Solution of the equations for nonlinear interaction of three damped waves

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Three-wave interaction is analyzed in a coherent-wave description with assumption of different linear damping (or growth) of the individual waves. It is demonstrated that when two of the coefficients of dissipation are equal, the set of equations can be reduced to a single equivalent equation, which in the nonlinearly unstable case where one wave is undamped, asymptotically takes the form of an equation defining the third Painlevé transcendent. It is then possible to find an asymptotic expansion near the time of explosion. This solution is of principal interest since it indicates that the solution of the general three-wave system, where the waves experience mutually different dissipations, belongs to a higher class of functions, which reduces to Jacobian elliptic functions only in the case where all waves experience the same damping.

I. INTRODUCTION

It is an important property of a plasma to sustain a great variety of different types of waves. These waves may be caused by external perturbations or generated by instabilities in the plasma itself. In particular, when a magnetic field is present, the number of possible types of waves becomes large. These various waves do not only propagate with their characteristic phase velocities, determined by the dispersive properties of the medium, they also experience different dissipative processes by which the wave energies of the individual modes change. Accordingly, the coefficients of the linear damping terms (and also of the nonlinear terms contributing to the attenuation) may be very different for different types of waves. For an electron plasma wave or an ion-acoustic wave the Landau damping differs appreciably from the collisional damping of a microwave or of laser radiation penetrating a plasma. When the amplitudes are small so that the equations can be linearized, the effect of dissipation on the uncoupled waves can easily be described. However, the problem of studying a system of nonlinearly coupled waves, which experience mutually different linear dampings, is a far from trivial problem.[1-6] In fact, even to lowest order in nonlinear resonant wave interaction, it does not seem possible to obtain a complete analytic solution for the general case where all three waves are damped differently. This situation is most frustrating since the experiments carried out on laser-plasma interaction and on heating of toroidal devices are now approaching power levels where nonlinear phenomena, as described by the above three-wave system, become essential.

With few exceptions[7,8] the nonlinear processes occurring in laser-plasma interaction and in the heating of magnetized plasmas have been treated in the parametric approximation, assuming the pump wave to be unchanged during the interaction process. If the parametric approach is applied, a difference between the linear damping constants of the excited waves, e.g., an ion-acoustic wave and an electromagnetic wave, causes no problem. Even if the parametric approximation provides valuable information on the thresholds, i.e., on the power levels needed to start nonlinear interactions, it fails to describe correctly the nonlinear evolution of the field amplitudes during the process of interaction. Since in current and future experiments the depletion of the pump wave is not a minor fraction of the incident wave, detailed studies which do not make use of the parametric approximation, but instead treat the full three-wave system with appropriate boundary conditions and take into account mutually different linear dampings of the interacting waves, are certainly well motivated.

For several of the three-wave interactions (e.g., the stimulated Brillouin and Raman scattering processes, where the incident radiation gives rise to either an ion-acoustic wave or an electron plasma wave and a scattered electromagnetic wave) two of the waves (i.e., the incident and scattered electromagnetic waves) experience the same linear damping. The third wave, which is a collective wave, experiences by Landau damping a linear dissipation, which is, however, different from that of the two other waves. For the process where an incident electromagnetic wave, in quantum language a photon, decays into two plasmons we have again a similar situation. As is well known, these processes are all of great interest when considering laser-plasma interaction. It is therefore worthwhile to study separately the case where two of the waves have the same linear damping,[9,10] an assumption which enables us to describe the dynamic system in a more tractable form which yields, in fact, a generalized equation for a damped nonlinear pendulum including an extra term accounting for initial conditions of the three-wave system.

For other processes than the above mentioned, e.g., the decay of a photon into a plasmon and a phonon, one is faced with the difficulty of different linear dampings of all three waves.

It is the purpose of the present paper to discuss the properties of the resonant nonlinear three-wave system with different linear dampings of the waves, in order to indicate the difficulties of the complete problem with all three linear dampings different, and to formulate an equation for the case where two of the dampings are equal but different from the third one (all three dampings being different from zero).

For the case where two of the dissipation constants are equal and where we consider nonlinearly unstable situations, certain asymptotic solutions can be obtained analytically. For a more complete description of the evolution of the system, including stable cases as well, one has to rely on computer results, or, under certain conditions, to use, for example, an approximate analytic description based on a generalization of the constants of motion.[4]

Our study, which is carried out within the framework of a coherent-wave description, elucidates the fact that the set of nonlinear equations with mutually different dampings (and also the case where two dampings are equal) corresponds mathematically to a class of functions having new properties which do not reduce to those of the Jacobian elliptic functions except for the case where all attenuation coefficients are equal.[11]

II. BASIC EQUATIONS OF THE THREE-WAVE SYSTEM

The basic system of equations describing resonant wave-wave interaction can be formulated in terms of normalized amplitudes u_i and mutually different linear damping ν_i,[12] as follows:

$$\frac{\partial u_0}{\partial t}+\nu_0 u_0=u_1u_2\cos(\Phi+\theta_{12}),$$

$$\frac{\partial u_1}{\partial t}+\nu_1 u_1=u_0u_2\cos(\Phi+\theta_{02}),$$

$$\frac{\partial u_2}{\partial t}+\nu_2 u_2=u_0u_1\cos(\Phi+\theta_{01}), \tag{1}$$

$$\frac{\partial \Phi}{\partial t}=-\frac{u_1u_2}{u_0}\sin(\Phi+\theta_{12})-\frac{u_0u_2}{u_1}\sin(\Phi+\theta_{02})-\frac{u_0u_1}{u_2}\sin(\Phi+\theta_{01}),$$

where we have assumed that the wave frequencies and wave vectors fulfill the matching conditions

$$\omega_0=\omega_1+\omega_2 \quad\text{and}\quad \vec{k}_0=\vec{k}_1+\vec{k}_2. \tag{2}$$

In Eqs. (1), Φ denotes the linear combination of phases of the individual waves $\Phi=\phi_0-\phi_1-\phi_2$ and the θ_{ij} represent the phase angles of the coupling constants $C_{ij}=|C_{ij}|e^{\theta_{ij}}$, where in (1) the absolute values $|C_{ij}|$ do not appear because of the normalization chosen for the amplitudes.[5,12]

The phase angles θ_{ij} of the coupling constants C_{ij} determine if the system is nonlinearly stable or not. A necessary[12] and sufficient[13] condition for the system to be nonlinearly (explosively) unstable (assuming that all ν_i are equal) is that the phases θ_{ij} define complex vectors which all point in the same half-plane. If this is the case all amplitudes have a tendency to grow as $u_j \sim (t_\infty - t)^{-1}$, where t_∞ is the time of "explosion."[12,14] However, if the complex vectors do not all point in the same half-plane (still assuming that all ν_i are equal), the solutions are stable. If the medium is free of dissipation (and has no linear growth), the coupling constants are real quantities.

Their signs depend on factors of the form $[\partial(\omega^2\epsilon)/\partial\omega]_{\omega=\omega_i}$, where $\epsilon(\omega,k)$ is the dielectric constant associated with the mode i. The phase angles θ_{ij} may then take the values 0 or π as determined by the sign of the related coupling constant.

In order to study particularly the dependence of the solutions of Eqs. (1) on the different attenuation coefficients ν_i, we may neglect the imaginary parts of the coupling coefficients; i.e., we may take for θ_{ij} the values 0 or π in spite of the fact that the coupling coefficients and therefore the θ_{ij} depend on the attenuation constants ν_i. We then have to consider the equations

$$\frac{\partial u_0}{\partial t}+\nu_0 u_0=s_{12}u_1u_2\cos\Phi, \tag{3}$$

$$\frac{\partial u_1}{\partial t}+\nu_1 u_1=s_{02}u_0u_2\cos\Phi, \tag{4}$$

$$\frac{\partial u_2}{\partial t}+\nu_2 u_2=s_{01}u_0u_1\cos\Phi, \tag{5}$$

$$\frac{\partial \Phi}{\partial t}=-\left(s_{12}\frac{u_1u_2}{u_0}+s_{02}\frac{u_0u_2}{u_1}+s_{01}\frac{u_0u_1}{u_2}\right)\sin\Phi. \tag{6}$$

In the system of equations (3)–(6) for the normalized amplitudes u_i and the phase variable Φ, the s_{ij} denote the signs associated with the phase angles θ_{ij} in (1), which are, in (3)–(6), 0 or π.

As is well known, Eqs. (3)–(6) can be solved exactly in terms of Jacobian elliptic functions in the case where all ν_i are zero and also in the case where the ν_i are different from zero but then only when they are all equal, i.e., $\nu_i = \nu$.[12]

Even in the case where all ν_i are different, the system of equations (3)–(6) has a constant of motion which can be expressed by the relation[1]

$$u_0 u_1 u_2 \sin\Phi \exp(\nu_0 + \nu_1 + \nu_2)t = \Gamma , \tag{7}$$

where Γ is a constant determined by the initial conditions, i.e., $\Gamma = u_0(0)u_1(0)u_2(0)\sin\Phi(0)$.

It should be noted that relation (7) is valid whatever the signs s_{ij} in Eqs. (3)–(6) may be.

III. CASE OF TWO EQUAL LINEAR ATTENUATION RATES: DERIVATION OF A SINGLE EQUIVALENT EQUATION

To start with, we note that for the special case where two of the ν_i are equal there exist only three independent possibilities with regard to the choice of signs s_{ij}. Owing to the symmetry of the equations, any other combination of signs and choice of ν_i, with two ν_i being equal, can by a change of signs in the normalization of the amplitudes be reduced to one of the three above-mentioned possibilities.

We here take $\nu_1 = \nu_2 = \nu$, assuming ν_0 different from ν, and consider the three following combinations of signs:

$$\begin{aligned} s_{12} &= +1, \quad +1, \quad -1; \\ s_{02} &= +1, \quad +1, \quad +1; \\ s_{01} &= +1, \quad -1, \quad +1. \end{aligned} \tag{8}$$

Case (a), $s_{12} = +1$, $s_{02} = +1$, $s_{01} = +1$

This case corresponds to a situation which is nonlinearly (explosively) unstable when all the ν_i are equal.[12]

For the case where $\nu_1 = \nu_2 = \nu \neq \nu_0$, it is convenient to introduce the new variable ψ and the constant Ω according to the transformations

$$u_1 = \Omega e^{-\nu t} \sinh\tfrac{1}{2}\psi , \tag{9}$$

$$u_2 = \Omega e^{-\nu t} \cosh\tfrac{1}{2}\psi , \tag{10}$$

where the real constant of motion Ω (which is, in the general case, different from zero) is given by

$$\Omega^2 = (u_2^2 - u_1^2)e^{2\nu t} = u_2^2(0) - u_1^2(0) , \tag{11}$$

and we assume, without lack of generality, that

$$u_2^2(0) > u_1^2(0) .$$

By making use of the transformations (9) and (10), we deduce from Eqs. (4) and (5)

$$u_0 \cos\Phi = \frac{1}{2}\frac{\partial\psi}{\partial t} . \tag{12}$$

Taking the derivative of Eq. (12) and using Eqs. (3)–(6), Eqs. (9) and (10), and Eq. (7) for the invariant Γ, we obtain the following differential equation for the single dependent variable ψ:

$$\frac{\partial^2\psi}{\partial t^2} + \nu_0 \frac{\partial\psi}{\partial t} - \Omega^2 e^{-2\nu t} \sinh\psi = \frac{16\Gamma^2}{\Omega^4} e^{-2\nu_0 t} \frac{\cosh\psi}{\sinh^3\psi} , \tag{13}$$

where the constant of motion Γ is again given by Eq. (7), in which we let $\nu_1 = \nu_2 = \nu$.

It should be noted that the right-hand side of Eq. (13) is proportional to the square of the ratio between the two constants of motion Γ and Ω^2.

Case (b), $s_{12} = +1$, $s_{02} = +1$, $s_{01} = -1$

This case corresponds to a situation which is nonlinearly stable when all the ν_i are equal.[12]

We here introduce the transformations

$$u_1 = \Omega e^{-\nu t} \sin\tfrac{1}{2}\psi , \tag{14}$$

$$u_2 = \Omega e^{-\nu t} \cos\tfrac{1}{2}\psi , \tag{15}$$

where now $\Omega^2 = u_2^2(0) + u_1^2(0)$, and we obtain in a similar manner as in case (a) the following differential equation:

$$\frac{\partial^2\psi}{\partial t^2} + \nu_0 \frac{\partial\psi}{\partial t} - \Omega^2 e^{-2\nu t} \sin\psi = \frac{16\Gamma^2}{\Omega^4} e^{-2\nu_0 t} \frac{\cos\psi}{\sin^3\psi} . \tag{16}$$

Equation (16) may be considered as an equation for a damped nonlinear motion of a pendulum with an exponentially changing length, the equation being supplemented by a nonlinear term which accounts for the initial conditions.

Case (c), $s_{12} = -1$, $s_{02} = +1$, $s_{01} = +1$

For the remaining case we again use the transformations (9) and (10) and the same constant of motion given by (11), and we obtain the equation

$$\frac{\partial^2\psi}{\partial t^2} + \nu_0 \frac{\partial\psi}{\partial t} + \Omega^2 e^{-2\nu t} \sinh\psi = \frac{16\Gamma^2}{\Omega^4} e^{-2\nu_0 t} \frac{\cosh\psi}{\sinh^3\psi} . \tag{17}$$

It should be emphasized that Eq. (17) differs from Eq. (13) only in the sign of the third term on the left-hand side of the equations. It turns out that the sign of this term is decisive for the stability or instability of the motion, where the stability of the motion described by Eq. (16) does not depend on the sign of the third term in the left-hand side of this equation, owing to the periodic nature of this term.

Even in the case where two of the coefficients of dissipation ν_i are equal but different from the third, it seems impossible to find complete analytic solutions to the corresponding equations (13), (16), and (17), also in the case where $\Gamma = 0$, which corresponds necessarily to $\Phi(0) = 0$ and therefore to $\Phi(t) = 0$ for all times. The reason is that even in the simplified form of the equations corresponding to $\Gamma = 0$, the remaining equations are in fact nonlinear differential second-order equations of polynomial class as can easily be seen by means of a change of variables $\cosh\psi = V$ in Eq. (13) or (17) and $\cos\psi = V$ in Eq. (16). The equations then obtained in terms of V do not belong to the class of nonlinear differential equations which are free from movable essential singularities. Accordingly, the general solution of Eqs. (13), (16), and (17) possesses certainly movable essential singularities (movable meaning that their location depends on the initial conditions). This fact, indeed, removes any hope one might have of obtaining a complete analytic solution. It is nevertheless possible to investigate some of the asymptotic properties of the equations. In the unstable case corresponding to Eq. (13), the explosive nature of the solution facilitates such an analytic description of the asymptotic solutions.

IV. ASYMPTOTIC SOLUTIONS OF THE NONLINEARLY UNSTABLE CASE

In the nonlinearly unstable case with $\nu_1 = \nu_2 = \nu \neq \nu_0$, governed by Eq. (13), we can assume without lack of generality that $\psi > 0$, since the equation is invariant with regard to the sign of ψ. Let us here investigate two particular cases, namely, (i) $\nu_0 = 0$ and explicitly taking into account the term in Γ, and (ii) $\nu \neq \nu_0 \neq 0$ but asymptotically neglecting the term in Γ.

In both cases we have in the limit when $\psi \to +\infty$, $\sinh\psi \to \frac{1}{2}e^{\psi}$ and also $\cosh\psi \to \frac{1}{2}e^{\psi}$.

Asymptotically we obtain the following equation:

$$\frac{\partial^2\psi}{\partial t^2} + \nu_0\frac{\partial\psi}{\partial t} - \frac{\Omega^2}{2}e^{\psi-2\nu t} = \frac{64\Gamma^2}{\Omega^4}e^{-2(\nu_0 t+\psi)} . \tag{18}$$

Case (i)

In this case, Eq. (18) becomes

$$\frac{\partial^2\psi}{\partial t^2} - \frac{\Omega^2}{2}e^{\psi-2\nu t} = \frac{64\Gamma^2}{\Omega^2}e^{-2\psi} . \tag{19}$$

Equation (19) can be shown, by means of the substitution

$$x = e^{-\nu t}, \quad Y = xe^{\psi} \tag{20}$$

to be of the form of the canonical third Painlevé transcendent (cf. Ref. 15):

$$\frac{\partial^2 Y}{\partial x^2} = \frac{1}{Y}\left(\frac{\partial Y}{\partial x}\right)^2 - \frac{1}{x}\frac{\partial Y}{\partial x} + \frac{\Omega^2}{2\nu^2}\frac{Y^2}{x} + \frac{64\Gamma^2}{\Omega^4\nu^2}\frac{1}{Y} . \tag{21}$$

As is well known, the main property of this trancendent is that it is free from movable essential singularities.

Equation (21) indicates what we may regard as the lower limit with regard to the complexity of an analytic solution to the original set of Equations (3)–(6), retaining at least two of the ν_i different from each other.

Here let us only remember the successive simplifications that we have used to obtain Eq. (21): (a) absence of frequency, or wave number, mismatch,[5,6]; (b) $\theta_{ij} = 0$, i.e., all nonlinear coupling constants are assumed real; (c) $\nu_i = \nu_j \neq \nu_k = 0$, i.e., two of the real linear attenuation coefficients are assumed equal and different from the third which is zero; (d) in the case where ψ is large, $\sinh\psi$ is approximated by $\frac{1}{2}e^{\psi}$.

Case (ii)

In this case Eq. (18) takes the form

$$\frac{\partial^2\psi}{\partial t^2} + \nu_0\frac{\partial\psi}{\partial t} - \frac{\Omega^2}{2}e^{\psi-2\nu t} = 0 , \tag{22}$$

where it is assumed that ψ is a large positive quantity. By the change of variables

$$e^{\psi-2\nu t} = Z(t) , \tag{23}$$

Eq. (22) becomes

$$Z\frac{\partial^2 Z}{\partial t^2} - \left(\frac{\partial Z}{\partial t}\right)^2 + \nu_0 Z\frac{\partial Z}{\partial t} + 2\nu_0\nu Z^2 - \frac{\Omega^2}{2}Z^3 = 0 . \tag{24}$$

We here note that in Eq. (24) the time no longer appears explicitly. This allows us to introduce the new change of variables

$$\frac{\partial Z}{\partial t} = Zp(Z) , \tag{25}$$

where $p(Z)$ is a function of Z only. We then obtain the following equation in p:

$$Zp\frac{dp}{dZ} + \nu_0 p + 2\nu_0\nu - \tfrac{1}{2}\Omega^2 Z = 0 . \tag{26}$$

Equation (26) is an Abelian-type first-order ordinary differential equation which cannot be integrated analytically.

We can, however, find an asymptotic solution for $Z \to +\infty$, by expanding the function $p(Z)$ as follows:

$$p(Z) = a_0 Z^{1/2} + a_1 + a_2 Z^{-1/2}\ln Z + a_3 Z^{-1/2} + a_4 Z^{-1}\ln Z + \cdots . \tag{27}$$

By substituting Eq. (27) and its derivative into Eq. (26) and identifying order by order the successive terms, we obtain for the coefficients a_i the following values:

$$a_0=\Omega,\quad a_1=-2\nu_0,\quad a_2=\frac{2\nu_0(\nu_0-\nu)}{\Omega},$$

$$a_3=\frac{2\nu_0(\nu_0-\nu)}{\Omega},\quad a_4=\frac{8}{3}\frac{\nu_0^2(\nu_0-\nu)}{\Omega^2}.$$

Limiting ourselves to the first three terms in $p(Z)$, we obtain from (25) the following asymptotic expression for the time derivative of Z:

$$\frac{\partial Z}{\partial t}=\Omega Z^{3/2}-2\nu_0 Z+\frac{2\nu_0(\nu_0-\nu)}{\Omega}Z^{1/2}\ln Z. \qquad (28)$$

We may then write the formal integral relation as follows:

$$\int_Z^\infty \frac{dz}{z^{3/2}[1-2\nu_0 z^{-1/2}/\Omega+2\nu_0(\nu_0-\nu)z^{-1}\ln z/\Omega^2]}=\Omega(t_\infty-t), \quad (29)$$

where t_∞ is the explosion time.

In order to evaluate integral (29), it is convenient to make a change of variables

$$Z^{-1/2}=V;$$

we then obtain the following integral expression:

$$\int_0^{Z^{-1/2}}\frac{dV}{1-(2\nu_0/\Omega)V-4[\nu_0(\nu_0-\nu)/\Omega^2]V^2\ln V}=\tfrac{1}{2}\Omega(t_\infty-t). \quad (30)$$

In the upper limit of the integral in (30) we note that Z is a large quantity so that in the integrand the quantity V takes only small values, and we obtain by expanding and integrating term by term

$$Z^{-1/2}+\frac{\nu_0}{\Omega}Z^{-1}-\frac{2}{3}\frac{\nu_0(\nu_0-\nu)}{\Omega^2}Z^{-3/2}\ln Z+\cdots=\frac{\Omega}{2}(t_\infty-t). \quad (31)$$

Within the accuracy of our expansion, which keeps only the three first terms, we can introduce the lowest-order approximation

$$Z^{-1/2}=\tfrac{1}{2}\Omega(t_\infty-t)$$

into the second and third terms of relation (31).

We then obtain the asymptotic solution

$$Z^{-1/2}=\tfrac{1}{2}\Omega(t_\infty-t)[1-\tfrac{1}{2}\nu_0(t_\infty-t)+\tfrac{1}{6}\nu_0(\nu_0-\nu)(t_\infty-t)^2\times\ln\tfrac{1}{2}\Omega(t_\infty-t)+\cdots]. \quad (32)$$

From Eqs. (9), (10), and (23) we then find the asymptotic solutions for u_1 and u_2:

$$\lim_{t\to t_\infty}u_1=\lim_{t\to t_\infty}u_2=\tfrac{1}{2}\Omega\exp(\tfrac{1}{2}\psi-\nu t)=\tfrac{1}{2}\Omega Z^{1/2}$$
$$=\frac{1}{t_\infty-t}[1+\tfrac{1}{2}\nu_0(t_\infty-t)-\tfrac{1}{6}\nu_0(\nu_0-\nu)(t_\infty-t)^2\times\ln\tfrac{1}{2}\Omega(t_\infty-t)+\cdots]. \quad (33)$$

Furthermore, we obtain for the remaining variables u_0 and Φ, from Eqs. (7), (12), and (32), the following two relations:

$$u_0\sin\Phi=\Gamma(t_\infty-t)^2\exp[-(\nu_0+2\nu)t]\times[1-\nu_0(t_\infty-t)+\tfrac{1}{3}\nu_0(\nu_0-\nu)(t_\infty-t)^2\times\ln\tfrac{1}{2}\Omega(t_\infty-t)+\cdots], \quad (34)$$

$$u_0\cos\Phi=\nu+\frac{1}{2}\frac{1}{Z}\frac{\partial Z}{\partial t}$$
$$=\frac{1}{t_\infty-t}[1+(\nu-\tfrac{1}{2}\nu_0)(t_\infty-t)+\tfrac{1}{3}\nu_0(\nu_0-\nu)(t_\infty-t)^2\times\ln\tfrac{1}{2}\Omega(t_\infty-t)+\cdots], \quad (35)$$

from which we find the asymptotic expressions for u_0 and $\tan\Phi$, namely,

$$u_0=\frac{1}{t_\infty-t}[1+(\nu-\tfrac{1}{2}\nu_0)(t_\infty-t)+\tfrac{1}{3}\nu_0(\nu_0-\nu)(t_\infty-t)^2\ln\tfrac{1}{2}\Omega(t_\infty-t)+\cdots], \quad (36)$$

$$\tan\Phi=\Gamma(t_\infty-t)^3\exp[-(\nu_0+2\nu)t]\times[1-(\nu+\tfrac{1}{2}\nu_0)(t_\infty-t)+\cdots]. \quad (37)$$

We note that to lowest order, but only then, u_0 has the same asymptotic behavior as u_1 and u_2; i.e., the effects of mutually different linear dissipations enter only in the higher-order terms. We also observe the fast approach to zero of the phase angle as t approaches the time of explosion, according to relation (37). Furthermore, expressions (33) and (36) clearly show that the influence of the initial conditions becomes more and more negligible close to the explosion time.

We have thus succeeded in obtaining satisfactory asymptotic solutions in the unstable case when two ν_i are equal. However, the problem of connecting the time of explosion with the initial conditions remains —a problem which in our opinion is equivalent to solving the whole problem, for which, as we have already discussed, the solution does not seem accessible.

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