On the other hand, it turns out to be sharper than the stationary solution of Mandel's Fokker-Planck equation when the laser is sufficiently above threshold. Hence the terms with derivatives of order higher than second introduce a shrinking effect which counteracts the typical broadening effect of the second-derivative term. Such a shrinking effect vanishes at threshold and becomes more important the higher the laser is above threshold. We deduce an exact relation which allows us to calculate all the moments of our steady-state distribution from the first one; the first moment is calculated within a negligible error by approximating the exact steady-state solution by a distribution which is Gaussian in intensity.

In Sec. II we recall the single-mode laser model and the time-evolution equation for $P(\beta,\beta^*,t)$ deduced in Ref. 4. In Sec. III we find the steady-state solution and in Sec. IV we list the exact results which can be deduced from it, including the photon distribution and the photoelectron distribution. The Gaussian approximation of the steady-state distribution is discussed in Sec. V. Sections VI–

VIII are devoted to a comparison of this distribution with the stationary solution of Risken's equation (Sec. VI), of Mandel's Fokker-Planck equation (Sec. VII), and with the stationary distribution deduced by Weidlich, Risken, and Haken (WRH) (Sec. VIII).[9]

Another point that we analyze in this paper is the connection with the quantum-mechanical single-mode laser theory of Scully and Lamb[1(a),10] in the high-intensity region. In fact it is well known that Risken's equation is essentially equivalent to the Scully-Lamb master equation in the so-called cubic approximation (i.e., fourth order in the coupling constant). On the basis of the closed equation for $P(\beta, \beta^*, t)$, deduced in Refs. 3 and 4 from the model of Ref. 1(b), we can now study the connection of this model with the Scully-Lamb strong-signal equation. Using a quite straightforward procedure we translate in Sec. IX the complete Scully-Lamb master equation (formulated in the photon-number representation) into the Glauber representation, obtaining a linear equation for $P(\beta, \beta^*, t)$ containing derivatives of all orders in β and β^*. From this equation the strong-signal Scully-Lamb steady-state distribution in the Glauber representation is immediately obtained. Again we find that by neglecting the terms with derivatives of order higher than second, one obtains a broadening of the stationary distribution above threshold.

Finally, in Sec. X we connect the parameters A, B, and C of the Scully-Lamb equation with the parameters of the model of Haken and co-workers, obtaining relations which generalize to the high-intensity region the relations given by Arecchi and DeGiorgio.[11] Using this connection, we show that there is perfect agreement between the strong-signal Scully-Lamb steady-state distribution and that deduced by WRH.

II. TIME-EVOLUTION EQUATION FOR $P(\beta, \beta^*, t)$

Let us first recall the basic features of the single-mode laser model as formulated by Haken.[1(b)] For our purposes we can limit our considerations to a "tuned" laser, i.e., a single-field mode of frequency ω interacting with N two-level atoms whose transition frequencies are exactly ω. The system field plus atoms $(F+A)$ is described in the interaction picture. The time evolution of the statistical operator $W(t)$ is then given by the following master equation:

$$\frac{dW(t)}{dt} = -iLW(t), \qquad L = L_{AF} + i\Lambda_A + i\Lambda_F, \quad \hbar = 1, \tag{2.1}$$

where the Liouville operator L takes into account the field damping and the atomic pump and spontaneous decay processes via the operators Λ_F and Λ_A, which have the form

$$\Lambda_F X = k([bX, b^\dagger] + [b, Xb^\dagger]),$$

$$\Lambda_A X = \sum_{i=1}^{N} \Lambda_{Ai} X, \tag{2.2}$$

$$\Lambda_{Ai} = \tfrac{1}{2}\gamma_\downarrow([r_i^-, X r_i^+] + [r_i^- X, r_i^+]) + \tfrac{1}{2}\gamma_\uparrow([r_i^+, X r_i^-] + [r_i^+ X, r_i^-]) - \tfrac{1}{2}\eta([r_i^- r_i^+, X r_i^+ r_i^-] + [r_i^+ r_i^- X, r_i^- r_i^+]);$$

furthermore, for any X

$$L_{AF} X = [H_{AF}, X],$$

where H_{AF} is the interaction Hamiltonian, obtained in the dipole and rotating-wave approximations, with a coupling constant g independent of the atoms (running wave);

$$H_{AF} = g \sum_{i=1}^{N} (b r_i^+ + b^\dagger r_i^-). \tag{2.3}$$

In Eqs. (2.2) and (2.3), b is the annihilation field mode operator, obeying boson commutation rules; $r_i^\pm$ are the spin-flip operators for the ith atom, satisfying fermion anticommutation rules; k is the damping constant of the field mode; $\gamma_\uparrow$, $\gamma_\downarrow$, and η are the atomic transition rates connected to the transverse $(\gamma_\perp)$ and longitudinal $(\gamma_\parallel)$ relaxation constants and to the unsaturated inversion (σ_0) as follows:

$$\gamma_\perp = \tfrac{1}{2}(\gamma_\uparrow + \gamma_\downarrow + \eta), \quad \gamma_\parallel = \gamma_\uparrow + \gamma_\downarrow, \qquad \sigma_0 = (\gamma_\uparrow - \gamma_\downarrow)/(\gamma_\uparrow + \gamma_\downarrow). \tag{2.4}$$

In the laser the experimentally detected system is the radiation field. In Ref. 4 the problem considered was to deduce from Eq. (2.1) a closed time-evolution equation for the reduced statistical operator $\rho(t)$ of the field mode alone:

$$\rho(t) = \mathrm{Tr}_A W(t) \tag{2.5}$$

(Tr_A stands for the partial trace over the atomic Hilbert space) or equivalently for the Glauber quasiprobability distribution function[5] $P(\beta, \beta^*, t)$ associated with $\rho(t)$:

$$\rho(t) = \int d_2\beta\, P(\beta, \beta^*, t)\, |\beta\rangle\langle\beta| . \tag{2.6}$$

The analysis of Ref. 4, which follows the general method of treating open systems developed by one of us,[12] leads to the following equation for $P(\beta, \beta^*, t)$:

$$\frac{\partial P(\beta,\beta^*,t)}{\partial t}=\Lambda P(\beta,\beta^*,t)\,,$$

$$\Lambda=\left(k+\frac{g^2N}{\gamma_\perp}\right)\left(\frac{\partial}{\partial\beta}\beta+\frac{\partial}{\partial\beta^*}\beta^*\right)-\frac{2g^2N}{\gamma_\perp}\left(\frac{\partial}{\partial\beta}\beta+\frac{\partial}{\partial\beta^*}\beta^*-\frac{\partial^2}{\partial\beta^*\partial\beta}\right)$$
$$\times\left[1-\frac{g^2}{\gamma_\perp\gamma_\parallel}\left(-4|\beta|^2+\beta\frac{\partial}{\partial\beta}+\beta^*\frac{\partial}{\partial\beta^*}\right)\right]^{-1}\left(\frac{1+\sigma_0}{2}+\frac{2g^2}{\gamma_\perp\gamma_\parallel}|\beta|^2\right). \tag{2.7}$$

Equation (2.7) has been obtained through the following approximations:

(a) Consideration of only the interaction of the field mode with the single atoms, neglecting all the interaction processes in which the mode interacts with two or more atoms successively. This approximation has the consequence that the operator Λ in Eq. (2.7) is simply proportional to the number of atoms N.

(b) Neglect of Λ_F with respect to Λ_A, on the basis of the assumption

$$k\ll\gamma_\perp,\gamma_\parallel\,. \tag{2.8}$$

By this approximation, the losses of the field enter into the final equation (2.7) only via the linear term $k[(\partial/\partial\beta)\beta+(\partial/\partial\beta^*)\beta^*]$.

(c) The Markoff (or adiabatic) approximation.[13]

Approximations (a) and (b) are automatically contained in the usual semiclassical equations. Furthermore, we stress that approximations (a)–(c) are contained also in the Scully-Lamb theory.[10]

Mandel[3] has previously obtained an equation more complicated but essentially equivalent to (2.7) through a much more cumbersome procedure using the Zwanzig projection technique.[14] In order to get a Fokker-Planck equation (i.e., an equation with first- and second-order derivatives in β and β^*) for $P(\beta,\beta^*,t)$, Mandel simplified his infinite-order equation by a well-known scaling argument[7] which, when applied to Eq. (2.7), goes as follows: Let us introduce the normalized field variable

$$\overline{\beta}=\epsilon^{1/4}\beta\,,\quad \epsilon=16k/N\gamma_\parallel\,, \tag{2.9}$$

where $\epsilon^{1/2}$ is essentially the inverse of the mean photon number at threshold and is therefore a small number ($\sim10^{-4}$). Then one evaluates

$$|\beta|^2=O(\epsilon^{-1/2})\,,\quad \beta\frac{\partial}{\partial\beta}+\beta^*\frac{\partial}{\partial\beta^*}=O(\epsilon^0)\,. \tag{2.10}$$

On the basis of Eq. (2.10), one neglects the term $\beta(\partial/\partial\beta)+\beta^*(\partial/\partial\beta^*)$ with respect to $4|\beta|^2$ in the denominator of Eq. (2.7), thus obtaining Mandel's Fokker-Planck equation[15]:

$$\frac{\partial P(\beta,\beta^*,t)}{\partial t}=\left[\left(\frac{\partial}{\partial\beta}\beta+\frac{\partial}{\partial\beta^*}\beta^*\right)\left(k-\frac{g^2N\sigma_0}{\gamma_\perp}+\frac{4g^4N\sigma_0}{\gamma_\perp^2\gamma_\parallel}\frac{|\beta|^2}{1+(4g^2/\gamma_\perp\gamma_\parallel)|\beta|^2}\right)\right.$$
$$\left.+\frac{\partial^2}{\partial\beta^*\partial\beta}\left(\frac{g^2N(1+\sigma_0)}{\gamma_\perp}-\frac{4g^4N\sigma_0}{\gamma_\perp^2\gamma_\parallel}\frac{|\beta|^2}{1+(4g^2/\gamma_\perp\gamma_\parallel)|\beta|^2}\right)\right]P(\beta,\beta^*,t)\,. \tag{2.11}$$

Equation (2.11) is directly linked to the semiclassical equations. In fact, neglecting the diffusion term, such an equation can be solved by the method of characteristics; it turns out that the resulting characteristic lines coincide with the semiclassical trajectories. Thus the term

$$\frac{g^2}{\gamma_\perp\gamma_\parallel}\left(\beta\frac{\partial}{\partial\beta}+\beta^*\frac{\partial}{\partial\beta^*}\right) \tag{2.12}$$

in the denominator of Eq. (2.7) is a purely quantum-mechanical effect. Risken's equation[6] is immediately obtained from Eq. (2.11) by (i) neglecting $(4g^2/\gamma_\perp\gamma_\parallel)|\beta|^2$ with respect to 1 in the denominator of the drift term, and (ii) keeping only the term $g^2N(1+\sigma_0)/\gamma_\perp$ in the diffusion term. One thus obtains

$$\frac{\partial P(\beta,\beta^*,t)}{\partial t}=\left[\left(\frac{\partial}{\partial\beta}\beta+\frac{\partial}{\partial\beta^*}\beta^*\right)\left(k-\frac{g^2N\sigma_0}{\gamma_\perp}+\frac{4g^4N\sigma_0}{\gamma_\perp^2\gamma_\parallel}|\beta|^2\right)+\frac{g^2N(1+\sigma_0)}{\gamma_\perp}\frac{\partial^2}{\partial\beta\partial\beta^*}\right]P(\beta,\beta^*,t)\,. \tag{2.13}$$

Approximation (i) holds only for $\sigma_0-\sigma_{\text{thr}}\ll\sigma_{\text{thr}}$, where σ_{thr} is the threshold inversion per atom,

$$\sigma_{\text{thr}}=k\gamma_\perp/Ng^2\,. \tag{2.14}$$

III. STEADY-STATE DISTRIBUTION

The advantage of the method used in Ref. 4 is that it allows one to find the exact stationary solution of Eq. (2.7) without resorting to any scaling argument to simplify the equation. Thus we can fully take into account the effect of the quantum term (2.12) in Eq. (2.7). Equation (2.7) has been obtained in Ref. 4 from the two coupled equations

$$\frac{\partial P(\beta,\beta^*,t)}{\partial t} = \left(k+\frac{g^2N}{\gamma_\perp}\right)\left(\frac{\partial}{\partial\beta}\beta+\frac{\partial}{\partial\beta^*}\beta^*\right)P(\beta,\beta^*,t) - \frac{2g^2N}{\gamma_\perp}\left(\frac{\partial}{\partial\beta}\beta+\frac{\partial}{\partial\beta^*}\beta^*-\frac{\partial^2}{\partial\beta\partial\beta^*}\right)W_1(\beta,\beta^*,t), \quad (3.1)$$

$$\left[1-\frac{g^2}{\gamma_\perp\gamma_\parallel}\left(\frac{\partial}{\partial\beta}\beta+\frac{\partial}{\partial\beta^*}\beta^*-2(1+2|\beta|^2)\right)\right]W_1(\beta,\beta^*,t) = \left(\frac{1+\sigma_0}{2}+\frac{2g^2}{\gamma_\perp\gamma_\parallel}|\beta|^2\right)P(\beta,\beta^*,t), \quad (3.2)$$

by simply eliminating $W_1(\beta,\beta^*,t)$ between Eqs. (3.1) and (3.2). $W_1(\beta,\beta^*,t)$ is an atom-field correlation function which is directly linked to the population of the upper level per atom; in fact the mean value of the quantity $(b^\dagger)^l b^n r^+ r^-$ (where r^+r^- refers to any one of the atoms) is given by[4]

$$\langle (b^\dagger)^l b^n r^+ r^-\rangle = \int d_2\beta(\beta^*)^l\beta^n W_1(\beta,\beta^*,t). \quad (3.3)$$

As shown in Ref. 4, Eqs. (3.1) and (3.2) can be considered as the operator analog in the Schrödinger picture of the semiclassical equations.

In the stationary situation [i.e., for $dP(\beta,\beta^*,t)/dt=0$ in Eq. (3.1)] the steady-state distributions $P^{(\mathrm{st})}$ and $W_1^{(\mathrm{st})}$ are functions of the modulus of β only. Then introducing the polar coordinates

$$\beta = re^{i\varphi}, \quad (3.4)$$

one obtains from Eqs. (3.1) and (3.2) the following coupled steady-state equations:

$$\left(k+\frac{g^2N}{\gamma_\perp}\right)rP^{(\mathrm{st})}(r)-\frac{2g^2N}{\gamma_\perp}rW_1^{(\mathrm{st})}(r)+\frac{1}{2}\frac{g^2N}{\gamma_\perp}\frac{dW_1^{(\mathrm{st})}(r)}{dr}=0, \quad (3.5)$$

$$W_1^{(\mathrm{st})}(r)=\frac{1+\sigma_0}{2}P^{(\mathrm{st})}(r)+\frac{g^2}{\gamma_\perp\gamma_\parallel}\left(2r^2P^{(\mathrm{st})}(r)-4r^2W_1^{(\mathrm{st})}(r)+r\frac{dW_1^{(\mathrm{st})}(r)}{dr}\right). \quad (3.6)$$

We note that the term $r\,dW_1^{(\mathrm{st})}(r)/dr$ in Eq. (3.6) comes from the quantum-mechanical term (2.12) in Eqs. (2.7) and (3.2).

If $dW_1^{(\mathrm{st})}/dr$ is derived from Eq. (3.5) and substituted into Eq. (3.6), the semiclassical terms $2r^2[P^{(\mathrm{st})}(r)-2W_1^{(\mathrm{st})}(r)]$ are exactly canceled, so that Eq. (3.6) take the simple form

$$W_1^{(\mathrm{st})}(r) = [\tfrac{1}{2}(1+\sigma_0)-(2k/N\gamma_\parallel)r^2]P^{(\mathrm{st})}(r). \quad (3.7)$$

If we now introduce the intensity variable

$$z=r^2 \quad (3.8)$$

and set

$$Z=N\gamma_\parallel/4k, \quad \hat{z}=Z(\sigma_0-\sigma_{\mathrm{thr}}), \quad \bar{z}=Z(1+\sigma_0); \quad (3.9)$$

we easily get from Eqs. (3.5) and (3.7) the equation for $P^{(\mathrm{st})}(z)$:

$$(\bar{z}-z)\frac{dP^{(\mathrm{st})}(z)}{dz}=2(\hat{z}+\tfrac{1}{2}-z)P^{(\mathrm{st})}(z). \quad (3.10)$$

Equation (3.10) has the following normalizable solution:

$$P^{(\mathrm{st})}(z)=\begin{cases}\mathfrak{N}e^{2z}(\bar{z}-z)^{2(\bar{z}-\hat{z})-1}, & \text{for } 0\le z\le\bar{z},\\ 0, & \text{for } z\ge\bar{z},\end{cases} \quad (3.11)$$

where $\mathfrak{N}$ is a suitable normalization constant, i.e., one has

$$\int_0^{\bar{z}} dz\,P^{(\mathrm{st})}(z)=1$$

for

$$\mathfrak{N}=[e^{2\bar{z}}2^{-2(\bar{z}-\hat{z})}\gamma(2(\bar{z}-\hat{z});2\bar{z})]^{-1}, \quad (3.12)$$

$\gamma(a;x)$ being the truncated γ function defined as

$$\gamma(a;x)=\int_0^x e^{-t}t^{a-1}\,dt.$$

As one sees, solution (3.11) vanishes for $z\ge\bar{z}$ and is therefore a nonanalytic function of z. We show in the Appendix that taking the field heat bath with finite temperature β^{-1} [instead of zero temperature as in Eq. (2.2)] one actually gets an *analytic* stationary solution, which for $\beta\to\infty$ approaches the nonanalytic function (3.11).

Distribution (3.11) has one maximum at

$$z=\hat{z}+\tfrac{1}{2} \quad (3.13)$$

if

$$\sigma_0\ge\sigma_{\mathrm{thr}}-1/2Z. \quad (3.14)$$

Since $(2Z)^{-1}\ll\sigma_{\mathrm{thr}}$ [see Eqs. (3.9) and (2.14)], the threshold condition (3.14) essentially coincides with the usual one, and the position of the maximum (3.13) coincides with the semiclassical value of the intensity above threshold $z=\hat{z}$.

IV. EXACT RESULTS

A. Photon distribution

Let us consider the photon distribution function $p(n)$, given by

$$p(n)=\int_0^\infty dz\, e^{-z}\frac{z^n}{n!}P^{(st)}(z)\,. \tag{4.1}$$

One has from Eq. (3.11)

$$p(n)=\frac{\Gamma(2(\bar{z}-\hat{z})+1)}{\Gamma(2(\bar{z}-\hat{z})+n+1)}\bar{z}^n e^{-\bar{z}} \times\frac{\phi(2(\bar{z}-\hat{z}),2(\bar{z}-\hat{z})+n+1;-\bar{z})}{\phi(2(\bar{z}-\hat{z}),2(\bar{z}-\hat{z})+1;-2\bar{z})}\,, \tag{4.2}$$

where $\phi(a,c;x)$ is the confluent hypergeometric function. The ϕ in the denominator comes from the normalization constant (3.12), recalling that $\gamma(a;x)=a^{-1}x^a\phi(a,a+1;-x)$.

B. Photoelectron distribution

Let us assume that the time interval T during which the photoelectrons emitted as a consequence of the arrival of laser light are counted is small with respect to the relaxation times of the laser. Then the probability $p(m,T)$ that m photoelectrons will be released in a time interval T is given by[7,16]

$$p(m,T)=\int_0^\infty dz\frac{(\alpha Tz)^m}{m!}e^{-\alpha Tz}P^{(st)}(z)\,, \tag{4.3}$$

where α takes into account the quantum efficiency of the detector, etc. Using Eq. (3.11), we obtain

$$p(m,T)=(\alpha T)^m\frac{\Gamma(2(\bar{z}-\hat{z})+1)}{\Gamma(2(\bar{z}-\hat{z})+m+1)}\bar{z}^m e^{-\alpha T\bar{z}} \times\frac{\phi(2(\bar{z}-\hat{z}),2(\bar{z}-\hat{z})+m+1;(\alpha T-2)\bar{z})}{\phi(2(\bar{z}-\hat{z}),2(\bar{z}-\hat{z})+1;-2\bar{z})}\,. \tag{4.4}$$

C. Moments of the distribution (3.11)

Let $\langle z^n\rangle$ be the nth moment of the distribution (3.11):

$$\langle z^n\rangle=\int_0^\infty dz\, z^n P^{(st)}(z)\,. \tag{4.5}$$

We deduce now some exact relations concerning these moments. Let us integrate both sides of Eq. (3.10) form 0 to $\bar{z}$; performing an integration by parts we obtain

$$\langle z\rangle=\hat{z}+\tfrac{1}{2}\bar{z}P^{(st)}(0)=\hat{z}+\tfrac{1}{2}\mathfrak{N}\bar{z}^{2(\bar{z}-\hat{z})}, \tag{4.6}$$

where $\mathfrak{N}$ is given by Eq. (3.12). Furthermore, if we multiply both sides of Eq. (3.10) by z^{n-1} and integrate from 0 to $\bar{z}$, we get

$$\langle z^n\rangle=[\hat{z}-\tfrac{1}{2}(n-1)]\langle z^{n-1}\rangle+\tfrac{1}{2}(n-1)\bar{z}\langle z^{n-2}\rangle\,,\quad n>1\,; \tag{4.7}$$

in particular, for $n=2$

$$\langle z^2\rangle=(\hat{z}-\tfrac{1}{2})\langle z\rangle+\tfrac{1}{2}\bar{z}\,. \tag{4.8}$$

Relations (4.7) and (4.8) allow one to calculate all the moments from the first one. The moments $\langle n^k\rangle$ of the photon distribution (4.2) can be computed by taking into account that $\langle z^k\rangle$ coincides with the kth factorial moment of the photon distribution. For example, one has

$$\langle n\rangle=\langle z\rangle\,,\quad \langle n^2\rangle=\langle z^2\rangle+\langle z\rangle\,,\quad \text{etc.} \tag{4.9}$$

D. Stationary inversion per atom

From Eq. (3.3) we find that the mean inversion per atom $\langle r_3\rangle(t)$ at time t is given by

$$\langle r_3\rangle(t)=\langle 2r^+r^- -1\rangle(t)=2\int d_2\beta\, W_1(\beta,\beta^*,t)-1\,. \tag{4.10}$$

Using Eqs. (4.10) and (3.7)–(3.9) we obtain the stationary inversion per atom,

$$\langle r_3\rangle_{st}=2\int dz\, W_1^{(st)}(z)-1=\sigma_0-\langle z\rangle/Z\,. \tag{4.11}$$

Equation (4.11) gives an exact relation between the stationary values of inversion and of photon numbers. Below threshold $\langle z\rangle\ll Z$, so that $\langle r_3\rangle_{st}\sim\sigma_0$; sufficiently high above threshold, as we shall see in Sec. V, $\langle z\rangle\sim\hat{z}$; hence by Eq. (3.9) $\langle r_3\rangle_{st}\sim\sigma_{thr}$, in agreement with the semiclassical result.

V. GAUSSIAN APPROXIMATION

Let us approximate $P(z)$ by a Gaussian peaked at $z=\hat{z}$:

$$P^{(st)}(z)\simeq\bar{\mathfrak{N}}e^{-(z-\hat{z})^2/q^2}\,, \tag{5.1}$$

where $\bar{\mathfrak{N}}$ is the normalization constant

$$\bar{\mathfrak{N}}=\{\tfrac{1}{2}\sqrt{\pi}\,q\,[1+\mathrm{sgn}(\hat{z})\,\mathrm{erf}(|\hat{z}|/q)]\}^{-1}\,. \tag{5.2}$$

Distributions of type (5.1) have been used extensively in the laser literature.[6,9,16-18]

From Eq. (3.11) we easily have

$$q^2=\bar{z}-\hat{z}=Z(1+\sigma_{thr})\,. \tag{5.3}$$

Thus it turns out that the width of the Gaussian approximating distribution (3.11) is pump independent. As we have seen in Sec. IV, the second and higher moments of the photon distribution can be calculated from the first one using Eq. (4.7). The first moment can be easily calculated using the Gaussian approximation (5.1):

$$\langle n\rangle = \hat{z} + \frac{q}{\sqrt{\pi}} \frac{e^{-\hat{z}^2/q^2}}{1+\operatorname{sgn}(\hat{z})\operatorname{erf}(|\hat{z}|/q)} . \tag{5.4}$$

One can verify that the Gaussian is indeed a good approximation for distribution (3.11). For example, if one substitutes Eq. (5.4) into the exact relation (4.8) to calculate $\langle z^2\rangle$, one obtains

$$\langle z^2\rangle = \hat{z}^2 + \tfrac{1}{2}(\bar{z}-\hat{z}) + \frac{1}{\sqrt{\pi}}(\hat{z}-\tfrac{1}{2})q \frac{e^{-\hat{z}^2/q^2}}{1+\operatorname{sgn}(\hat{z})\operatorname{erf}(|\hat{z}|/q)} . \tag{5.5}$$

On the other hand, using the Gaussian approximation (5.1), directly one obtains

$$\langle z\rangle^2 = \hat{z}^2 + \tfrac{1}{2}(\bar{z}-\hat{z}) + \frac{1}{\sqrt{\pi}}\hat{z}q \frac{e^{-\hat{z}^2/q^2}}{1+\operatorname{sgn}(\hat{z})\operatorname{erf}(|\hat{z}|/q)} ; \tag{5.5$'$}$$

clearly the difference between Eqs. (5.5) and (5.5′) is immaterial. When the laser is sufficiently above threshold, i.e., for $(\hat{z}^2/q^2) \gg 1$, Eqs. (5.4) and (5.5) show that

$$\langle n\rangle - \hat{z} \ll 1, \quad \langle z^2\rangle \simeq \hat{z}^2 + \tfrac{1}{2}(\bar{z}-\hat{z}) . \tag{5.6}$$

Hence from Eqs. (4.9) and (5.6) the mean-square deviation Δ^2 of the photon distribution is given by

$$\Delta^2 = \langle n^2\rangle - \langle n\rangle^2 \simeq \tfrac{1}{2}(\bar{z} + \langle n\rangle) = \tfrac{1}{2}[Z(1+\sigma_0) + \langle n\rangle] . \tag{5.7}$$

Taking into account that by Eqs. (3.9) and (5.6) $Z \simeq \langle n\rangle/(\sigma_0 - \sigma_{\text{thr}})$, we conclude that

$$\Delta^2 \simeq \tfrac{1}{2}[(1+\sigma_0)/(\sigma_0-\sigma_{\text{thr}}) + 1]\langle n\rangle . \tag{5.8}$$

We see that high above threshold Δ^2 becomes of the order of magnitude of $\langle n\rangle$, as expected. However, the distribution does not become a perfectly Poisson one even for very high pumping; in fact, for $\sigma_0 = 1$, one has from Eq. (5.8) $\Delta^2/\langle n\rangle = \frac{3}{2}$.

VI. COMPARISON WITH THE RISKEN OR "CUBIC" SCULLY-LAMB STATIONARY SOLUTION

The stationary distribution of Risken's equation (2.13) is given exactly by a Gaussian:

$$P_R(z) = \mathfrak{N}_R e^{-(z-z_{\max})^2/q_R^2} , \tag{6.1}$$

with

$$z_{\max} = (\gamma_\perp\gamma_\parallel/4g^2\sigma_0)(\sigma_0-\sigma_{\text{thr}}), \quad q_R^2 = Z(1+\sigma_0)(\sigma_{\text{thr}}/\sigma_0) . \tag{6.2}$$

From Eqs. (2.14), (3.9), and (5.3) we see that in the region of validity of Risken's equation, i.e., $\sigma_0 - \sigma_{\text{thr}} \ll \sigma_{\text{thr}}$, the two distributions (5.1) and (6.1) coincide. On the other hand, as one might expect, the two distributions become different beyond this region. In fact, (i) $\hat{z}$ depends linearly on σ_0, whereas $z_{\max}$ does not; for $\sigma_0 \gg \sigma_{\text{thr}}$, $z_{\max} \ll \hat{z}$, and (ii) distribution (6.1) shrinks with increasing σ_0, whereas distribution (5.1) does not.

In the literature [see, e.g., Ref. 1(b)] one often finds Risken's equation written in a form different from (2.13); namely, the σ_0 in the nonlinear drift term and in the diffusion term is replaced by σ_{thr}. This is quite correct in the region $\sigma_0 - \sigma_{\text{thr}} \ll \sigma_{\text{thr}}$, but of course it changes the situation when one extrapolates distribution (6.1) beyond this region, as we have done before. Curiously enough, performing the replacement one obtains $z_{\max} = \hat{z}$ and $q_R^2 = q^2$, so that distributions (5.1) and (6.1) become identical for all values of σ_0. However, this coincidence is completely casual. In fact, Eq. (5.1) is the Gaussian approximation of distribution (3.11), which incorporates the saturation effects arising in high-intensity lasers; these effects are completely neglected in Risken's equation.

VII. COMPARISON WITH MANDEL'S STATIONARY SOLUTION

The steady-state solution of Mandel's equation (2.11) is given by[3]

$$P_M(z) = \mathfrak{N}_M\left(1 + \frac{4g^2}{\gamma_\perp\gamma_\parallel}z\right)e^{-2z\sigma_{\text{thr}}}[z + \bar{z}\sigma_{\text{thr}}]^{2\sigma_{\text{thr}}(\hat{z}+\bar{z}\sigma_{\text{thr}})-1}$$
$$\equiv \mathfrak{N}_M' \frac{1+(4g^2/\gamma_\perp\gamma_\parallel)z}{1+\sigma_0+(4g^2/\gamma_\perp\gamma_\parallel)z} \times e^{-2z\sigma_{\text{thr}}}[z+\bar{z}\sigma_{\text{thr}}]^{2\sigma_{\text{thr}}(\hat{z}+\bar{z}\sigma_{\text{thr}})} , \tag{7.1}$$

where $\mathfrak{N}_M$ and $\mathfrak{N}_M'$ are suitable normalization constants. The last step in Eq. (7.1) is explained as follows: The factor in brackets has one maximum at $z = \hat{z}$; this maximum is very slightly shifted by the other factor

$$\frac{1+(4g^2/\gamma_\perp\gamma_\parallel)z}{1+\sigma_0+(4g^2/\gamma_\perp\gamma_\parallel)z} .$$

Neglecting the latter, slowly varying factor, the Gaussian approximation of distribution (7.1) is given by

$$P_M(z) \simeq \bar{\mathfrak{N}}_M e^{-(z-\hat{z})^2/q_M^2} , \quad q_M^2 = Z(1+\sigma_{\text{thr}})(\sigma_0/\sigma_{\text{thr}}) . \tag{7.2}$$

One sees that (i) for $\sigma_0 \sim \sigma_{\text{thr}}$, distributions (5.1) and (7.2) coincide, and (ii) for $\sigma_0 \gg \sigma_{\text{thr}}$, distribution (7.2) gets broadened, until for $\sigma_0 = 1$ one has

$$q_M^2 = Z(1+\sigma_{\rm thr})/\sigma_{\rm thr} \gg Z(1+\sigma_{\rm thr}) .$$

Therefore the quantum term (2.12) has mainly the effect of eliminating the broadening that appears in distribution (7.2). Hence the scaling argument (2.10) seems more valid in the threshold region than in the region far above threshold.

VIII. COMPARISON WITH THE WRH STATIONARY SOLUTION

WRH[9] have treated the single-mode laser model (2.1) without using approximations (a) and (b) described in Sec. II, but introducing a suitable factorization ansatz which assumes that the atoms are completely uncorrelated. This ansatz translates the original master equation (2.1) into a set of four nonlinear equations. Within this approach, it is practically impossible to obtain a closed equation for the field distribution such as (2.7). However, as WRH have shown,[9] in the stationary situation these equations become linear, so that one can discuss them fully. In fact these authors have obtained, via suitable approximations, the following steady-state solution:

$$P_{\rm WRH}(z) = \begin{cases} \mathfrak{N}_{\rm WRH} e^{z} (\frac{1}{2}\bar{z} - z)^{\bar{z}/2 - \hat{z}}, & z \le \frac{1}{2}\bar{z} , \\ 0, & z \ge \frac{1}{2}\bar{z} , \end{cases} \quad (8.1)$$

where $\mathfrak{N}_{\rm WRH}$ is the normalization constant.

The Gaussian approximation of distribution (8.1) is given by

$$P_{\rm WRH}(z) \simeq \bar{\mathfrak{N}}_{\rm WRH} e^{-(z-\hat{z})^2/q^2_{\rm WRH}} ,$$
$$q^2_{\rm WRH} = Z(1-\sigma_0 + 2\sigma_{\rm thr}) . \quad (8.2)$$

One verifies that (i) distributions (5.1) and (8.2) practically coincide for all values $\sigma_0 \ll 1$, and (ii) for $\sigma_0 \sim 1$, distribution (8.2) gets sharper until for $\sigma_0 = 1$ one has $q^2_{\rm WRH} = 2Z\sigma_{\rm thr} \ll Z(1+\sigma_{\rm thr})$.

Curiously enough, for $\sigma_0 = 1$ the value of $q^2_{\rm WRH}$ coincides with the width q_R^2 of the extrapolated Risken distribution [cf. (6.2) for $\sigma_0 = 1$]. The shrinking of distribution (8.2) for extremely strong pumping (i.e., $\sigma_0 \sim 1$) leads within an excellent approximation to a Poisson photon distribution.[9]

In conclusion, distributions (3.11) and (8.1) differ appreciably only in the extremely high pumping region $\sigma_0 \sim 1$.

IX. GLAUBER REPRESENTATION TREATMENT OF THE SCULLY-LAMB STRONG-SIGNAL MASTER EQUATION

Let us consider the Scully-Lamb master equation[1(a)]

$$\frac{d\rho_{nn'}}{dt} = -\frac{A\mathfrak{N}'_{nn'}}{1+(B/A)\mathfrak{N}_{nn'}}\rho_{nn'} + \frac{(nn')^{1/2}A}{1+(B/A)\mathfrak{N}_{n-1,n'-1}}\rho_{n-1,n'-1} + \tfrac{1}{2}C\{2[(n+1)(n'+1)]^{1/2}\rho_{n+1,n'+1} - (n+n')\rho_{n,n'}\} , \quad (9.1)$$

where $\rho_{nn'}$ are the matrix elements of the statistical operator of the field in the photon number representation, A, B, and C are the so-called gain, saturation, and loss parameters, respectively, and

$$\mathfrak{N}'_{nn'} = \tfrac{1}{2}(n+1+n'+1) + \tfrac{1}{8}(B/A)(n-n')^2 ,$$
$$\mathfrak{N}_{nn'} = \tfrac{1}{2}(n+1+n'+1) + \tfrac{1}{16}(B/A)(n-n')^2 . \quad (9.2)$$

Introducing the auxiliary quantities

$$\overline{W}_{nn'}(t) = [1+(B/A)\mathfrak{N}_{nn'}]^{-1}\rho_{nn'}(t) , \quad (9.3)$$

one sees that Eq. (9.1) is equivalent to the following coupled equations:

$$\frac{d\rho_{nn'}}{dt} = -A[\tfrac{1}{2}(n+1+n'+1)\overline{W}_{nn'} + \tfrac{1}{8}(n-n')^2(B/A)\overline{W}_{nn'} - (nn')^{1/2}\overline{W}_{n-1,n'-1}] + \tfrac{1}{2}C\{2[(n+1)(n'+1)]^{1/2}\rho_{n+1,n'+1} - (n+n')\rho_{nn'}\} , \quad (9.4)$$

$$[1+\tfrac{1}{2}(n+1+n'+1)(B/A) + \tfrac{1}{16}(n-n')^2(B^2/A^2)]\overline{W}_{nn'} = \rho_{nn'} . \quad (9.5)$$

Using the relationship

$$\rho_{nn'}(t) = \int d_2\beta\, e^{-|\beta|^2} \frac{\beta^n}{\sqrt{n!}} \frac{\beta^{*n'}}{\sqrt{n'!}} P(\beta,\beta^*,t) , \quad (9.6)$$

one easily translates Eqs. (9.4) and (9.5) into the Glauber representation, obtaining[19]

$$\frac{\partial P(\beta,\beta^*,t)}{\partial t} = -\frac{A}{2}\left[\frac{\partial}{\partial\beta}\beta + \frac{\partial}{\partial\beta^*}\beta^* - 2\frac{\partial^2}{\partial\beta^*\partial\beta} + \frac{1}{4}\frac{B}{A}\left(\frac{\partial}{\partial\beta}\beta - \frac{\partial}{\partial\beta^*}\beta^*\right)^2\right]\overline{W}(\beta,\beta^*,t) + \frac{C}{2}\left(\frac{\partial}{\partial\beta}\beta + \frac{\partial}{\partial\beta^*}\beta^*\right)P(\beta,\beta^*,t) , \quad (9.7)$$

$$\left[1 - \frac{B}{2A}\left(\frac{\partial}{\partial\beta}\beta + \frac{\partial}{\partial\beta^*}\beta^* - 2(1+|\beta|^2)\right) + \frac{1}{16}\frac{B^2}{A^2}\left(\frac{\partial}{\partial\beta}\beta - \frac{\partial}{\partial\beta^*}\beta^*\right)^2\right]\overline{W}(\beta,\beta^*,t) = P(\beta,\beta^*,t) . \quad (9.8)$$

Eliminating $\overline{W}(\beta, \beta^*, t)$ from Eqs. (9.7) and (9.8) one obtains the Scully-Lamb equation in the Glauber representation:

$$\frac{\partial P(\beta,\beta^*,t)}{\partial t} = -\frac{A}{2}\left[\frac{\partial}{\partial\beta}\beta + \frac{\partial}{\partial\beta^*}\beta^* - 2\frac{\partial^2}{\partial\beta^*\partial\beta} + \frac{1}{4}\frac{B}{A}\left(\frac{\partial}{\partial\beta}\beta - \frac{\partial}{\partial\beta^*}\beta^*\right)^2\right]$$
$$\times\left[1 - \frac{B}{2A}\left(-2|\beta|^2 + \beta\frac{\partial}{\partial\beta} + \beta^*\frac{\partial}{\partial\beta^*}\right) + \frac{1}{16}\frac{B^2}{A^2}\left(\frac{\partial}{\partial\beta}\beta - \frac{\partial}{\partial\beta^*}\beta^*\right)^2\right]^{-1}$$
$$\times P(\beta,\beta^*,t) + \frac{C}{2}\left(\frac{\partial}{\partial\beta}\beta + \frac{\partial}{\partial\beta^*}\beta^*\right)P(\beta,\beta^*,t) \ . \tag{9.9}$$

Treating Eqs. (9.7) and (9.8) exactly as we have treated Eqs. (3.1) and (3.2) we get the strong-signal steady-state solution of the Scully-Lamb equation in the Glauber representation ($z = \beta\beta^*$):

$$P_{SL}(z) = \begin{cases} \mathfrak{N}_{SL} e^z (A^2/BC - z)^{A/B-1}, & z \le A^2/BC \ , \\ 0, & z \ge A^2/BC \ , \end{cases} \tag{9.10}$$

with

$$\mathfrak{N}_{SL}^{-1} = \frac{B}{A}\left(\frac{A^2}{BC}\right)^{A/B}\phi\left(1, \frac{A}{B}+1; \frac{A^2}{BC}\right),$$

ϕ being the confluent hypergeometric function. Again, the presence of derivatives of all orders in β and β^* in Eq. (9.9) produces a steady state which is a nonanalytic function of z.

Furthermore, by the same procedure which has led to Eq. (4.7) we can deduce the following relations, which allow one to calculate all the moments of distribution (9.10) from the first one:

$$\langle z^n\rangle = \left(\frac{A^2}{BC} - \frac{A}{B} - (n-1)\right)\langle z^{n-1}\rangle + \frac{A^2}{BC}(n-1)\langle z^{n-2}\rangle, \quad n > 1 \ . \tag{9.11}$$

The Gaussian approximation of distribution (9.10) is given by

$$P_{SL}(z) \simeq \overline{\mathfrak{N}}_{SL}\exp\left(-\frac{[z - (A/BC)(A-C)]^2}{q_{SL}^2}\right),$$
$$q_{SL}^2 = 2A/B \ . \tag{9.12}$$

If we neglect the terms with derivatives of order higher than second in Eq. (9.9), replacing by $[1 + (B/A)|\beta|^2]$ the denominator in Eq. (9.9), we obtain instead of (9.10) the following steady-state distribution:

$$\tilde{\mathfrak{N}}\left(1 + \frac{B}{A}z\right)\exp\left[-\left(z - \frac{A}{BC}(A-C)\right)^2\frac{BC}{2A^2}\right] . \tag{9.13}$$

Since the factor $1 + Bz/A$ varies slowly, we see that distribution (9.13) is broader than (9.12) when $A > C$, i.e., when the laser is above threshold. Thus we have found again that the quantum terms with derivatives of order higher than second have a shrinking effect on the stationary solution, and that this effect vanishes at threshold.

X. CONNECTION BETWEEN SINGLE-MODE LASER MODELS

To connect the Scully-Lamb equation (9.1) with the model of Haken and co-workers we must express the parameters A, B, and C in terms of the parameters k, g, $\gamma_\perp$, $\gamma_\parallel$, and σ_0. Since the description of the pumping mechanism underlying Eq. (9.1) is different from that underlying Eq. (2.1), such expressions cannot be found on the basis of the microscopic definition of A, B, and C given in Refs. 1(a) and 10. Hence the connection must have a phenomenological character, and must be based on the comparison between the predictions of the two theories.[11] Specifically, we impose the following conditions.

(a) Let us consider the *linear* regime of the laser (i.e., $\sigma_0 \ll \sigma_{thr}$, $A \ll C$). In such a situation, Eqs. (2.7) and (9.9) reduce to standard Fokker-Planck equations with constant coefficients [obtained by keeping only the terms up to second order in g in Eq. (2.7), and putting $B=0$ in Eq. (9.9) respectively]. We require that the coefficients of both the drift and the diffusion terms in the two Fokker-Planck equations coincide, obtaining the two relations

$$\tfrac{1}{2}(A - C) = g^2N\sigma_0/\gamma_\perp - k \ ,$$
$$A = (g^2N/\gamma_\perp)(1+\sigma_0) \Rightarrow C = (g^2N/\gamma_\perp)(1 - \sigma_0 + 2\sigma_{thr}) \ . \tag{10.1}$$

(b) We require that the two models predict the same value for the mean photon number in the stationary state above threshold. This condition gives

$$(A/BC)(A - C) = Z(\sigma_0 - \sigma_{thr}) \ ; \tag{10.2}$$

from Eq. (10.1) we obtain

$$B = (8g^2k/\gamma_\parallel\gamma_\perp)(1+\sigma_0)/(1 - \sigma_0 + 2\sigma_{thr}) \ . \tag{10.3}$$

The values (10.1) of A and C coincide with those given in Ref. 11; the value (10.3) of B reduces to that of Ref. 11 when $\sigma_0 \sim \sigma_{\text{thr}}$ $(A \sim C)$. Using the values (10.1) and (10.3) we immediately see that apart from the -1 in the exponent distributions (9.10) and (8.1) *coincide*. Such a perfect agreement may seem a bit surprising if one remembers that the stationary solution of WRH is obtained via approximations quite different from those used in the derivation of the Scully-Lamb equation; however, one must also take into account the phenomenological character of the connection established by relations (10.1) and (10.3).

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APPENDIX

In Eq. (2.2) the field has been assumed to have zero temperature. Now we remove this hypothesis and look for the stationary solution in the more general case of finite inverse temperature β; of course, such a solution must be shown to approach distribution (3.11) for $\beta \to \infty$.

The effect of thermal excitation is simply taken into account by considering the damping operator $\Lambda_F^{(\beta)}$, which in the Glauber representation is given by

$$\Lambda_F^{(\beta)} X = k\left(\frac{\partial}{\partial\beta}\beta + \frac{\partial}{\partial\beta^*}\beta^* + 2\bar{n}\frac{\partial^2}{\partial\beta^*\partial\beta}\right)X , \quad \text{(A1)}$$

where $\bar{n}$ is the mean photon number at temperature β^{-1}:

$$\bar{n} = 1/(e^{\beta\omega} - 1) . \quad \text{(A2)}$$

Thus we have only to add a further term; i.e.,

$$2k\bar{n}\left(\frac{\partial^2 P(\beta,\beta^*,t)}{\partial\beta^*\partial\beta}\right)$$

in the first of the two coupled equations (3.1) and (3.2) where field losses appear, leaving everything else unchanged. Such a term vanishes when $\beta \to \infty$. In the stationary situation, using polar coordinates [Eq. (3.4)], we have instead of Eqs. (3.5) and (3.6)

$$\left(k + \frac{g^2N}{\gamma_\perp}\right) r P^{(\text{st})}(r) + \frac{k\bar{n}}{2}\frac{dP^{(\text{st})}(r)}{dr} - \frac{2g^2N}{\gamma_\perp} r W_1^{(\text{st})}(r) + \frac{1}{2}\frac{g^2N}{\gamma_\perp}\frac{dW_1^{(\text{st})}(r)}{dr} = 0 , \quad \text{(A3)}$$

$$W_1^{(\text{st})}(r) = \frac{1+\sigma_0}{2} P^{(\text{st})}(r) + \frac{g^2}{\gamma_\perp\gamma_\parallel}\left(2r^2 P^{(\text{st})}(r) - 4r^2 W_1^{(\text{st})}(r) + r\frac{dW_1^{(\text{st})}(r)}{dr}\right) . \quad \text{(A4)}$$

Now Eqs. (A3) and (A4) yield the following second-order differential equation for the field stationary distribution $P^{(\text{st})}(r)$:

$$\frac{d^2P^{(\text{st})}(r)}{dr^2} - \left(ar - \frac{b}{r}\right)\frac{dP^{(\text{st})}(r)}{dr} + (c + dr^2)P^{(\text{st})}(r) = 0 , \quad \text{(A5)}$$

where

$$a = 2\left(2 - \frac{1}{\bar{n}}\right), \quad b = 1 - \gamma_\parallel\left(\frac{\gamma_\perp}{g^2} + \frac{(1+\sigma_0)N}{2k\bar{n}}\right),$$
$$c = \frac{2}{\bar{n}}\left[2 - \gamma_\parallel\left(\frac{\gamma_\perp}{g^2} - \frac{\sigma_0 N}{k}\right)\right], \quad d = -\frac{8}{\bar{n}} . \quad \text{(A6)}$$

If we put

$$P^{(\text{st})}(r) = e^{2r^2} g[-(1/\bar{n} + 2)r^2] , \quad \text{(A7)}$$

we get from Eq. (A5) a hypergeometric confluent equation for $g[-(1/\bar{n}+2)r^2]$, which gives

$$P^{(\text{st})}(r) = e^{2r^2}\{\phi(D,E;\ -(1/\bar{n}+2)r^2) + \lambda((1/\bar{n}+2)r^2)^{1-c}\phi(D-E+1, 2-E; -(1/\bar{n}+2)r^2)\} , \quad \text{(A8)}$$

where

$$D = \frac{1}{1+2\bar{n}}\left[\left(2 - \frac{\gamma_\perp\gamma_\parallel}{g^2}\right)\left(\frac{1+2\bar{n}}{2}\right) - \frac{\gamma_\parallel N}{2k}\right],$$
$$E = 1 - (\gamma_\parallel N/2k)[\sigma_{\text{thr}} + (1+\sigma_0)/2\bar{n}] , \quad \text{(A9)}$$

and where in order that $P(r) \to 0$ for $r \to \infty$, the constant λ must take the value

$$\lambda = -[\Gamma(E)/\Gamma(E-D)]\Gamma(1-D)/\Gamma(2-E) . \quad \text{(A10)}$$

Let us now pass to the limit $\beta \to \infty$, i.e., $\bar{n} \to 0$, in Eq. (A8). In such a limit one has

$$D \simeq 1 - \tfrac{1}{2}\gamma_\parallel(\gamma_\perp/g^2 + N/k) \equiv -[2(\bar{z} - \hat{z}) - 1] ,$$
$$E \simeq -\frac{\gamma_\parallel N}{4k}\frac{1+\sigma_0}{\bar{n}} \equiv -\frac{\bar{z}}{\bar{n}} , \quad \text{(A11)}$$
$$-\left(\frac{1}{\bar{n}} + 2\right)r^2 \simeq -\frac{1}{\bar{n}}r^2 \equiv -\frac{z}{\bar{n}}$$

[see Eqs. (3.8), (3.9)].

Then from Eq. (A11) it follows that

$$\phi(D-E+1,2-E;\,-(1/\bar{n}+2)r^2) \overset{\beta\to\infty}{\sim} \phi(-E,-E;-z/\bar{n}) = e^{-z/\bar{n}} \to 0, \quad \text{for } \bar{n}\to 0\,, \tag{A12}$$

and in this limit the constant λ given in Eq. (A10) vanishes too; therefore Eq. (A8) reduces to

$$P^{(st)}(z) = e^{2z}\,\phi(-[2(\bar{z}-\hat{z})-1],\,-\bar{z}/\bar{n};\,-z/\bar{n})\,. \tag{A13}$$

The study of the limit $T\to 0$ for the function

$$\phi(-[2(\bar{z}-\hat{z})-1],\,-\bar{z}/\bar{n};\,-z/\bar{n}) \equiv \phi(a,c;x) \tag{A13$'$}$$

is performed in two separate steps.

In fact, in the case $z<\bar{z}$ we find a situation where a is finite, $c, x\to\infty$, and $|x/c|<1$, and whose asymptotic approximation is therefore[20]

$$\phi(a,c;x) \sim (1-|x/c|)^{-a} \equiv (1-z/\bar{z})^{2(\bar{z}-\hat{z})-1}\,. \tag{A14}$$

Combining Eq. (A13) with Eq. (A14) we get in the limit $\beta\to\infty$ the expected result

$$P^{(st)}(z) = \text{const}\times e^{2z}(\bar{z}-z)^{2(\bar{z}-\hat{z})-1}\,, \quad z<\bar{z}\,. \tag{A15}$$

In the case $z\geq\bar{z}$, the above argument does not hold, but one can use the integral representation[20] for the function (A13$'$); i.e.,

$$\phi(a,c;x) = \text{const}\times\int^{(1^+,0^+,1^-,0^-)} e^{xt}t^{a-1}(1-t)^{c-a-1}\,dt\,, \tag{A16}$$

where the symbol $(1^+,0^+,1^-,0^-)$ indicates that the integral must be taken along a path which starts at any point between $t=0$ and $t=1$, turns around at both these points for two times with the prescribed order and sense, and at last returns to the starting point.

When $\bar{n}\to 0$, by Eq. (A13$'$) we have in Eq. (A16)

$$e^{xt}t^{a-1}(1-t)^{c-a-1} \sim e^{-(z/\bar{n})t}\,t^{-2(\bar{z}-\hat{z})}(1-t)^{-\bar{z}/\bar{n}}\,. \tag{A17}$$

The integration path in Eq. (A16) can be chosen as composed by the segment on the real axis from $t=+\epsilon$ to $t=1-\delta$, $\epsilon,\delta>0$, and the two circles $|t|=\epsilon$, $|1-t|=\delta$, with ϵ infinitesimal and δ such that

$$1-\delta > -\ln\delta\,. \tag{A18}$$

Then if we consider the temperature-dependent part

$$F(t;\bar{n}) = \exp[-(z/\bar{n})t - (\bar{z}/\bar{n})\ln(1-t)]\,, \tag{A19}$$

we have from (A18) that

$$|F(t;\bar{n})| = \exp[-(z/\bar{n})\,\text{Re}(t) - (\bar{z}/\bar{n})\ln|1-t|] \to 0\,, \quad \text{for } \bar{n}\to 0\,,\quad z\geq\bar{z}\,. \tag{A20}$$

Therefore also in the case $z\geq\bar{z}$ the zero-temperature result (3.11) is regained.

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