Inelastic electron scattering off of simple atoms for large-momentum-transfer collisions*

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Inelastic electron scattering off of simple atoms is studied for collisions where the momentum transferred (q) from the incident electron to the atom is larger than the normal momenta of the bound electrons. We discuss the direct part of the scattering for collisions where the atom is excited to another bound state. In lowest order the second Born term, which dominates the scattering amplitude, factors into two parts. One part ($1/q^2$) displays a Rutherford-like scattering off the nucleus; the other factor gives the response of the atom to the presence of the impinging electron. The second Born scattering amplitudes for $1s \to 2s$ and $1s \to 2p$ electron-hydrogen scattering and $1\,{}^1S_0 \to 2\,{}^1P_1$ electron-helium scattering for large-momentum-transfer collisions are calculated.

I. INTRODUCTION

There has been recent interest[1] in various approximations to describe inelastic electron scattering off of hydrogen and helium at high but nonrelativistic energies and large momentum transfers. The reasons for the interest include the possibility of future measurements and previous work which has suggested that the second Born term dominates over the first in the region.[2] Inasmuch as the second Born term has not been calculated exactly, it seems reasonable to reexamine electron-hydrogen scattering for large-momentum-transfer collisions to determine which parts of the second Born term are most important and to calculate them without the use of closure or similar approximations. The application of the techniques, developed in Sec. II for other simple atoms, is discussed in Sec. III.

II. INELASTIC ELECTRON-HYDROGEN SCATTERING

A. General problem

Inelastic electron-hydrogen scattering may be represented up through and including the second Born term by Figs. 1 and 2(a)–(c). We are concerned with processes in which the impinging electron with initial momentum $\vec{k}_i$ encounters a hydrogen atom in state u_i, excites the atom to discrete state u_f, and then departs with final momentum $\vec{k}_f$. Figure 1 is the first Born term for inelastic scattering and Figs. 2(a)–2(c) are the second Born term. Additional diagrams consisting entirely of the Coulomb field of the nucleus interacting with the impinging electron do not contribute to inelastic scattering and are therefore neglected. Exchange graphs are expected to contribute little for high-energy scattering and are not considered here.

The work that follows has general applicability for high-energy, large-momentum-transfer collisions where the approximation of nonrelativistic scattering and the neglect of recoil remain valid. For purposes of comparing sizes of the contribution of the graphs, it is useful to employ the following power-counting assignments:

$$E_i, E_f \sim \alpha^2 m$$
$$k_i, k_f, q \sim \Lambda e m\,. \tag{1}$$

The electron momentum transfer $\vec{q}$ is defined to be $\vec{k}_i - \vec{k}_f$. E_i and E_f are the energies of the initial and final state, respectively. Λ is a number we choose subject to the condition $e \ll \Lambda \ll 1/e$. We work in the set of units where $\hbar = c = 1$ and $e = (\alpha)^{1/2}$ where α is the fine-structure constant. A possible choice for Λ is 1 which would lead the scattering amplitude to be written as a power series in e and $\ln(e)$ in much the same way that conventional bound-state quantum electrodynamics for hydrogen writes out its answers in terms of a power series in α and $\ln(\alpha)$.

The scattering amplitude for Fig. 1 is B_1 which may be calculated from the following expression:

$$B_1 = -(2\pi)^2 m \langle u_f, \vec{k}_f | V_{12} | u_i, \vec{k}_i \rangle\,. \tag{2}$$

In this quantity and all the expressions to follow, electron 1 is the electron in hydrogen and electron 2 is the impinging electron. Thus, r_{12} is the distance of the free electron from the bound electron.

The wave functions for free electrons are given in coordinate space by

$$\phi_{\vec{k}}(\vec{r}_2) = (2\pi)^{-3/2} \exp(i\vec{k} \cdot \vec{r}_2)\,. \tag{3}$$

We perform the integrals over $\vec{r}_1$ and $\vec{r}_2$ in Eq. (2):

$$B_1 = \frac{-2^4 \pi e^2 m}{q^2} D(\mu, \vec{\gamma}) \left[\frac{\mu}{[\mu^2 + (\vec{q} + \vec{\gamma})^2]^2} \right] \Bigg|_{\vec{\gamma}=0} \tag{4}$$

where $D(\mu, \vec{\gamma})$ is the required linear combination of derivatives of μ and $\vec{\gamma}$ which constructs $u_f^* u_i$

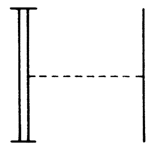

FIG. 1. First Born term for inelastic electron-hydrogen scattering. The double vertical lines crossed at the top and the bottom in this graph represent an electron in a hydrogen atom. The lower horizontal line designates the initial discrete state u_i and upper horizontal line designates the final discrete state u_f. The single vertical line is the impinging electron coming in with momentum $\vec{k}_i$ and leaving with momentum $\vec{k}_f$. The dashed line between the electron in hydrogen and the free electron denotes an electron-electron interaction.

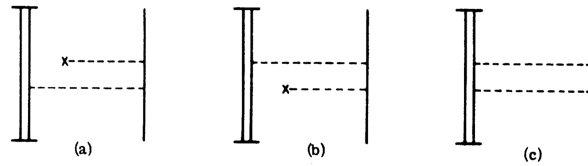

FIG. 2. Second Born term for inelastic electron-hydrogen scattering. The notation is the same as in Fig. 1, with the addition that the dashed line with a × at the end denotes an interaction of the free electron with the Coulomb field of the nucleus of hydrogen.

from $\exp(-\mu r_1 + i\vec{\gamma}\cdot\vec{r}_1)$.

It is clear for q greater than μ, which is of order αm, that B_1 decreases inversely with q at least as the sixth power. Inasmuch as it will be shown that the second Born term decreases at most as $1/q^2$, it is not surprising that the first Born term becomes smaller than the second Born term at large momentum transfer.

Figure 2(a) gives rise to B_{2a}:

$$B_{2a} = -(2\pi)^2 m \int d^3k \sum_n \frac{\langle u_f, \vec{k}_f | V_2 | u_n, \vec{k}\rangle \langle u_n, \vec{k} | V_{12} | u_i, \vec{k}_i\rangle}{E_i + k_i^2/2m - E_n - k^2/2m + i\epsilon} . \tag{5}$$

The sum over n in Eq. (7) is a sum over all the states of hydrogen. E_n is the energy of the nth state.

Next, we perform the integration over the plane-wave position vectors:

$$B_{2a} = \frac{e^4 m}{\pi^2} \int \frac{d^3k}{E_i - E_f + k_i^2/2m - k^2/2m + i\epsilon} \frac{1}{(\vec{k} - \vec{k}_f)^2} \frac{\langle u_f | e^{i(\vec{k}_i - \vec{k})\cdot\vec{r}_1} | u_i\rangle}{(\vec{k}_i - \vec{k})^2} . \tag{6}$$

B_{2a} can be reduced to a single integral using the techniques of Dalitz.[3] For the case of $ns \to ms$ scattering B_{2a} may be written in terms of parametric derivatives of the general three-denominator integral for which Lewis[4] gives an analytic answer.

We are interested in this paper in the behavior of B_{2a} where $q, k_i, k_f \gg \alpha m$. The exponential term $\exp[(\vec{k}_i - \vec{k})\cdot\vec{r}_1]$ in the matrix element controls the principal contribution in the integral over $\vec{k}$ to be in the neighborhood of $\vec{k}_i$ such that $|\vec{k}_i - \vec{k}| \lesssim \alpha m$. If we define $\vec{q}_i = \vec{k}_i - \vec{k}$, then in the region of interest $q_i/q \ll 1$ and $(\vec{k} - \vec{k}_f)^2$ may be expanded in the following power series:

$$1/(\vec{k} - \vec{k}_f)^2 = (1/q^2)(1 + 2\vec{q}_i\cdot\vec{q}/q^2 + \cdots) . \tag{7}$$

Equation (7) is used to simplify Eq. (6):

$$B_{2a} = \frac{e^4 m}{\pi^2} \int \frac{d^3q_i}{E_i - E_f + \vec{k}_i\cdot\vec{q}_i/m - q_i^2/2m + i\epsilon} \frac{\langle u_f | e^{i\vec{q}_i\cdot\vec{r}_1} | u_i\rangle}{q_i^2} \frac{1}{q^2}\left(1 + \frac{2\vec{q}_i\cdot\vec{q}}{q^2} + \cdots\right) . \tag{8}$$

If we keep the lowest-order term in $\alpha m/q$, we obtain a simple relation for B_{2a} valid for large-momentum-transfer collisions:

$$B_{2a} \approx \frac{e^4 m}{\pi^2 q^2} \int \frac{d^3q_i}{q_i^2} \frac{\langle u_f | e^{i\vec{q}_i\cdot\vec{r}_1} | u_i\rangle}{E_i - E_f + \vec{k}_i\cdot\vec{q}_i/m - q_i^2/2m + i\epsilon} . \tag{9}$$

Equation (9) is easily calculated. In lowest order B_{2a} factors into a e^2/q^2 part which corresponds to the impinging electron scattering off the nucleus and a part which contains the response of the atomic electron to the presence of the incident electron. This interpretation is supported by the appearance in the analysis of the response part of Eq. (9) of integrals which are similar to those encountered in the calculation of the density fluctuation about a charged static impurity in a neutral electron gas.[5] An example of this form of integral is evaluated in the Appendix.

The steps in treating Fig. 2(a) and its amplitude B_{2a} are used to examine Fig. 2(b) and its amplitude B_{2b}:

$$B_{2b} = -(2\pi)^2 m \int d^3k \sum_n \frac{\langle u_f, \vec{k}_f | V_{12} | u_n, \vec{k}\rangle \langle u_n, \vec{k} | V_2 | u_i, \vec{k}_i\rangle}{E_i + k_i^2/2m - E_n - k^2/2m + i\epsilon} \tag{10}$$

$$= \frac{e^4 m}{\pi^2} \int \frac{d^3k}{E_f - E_i + k_f^2/2m - k^2/2m + i\epsilon} \frac{1}{(\vec{k}_i - \vec{k})^2} \frac{\langle u_f | e^{-i(\vec{k}_f - \vec{k})\cdot \vec{r}_1} | u_i\rangle}{(\vec{k}_f - \vec{k})^2}. \tag{11}$$

Equation (11) is very similar to Eq. (6). We concentrate on the simplifications which arise in large-momentum-transfer collisions. Using the steps analogous to treating B_{2a} and the definition $\vec{q}_f = \vec{k}_f - \vec{k}$, we find an expression for B_{2b} in the $q, k_i, k_f \gg \alpha m$ region:

$$B_{2b} = \frac{e^4 m}{\pi^2} \int \frac{d^3 q_f}{E_f - E_i + \vec{k}_f \cdot \vec{q}_f/m - q_f^2/2m + i\epsilon} \frac{\langle u_f | e^{-i\vec{q}_f \cdot \vec{r}_1} | u_i\rangle}{q_f^2} \frac{1}{q^2}\left(1 - \frac{2\vec{q}\cdot\vec{q}_f}{q^2} + \cdots\right). \tag{12}$$

Keeping only the lowest-order term in $\alpha m/q$, we obtain a simple expression for B_{2b}:

$$B_{2b} = \frac{e^4 m}{\pi^2 q^2} \int \frac{d^3 q_f}{q_f^2} \frac{1}{E_f - E_i + \vec{k}_f \cdot \vec{q}_f/m - q_f^2/2m + i\epsilon} \langle u_f | e^{-i\vec{q}_f \cdot \vec{r}_1} | u_i\rangle. \tag{13}$$

In lowest order B_{2b} factors into an e^2/q^2 part which corresponds to the incident electron scattering off the nucleus and a part which contains the response of the atomic electron to the presence of the outgoing scattered electron.

The remaining diagram, Fig. 2(c), is the most cumbersome with which to deal. The expression B_{2c} corresponding to it is given below:

$$B_{2c} = -(2\pi)^2 m \int d^3k \sum_n \frac{\langle u_f, \vec{k}_f | V_{12} | u_n, \vec{k}\rangle \langle u_n, \vec{k} | V_{12} | u_i, \vec{k}_i\rangle}{E_i + k_i^2/2m - E_n - k^2/2m + i\epsilon}. \tag{14}$$

The fact that the Coulomb Green's function is known and methods[6] to deal with it have been developed and are adaptable for application to B_{2c} is not much comfort. The result of using these techniques is that B_{2c} may be expressed in the form of two real and one contour integrations over a considerable number of hypergeometric functions. The labor of performing this calculation even for the simplest choices of states i and f would be sizeable.

In this paper we circumvent the need for an exact evaluation by looking at this graph for large-momentum-transfer scattering. Here, we will find the contribution of B_{2c} is negligible compared to B_{2a} and B_{2b}, and thus may be neglected in calculating the first several orders in the total scattering amplitude.

First, we perform the integrals over the plane-wave position vectors:

$$B_{2c} = -\frac{e^4}{\pi^2} m \int \frac{d^3k}{(\vec{k} - \vec{k}_f)^2 (\vec{k} - \vec{k}_i)^2} \sum_n \frac{\langle u_f | e^{-i(\vec{k}_f - \vec{k})\cdot \vec{r}_1} | u_n\rangle \langle u_n | e^{i(\vec{k}_i - \vec{k})\cdot \vec{r}_1} | u_i\rangle}{E_i + k_i^2/2m - E_n - k^2/2m + i\epsilon}. \tag{15}$$

We consider Eq. (15) in momentum space and sum over the intermediate states, which turns the energy denominator into the Coulomb Green's operator where H is the Hamiltonian of hydrogen:

$$B_{2c} = -\frac{e^4 m}{\pi^2} \int \frac{d^3k}{(\vec{k} - \vec{k}_f)^2 (\vec{k} - \vec{k}_i)^2} \langle u_f(\vec{p}_1 - \vec{q}_f) | \frac{1}{E_i + k_i^2/2m - H - k^2/2m + i\epsilon} | u_i(\vec{p}_1 - \vec{q}_i)\rangle. \tag{16}$$

In Eq. (16) we use the well-known property of translation operators in momentum space:

$$e^{i\vec{k}\cdot\vec{r}_1} | u(\vec{p}_1)\rangle = | u(\vec{p}_1 - \vec{k})\rangle. \tag{17}$$

Since $|\vec{k}_i - \vec{k}_f|$ is larger than the normal atomic momenta, then either $u_i(\vec{p}_1 - \vec{q}_i)$ or $u_f(\vec{p}_1 - \vec{q}_f)$ is much smaller than its nominal value at normal atomic momenta by a factor of order $(\alpha m/q)^4$.[7] A consequence of this effect is that B_{2c} is reduced in size with respect to B_{2a} and B_{2b} by a factor of order $(\alpha m/q)^4$. For q of order $\Lambda e m$, $B_{2c} \sim \Lambda^{-4}\alpha^2 B_{2a}$, $\Lambda^{-4}\alpha^2 B_{2b}$. Thus, B_{2c} may be neglected in large-momentum-transfer collisions.

B. Calculations

1. 1s→2s excitation

To obtain scattering amplitudes for $1s \to 2s$ transitions for large-momentum-transfer collisions, we substitute the appropriate atomic parameters into B_1 in Eq. (4), B_{2a} in Eq. (8), and B_{2b} in Eq. (12), perform the necessary integrations and derivatives, and sum the contributions. As an example of the procedure, we give in the Appendix the evaluation of the lowest-order part of B_2 for the $1s \to 2s$ excitation.

The approximate scattering amplitude for large-momentum-transfer collisions is given by

$$f_2(1s\to 2s;\vec{q}) \approx B_1(1s\to 2s)+B_{2a}(1s\to 2s)+B_{2b}(1s\to 2s)\,. \quad (18)$$

We give in Eq. (19) the value of the first Born term and in Eqs. (20) and (21) the two lowest orders in e/Λ for the second Born part using the power counting assignments in Eq. (1):

$$B_1=-2^3\sqrt{2}e^{10}m^5\,(q^2+\tfrac{9}{4}e^4m^2)^{-3} \quad (19)$$

$$B_{2a}\approx\frac{-2^5\sqrt{2}e^4m^2}{3^4k_i}\left\{\frac{1}{q^2}\left[\left(i-\frac{g_i}{\alpha m}\right)+\frac{3\alpha m}{2^3k_i}\right]-\frac{3}{2}\,\frac{\vec{q}\cdot\hat{k}_i}{q^4}\,(\alpha m)\right\}, \quad (20)$$

$$B_{2b}\approx\frac{-2^5\sqrt{2}e^4m^2}{3^4k_f}\left\{\frac{1}{q^2}\left[\left(i-\frac{g_f}{\alpha m}\right)+\frac{3(\alpha m)}{2^3k_f}\right]+\frac{3}{2}\,\frac{\vec{q}\cdot\hat{k}_f(\alpha m)}{q^4}\right\}, \quad (21)$$

$$g_i=(m/k_i)(E_i-E_f+i\epsilon)\,, \quad (22)$$

$$g_f=(m/k_f)(E_f-E_i+i\epsilon)\,. \quad (23)$$

Employing the power-counting scheme described in Eq. (1), we find that $B_{2a}(1s\to 2s)$ and $B_{2b}(1s\to 2s)$ dominate $B_{1a}(1s\to 2s)$ by a factor of order $(e/\Lambda)^3$. Therefore, we may neglect the first Born term in the large-momentum-transfer region.

We combine Eqs. (20) and (21) and keep the two lowest-order terms

$$f_2(1s\to 2s;\vec{q})\approx\frac{-2^{13/2}e^4m^2}{3^4k_iq^2}\left(i-\frac{3}{8k_i}\,(\alpha m)\right). \quad (24)$$

2. *1s→2p excitation*

In this subsection we give the approximate scattering amplitudes for $1s\to 2p_x$ and $1s\to 2p_z$ large-momentum-transfer collisions. As in the $1s\to 2s$ example we keep only the two lowest-order terms in e/Λ for the second Born parts B_{2a} and B_{2b}:

$$B_1=\frac{-i2^2 3\sqrt{2}e^{12}m^6q_j}{q^2(\tfrac{9}{4}e^4m^2+q^2)^3}\,, \quad (25)$$

$$B_{2a}\approx\frac{i2^{9/2}e^4m^2}{3^3k_iq^2}\,(\hat{j}\cdot\hat{k}_i)\times\left\{1+\frac{\alpha m}{3k_i}\left[-\pi-4i\,\ln 2+2i\ln\left(\frac{\alpha m}{k_i}\right)\right]\right\}, \quad (26)$$

$$B_{2b}\approx\frac{-i2^{9/2}e^4m^2}{3^3k_fq^2}\,(\hat{j}\cdot\hat{k}_f)\times\left\{1+\frac{\alpha m}{3k_f}\left[-\pi-3i+4i\ln 2-2i\ln\left(\frac{\alpha m}{k_f}\right)\right]\right\}, \quad (27)$$

where $\hat{j}$ is either $\hat{x}$ or $\hat{z}$. The $\hat{z}$ direction is parallel to $\hat{k}_i$; $\hat{x}$ is perpendicular to $\hat{z}$ and is in the scattering plane.

As in the previous example the second Born parts, B_{2a} and B_{2b}, dominate the first Born term, B_{1a}, in the large-momentum-transfer region. We combine Eq. (29) and Eq. (30) and keep the two lowest-order terms:

$$f_2(1s\to 2p_x;\vec{q})\approx\frac{i2^{9/2}e^4m^2}{3^3k_iq^2}\,(\hat{k}_f\cdot\hat{x})\left\{-1+\frac{\alpha m}{3k_i}\left[3i+\pi-4i\ln 2+2i\ln\left(\frac{\alpha m}{k_i}\right)\right]\right\}, \quad (28)$$

$$f_2(1s\to 2p_z;\vec{q})\approx\frac{i2^{9/2}e^4m^2}{3^3k_iq^2}\left\{(1-\hat{z}\cdot\hat{k}_f)\left[1-\frac{\pi}{3}\left(\frac{\alpha m}{k_i}\right)\right]+(1+\hat{z}\cdot\hat{k}_f)\frac{\alpha m}{3k_i}\left[-4i\ln 2+2i\ln\left(\frac{\alpha m}{k_i}\right)\right]+i\left(\frac{\alpha m}{k_i}\right)\hat{z}\cdot\hat{k}_f\right\}. \quad (29)$$

A question which remains is to what order do the third Born terms give a contribution to the scattering amplitude. Power counting says that the third Born part is, nominally, an order e/Λ correction to the lowest-order second Born part. Thus, unless there is some unexpected cancellation, the lowest-order third Born contribution must be added to the second-order parts of Eqs. (24), (28), and (29) to determine these total scattering amplitudes accurate through two orders.

III. INELASTIC ELECTRON-HELIUM SCATTERING

The simplifications to the scattering amplitude which occur for large-momentum-transfer inelastic electron-hydrogen collisions are based on general properties of the momenta distribution of atomic systems. The extension of this work to other simple atoms is straightforward. We confine our efforts here to the $1\,^1S_0\to 2\,^1P_{1j}$ transitions where $\hat{j}$ is $\hat{x}$ or $\hat{z}$. Our coordinate system is set up so that $\hat{z}$ is parallel to the incident momentum $\vec{k}_i$ and $\hat{x}$ is in the scattering plane and is perpendicular to $\hat{z}$.

We utilize the simplest helium wave functions[8]

$$u_{1\,^1S_0}(\vec{r}_1,\vec{r}_2)=(Z_G^3/\pi)\exp(-Z_Gr_1-Z_Gr_2)\,, \quad (30)$$

$$u_{2\,^1P_{1j}}(\vec{r}_1,\vec{r}_2)=(Z_a^{5/2}Z_i^{3/2}/2^3\pi)S(1\longleftrightarrow 2)r_{1j}\times\exp(-\tfrac{1}{2}Z_ar_1-Z_ir_2)\,, \quad (31)$$

where $Z_a = 0.97\alpha m$, $Z_i = 2.0\alpha m$, and $Z_G = 1.69\alpha m$. $S(1 \longleftrightarrow 2)$ is an operator which ensures the symmetry between electron 1 and 2.

In the lowest order the $1\,^1S_0 \to 2\,^1P_{1j}$ scattering amplitude for large-momentum collisions is

$$f_2(1\,^1S_0 \to 2\,^1P_{1j}) \approx \frac{i3\times 2^5(e^4m^2)}{q^2\mu_1^4\mu_2^3}\, C\left(\frac{\hat{j}\cdot\hat{k}_i}{k_i} - \frac{\hat{j}\cdot\hat{k}_f}{k_f}\right), \tag{32}$$

where $C = Z_G^3 Z_a^{5/2} Z_i^{3/2}$, $\mu_1 = \frac{1}{2}Z_a + Z_G$, and $\mu_2 = Z_i + Z_G$.

Substituting the appropriate atomic parameters into Eq. (32), we find

$$f_2(1\,^1S_0 \to 2\,^1P_{1j}) \approx \frac{i(1.09)m^2e^4}{q^2k_i}\, \hat{j}\cdot(\hat{k}_i - \hat{k}_f). \tag{33}$$

Using the Mott and Massey wave function[9] for the ground state, we find

$$f_2(1\,^1S_0 \to 2\,^1P_{1j}) \approx \frac{i(1.21)m^2e^4}{q^2k_i}\, \hat{j}\cdot(\hat{k}_i - \hat{k}_f). \tag{34}$$

The features of this electron-helium scattering amplitude are the same as the electron-hydrogen scattering amplitudes which were discussed previously. The dominant part for large q has a $2e^2/q^2$ term which comes from the electron scattering off the nucleus multiplied by a response factor which describes the way the helium atom reacts to the presence of the impinging electron both before and after scattering.

IV. CONCLUSION

In this paper we examined the problem of direct inelastic electron scattering between bound states of simple atoms where the momentum transferred between the free electron and the atom is much larger than the normal momenta of the bound electrons. We note that the atom resolves the physical dilemma of accepting the large momentum transfer without the bound electrons being ionized by having the nucleus receive most of the momentum.

Further, we find that the parts of the first and second Born terms which contribute the most are those which contain a free electron-nucleus interaction. Thus, the parts of the second Born term with a free electron-nucleus interaction dominate over the first Born term and also the other parts of the second Born term which have only free electron-bound electron interactions. As a consequence, in lowest order the scattering amplitudes for large-momentum-transfer collisions factor into a $1/q^2$ part which displays a Rutherford-like scattering off the nucleus and a factor which gives the response of the atom to the presence of the impinging electron.[10]

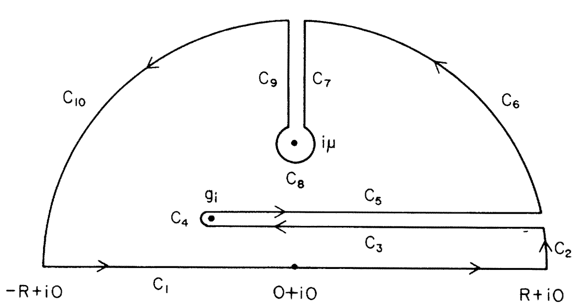

FIG. 3. Contour for the integral I in the calculation of the lowest-order part of B_{2a} for $1s \to 2s$ excitations.

APPENDIX

This Appendix gives as an example the calculation of the lowest-order part of B_2 for $1s \to 2s$ large-momentum-transfer excitations. For this transition the lowest-order part of B_{2a} and B_{2b} are equal so we concentrate on the calculation of B_{2a}. The mathematics for other transitions is quite similar.

Plugging the $1s$ and $2s$ wave functions into the matrix element in Eq. (9) and performing the integration over $\vec{r}_1$ and the angular parts of $\vec{q}_i$, we obtain

$$B_{2a} \approx \frac{-2\sqrt{2}e^{10}m^5}{\pi q^2 k_i}\left(1 + \frac{\mu}{3}\frac{\partial}{\partial\mu}\right)\frac{\partial}{\partial\mu}(\mathcal{G})\Big|_{\mu=3\alpha m/2} \tag{A1}$$

in which

$$\mathcal{G} = \int_0^\infty \frac{dq_i}{(q_i^2+\mu^2)}\,\frac{1}{q_i} \times \ln\left(\frac{2m(E_i - E_f) + 2k_iq_i - q_i^2 + i\epsilon}{2m(E_i - E_f) - 2k_iq_i - q_i^2 + i\epsilon}\right). \tag{A2}$$

The q_i^2 addends inside the logarithm do not contribute to lowest order, and therefore, we replace $\mathcal{G}$ by I where we have extended the range of integration from $-\infty$ to $+\infty$.

$$I = \frac{1}{2}\int_{-\infty}^{\infty} \frac{dq_i}{q_i(q_i^2+\mu^2)}\ln\left(\frac{g_i + q_i}{g_i - q_i}\right). \tag{A3}$$

The complex quantity g_i is given in Eq. (22).

The integrand has poles at $\pm i\mu$ and branch cuts at $[-\infty - i\epsilon, -g_i]$ and $[g_i, \infty + i\epsilon]$. We deform a portion $(-R, R)$ of the original path $(-\infty, \infty)$ in the upper half-plane as is shown in Fig. 3. Using Cauchy's theorem, the integral over section C_1 is minus the sum of the rest of the path.

$$I_1 = -(I_2+I_3+I_4+I_5+I_6+I_7+I_8+I_9+I_{10}). \tag{A4}$$

In the limit of the outer semicircle becoming large, the inner semicircle about g_i becoming small, and the paths C_7 and C_9 coalescing, $I = I_1$,

$I_7 = -I_9$, and $I_2 = I_6 = I_{10} = 0$. Thus,

$$I = -\lim_{R\to\infty}(I_3 + I_5 + I_8)\,. \tag{A5}$$

C_8 is a clockwise circle about the pole at $q_i = i\mu$:

$$I_8 = \frac{\pi i}{2\mu^2}\ln\left(\frac{g_i + i\mu}{g_i - i\mu}\right)\,. \tag{A6}$$

C_3 and C_5 are paths in opposite directions below and above the branch cut of the natural logarithmic function. Using the relation

$$\ln(-|X| + i\eta) - \ln(-|X| - i\eta) = 2\pi i$$

for η tending to zero, the sum $I_3 + I_5$ for R going to positive infinity is given by

$$\lim_{R\to\infty}(I_3 + I_5) = -\frac{\pi i}{2\mu^2}\ln\left(\frac{g_i^2}{g_i^2 + \mu^2}\right)\,. \tag{A7}$$

Finally, after a small amount of algebraic manipulation, we obtain

$$I = \frac{\pi i}{2\mu^2}\ln\left(\frac{g_i^2}{(g_i + i\mu)^2}\right)\,. \tag{A8}$$

After discarding higher-order terms in Eq. (A8), replacing it in Eq. (A1), and performing the required derivatives, we find the lowest-order term for B_{2a}. B_{2b} is obtained in a similar manner. For large-momentum-transfer collisions the dominant part of the scattering amplitude is the sum of B_{2a} and B_{2b}.

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[1]S. Geltman and M. B. Hidalgo, J. Phys. B 4, 1299 (1971); J. Gau and J. Macek, Phys. Rev. A 10, 522 (1974); E. J. Kelsey and J. Macek, Phys. Rev. A 12, 1127 (1975); B. R. Junker, Phys. Rev. A 11, 1552 (1974); F. W. Byron, Jr. and C. J. Joachain, J. Phys. B 8, L284 (1975).

[2]C. Quigg and C. J. Joachain, Rev. Mod. Phys. 46, 300 (1974).

[3]R. H. Dalitz, Proc. R. Soc. Lond. A 206, 509 (1951).

[4]R. R. Lewis, Phys. Rev. 102, 537 (1955).

[5]A. L. Fetter and J. D. Walecka, *Quantum Theory of Many Particle Systems* (McGraw-Hill, New York, 1971), p. 178.

[6]C. Fronsdal, Phys. Rev. 179, 1513 (1969); E. J. Kelsey and J. Macek, J. Math. Phys. (to be published).

[7]H. A. Bethe and E. E. Salpeter, *Quantum Mechanics of One- and Two-Electron Atoms* (Springer-Verlag, Berlin, 1957), p. 83.

[8]See Ref. 7, p. 159.

[9]N. F. Mott and H. S. W. Massey, *The Theory of Atomic Collisions* (Oxford U. P., England, 1965), p. 457.

[10]This response factor may contain a $\vec{q}$ dependence. Note, for example, in Eq. (34) $f_2(1\,{}^1S_0 \to 2\,{}^1P_{1j})$ may be written as $i(1.21)m^2c^4(\vec{q}\cdot\hat{j}/q^2k_1^2)$ to the order of accuracy desired.