Elastic scattering of photons by K-shell electrons at high energies*

V. Florescu and M. Gavrila
Department of Physics, University of Pittsburgh, Pittsburgh, Pennsylvania 15260
and Department of Physics, University of Bucharest, Bucharest, Romania
(Received 29 December 1975)

The matrix element for Rayleigh scattering by atomic K-shell electrons is evaluated in the limit of high photon energies at finite momentum transfers Δ. The limiting form of the matrix element $\overline{\mathfrak{M}}$ is derived from its relativistic expression at finite energies $\mathfrak{M}$ in which the one-electron Green's function G is replaced by its nonrelativistic approximation G_0 with adequately modified parameters. The expression obtained for $\overline{\mathfrak{M}}$ is exact in Δ and the atomic number Z, and is equivalent to the one found by Goldberger and Low. The evaluation of the matrix element is carried out in momentum space for the case of a Coulomb atomic field. Exact integral representations are used for G_0 and the ground-state eigenspinors. The integration of $\mathfrak{M}$ is carried out analytically as far as possible and at one stage the high-energy limit is taken. For one K-shell electron, when electron spin-flip is possible, $\overline{\mathfrak{M}}$ is expressed in terms of three real amplitudes H, K, and L, whereas the matrix element for the closed K shell, $\overline{\mathfrak{M}}_K$, depends only on H. The amplitudes are obtained as one-dimensional integrals over Appell functions F_1. They simplify considerably in the case of forward scattering, and the connection of their imaginary parts with absorption cross sections is discussed. Expansions of the forward amplitudes in powers of $a = \alpha Z$ are also given. Further, a connection between H and the nonrelativistic form factor is established, and the asymptotic behavior of H with respect to Δ is derived. Then, a description is given of the numerical methods used. H is computed for $0 \leq \Delta/am \leq 15$ and $1 \leq Z \leq \alpha^{-1}$. Of the electron spin-flip amplitudes only K is computed for forward scattering (L vanishes in this case). The numerical results are discussed and comparison is made with other works. The validity of the high-energy result $\overline{\mathfrak{M}}_K$ at lower photon energies is considered. Finally, the magnitude of the Rayleigh matrix element $\overline{\mathfrak{M}}_K$ is compared with the one for Delbrück scattering.

I. INTRODUCTION

Several elementary processes contribute to the elastic scattering of a photon by an atom: Rayleigh scattering by its bound electrons, nuclear Thomson scattering, Delbrück scattering by the electrostatic field of the nucleus, nuclear resonance scattering, and higher-order processes. The matrix elements of these processes have to be added coherently in order to obtain the matrix element for elastic scattering by the atom as a whole.

We are interested here in Rayleigh scattering at high energies. In the independent-electron approximation, in which all electrons are assumed to be under the influence of the *same* central atomic field, the matrix element for the whole system reduces to the coherent sum of their individual contributions. This should be a good approximation for the case of inner-shell electrons, even if the atomic field is taken to be of Coulomb form.

The one-electron matrix element for relativistic Rayleigh scattering was derived by Waller.[1] It contains a summation over the complete set of (positive and negative) energy eigenstates of the electron in the atomic field. Because of its complexity, the matrix element cannot be evaluated analytically exactly even for a Coulomb field. (This is contrary to what happens in the nonrelativistic case, where the matrix element can be expressed in terms of known transcendental functions.[2,3]) Consequently some simple analytic approximations have been sought, such as the relativistic form factor,[4] along with small-angle corrections to it,[5] and a first-order Born approximation for the sum over intermediate states.[6]

Major progress was achieved when a numerical computation of the matrix elements became feasible. This was first done by G. E. Brown and collaborators for the case of the K shell, in the 1950's.[7] The method is based on a partial-wave analysis of the matrix element. Because of the limited possibilities of the computers of the time, angular distributions were obtained only for $Z = 80$ and photon energies $\kappa/m = 0.32$, 0.64, 1.28, 2.56. The same method was subsequently applied to derive the angular distribution for $Z = 80$ and $\kappa/m = 5.12$ by Cornille and Chapdelaine.[8] Until recently, these were the only accurate results at relativistic energies which could be used for comparison. However, recently a new partial-wave calculation has been completed by Lin, Cheng, and Johnson.[9]

Other relativistic calculations have been carried out for the special case of forward scattering[10] or for evaluating the index of refraction,[11] a problem which is directly related to forward scattering.

Some time ago Goldberger and Low considered the case of Rayleigh scattering at high energies.[12]

Practically all photons are then scattered close to the forward direction. The higher the energy of the photon κ, the smaller the relevant scattering angles θ will be. In this case it is convenient to replace θ by the photon momentum transfer $\Delta = 2\kappa \sin\frac{1}{2}\theta$. Thus, in the high-energy limit, the matrix element will be nonvanishing only for finite momentum transfers Δ. In the following we will therefore express the matrix element $\mathfrak{M}$ in terms of the variables κ, Δ, Z (the nuclear charge).

The limit considered by Goldberger and Low is $\kappa \to \infty$, with Δ kept fixed. They obtained a finite result of the form

$$\lim_{\kappa\to\infty} \mathfrak{M}(\kappa, \Delta, Z) = \overline{\mathfrak{M}}(\Delta, Z), \tag{1}$$

and gave a formula for $\overline{\mathfrak{M}}(\Delta, Z)$ which, however, they did not evaluate. The consequences of the application of dispersion relations to the problem were also discussed.

In this connection, in earlier days there have been some uncertainties and inaccuracies. For example, Gell-Mann, Goldberger, and Thirring[13] suggested that in the high-energy limit and for forward scattering, $\Delta = 0$, the matrix element $\overline{\mathfrak{M}}$ of Eq. (1) would be independent of the binding of the electron (i.e., of Z) and would have the same value as if it were free ($\overline{\mathfrak{M}} = 1$, in our notation).[14,15] This suggestion was questioned soon afterwards by Levinger and Rustgi[10] who considered the specific case of lead and found from a numerical evaluation of the dispersion relation for a K-shell electron that $\overline{\mathfrak{M}}$ was smaller than 1. The formula derived by Goldberger and Low for $\overline{\mathfrak{M}}$ clearly displays at $\Delta = 0$ the Z dependence and hence the inaccuracy of the previous suggestion.

The relative magnitude of the matrix elements of the processes contributing to the elastic scattering of a photon by an atom depends on the energy of the photon, on the scattering angle, on the nuclear charge Z, on the number of electrons and their configuration, and, at higher energies, on the nuclear structure. At energies up to about 1 MeV, Rayleigh scattering is dominant for all Z. For higher energies, especially for high-Z atoms and large angles, Delbrück scattering becomes increasingly important and finally completely obscures Rayleigh scattering. In the energy range between 1 and 10 MeV there are interference effects between the two.

There has been considerable experimental interest lately in the detection of the Delbrück effect, and a new theoretical calculation was made by Papatzacos and Mork.[16] The effect has finally been established experimentally at energies higher than 10 MeV and for angles where Rayleigh scattering is negligible. However, the agreement between theory and experiment is rather poor in the energy range 1–10 MeV and at small angles, where both effects contribute. This is partly due to uncertainties in the theoretical knowledge of Rayleigh scattering, which has been estimated on the basis of semiempirical formulas. Whereas the energies in this range are low from the point of view of Delbrück scattering, we shall see that they may be considered high for Rayleigh scattering. Consequently, one will be able to use for the latter the high-energy-limit matrix element $\overline{\mathfrak{M}}(\Delta, Z)$, pertaining to the whole system of atomic electrons. At not too small momentum transfers Δ this is close to that of the K shell, as can be inferred from the approximate form-factor calculations for the higher shells.

In this paper we present the evaluation of the high-energy-limit matrix element $\overline{\mathfrak{M}}(\Delta, Z)$ for the case of a K-shell electron. The atomic field is assumed to be of Coulomb form. The integration of the matrix element is carried out in momentum space by a relativistic extension of the Green's-function method used previously by one of the authors in the nonrelativistic case.[2,3] We start from the matrix element at finite energies $\mathfrak{M}(\kappa, \Delta, Z)$, which we express in terms of the relativistic Coulomb Green's function, the properties of which were studied by Hostler.[17] Whereas in the nonrelativistic case exact closed-form expressions are known for the Coulomb Green's function, this does not hold in the relativistic case. Nevertheless, we show that in order to obtain the exact Δ and Z dependence of $\mathfrak{M}(\Delta, Z)$ it is sufficient to use the *nonrelativistic* Green's function, with parameters modified so as to take relativity into account. For this we use the Schwinger integral representation in momentum space.[18] Exact integral representations are used for the Dirac ground-state spinors. Thus the matrix element involves integrals over the momentum-space variables $\vec{p}_1$ and $\vec{p}_2$, over the two auxiliary variables introduced by the representations of the initial and final bound-state spinors, and over the auxiliary variable of the representation of the Green's function. After having integrated over $\vec{p}_1$ and $\vec{p}_2$ we take the limit $\kappa \to \infty$, with Δ kept fixed. The analytic calculation of the remaining integrals is pursued as far as possible. We are left in the end with a one-dimensional integral containing Appell functions F_1. In order to proceed further, numerical methods have to be used.

Section II contains the derivation by our method of the high-energy matrix element $\overline{\mathfrak{M}}(\Delta, Z)$. We treat the one-electron case, when electron spin flip is possible. The equivalence of our result with that of Goldberger and Low is proved. In Sec. III we give the decomposition of $\overline{\mathfrak{M}}(\Delta, Z)$ in terms

of rotation-invariant amplitudes. Section IV contains the analytic calculation of these amplitudes. For forward scattering the results simplify considerably and the connection of their imaginary parts with absorption cross sections is discussed in Sec. V. Agreement is shown to exist with the high-energy cross section of the photoeffect, studied by Hall[19] and Pratt.[20] Expansions in $a=\alpha Z$ of the amplitudes for forward scattering are also given. In Sec. VI it is shown that for sufficiently small a and not too large Δ/am the matrix element for the K shell, $\overline{\mathfrak{M}}_K(\Delta, Z)$, is closely approximated by the *nonrelativistic* form factor. The asymptotic behavior of $\overline{\mathfrak{M}}_K(\Delta, Z)$ with respect to Δ is discussed. The methods we have used in the numerical computation are presented briefly in Sec. VII. We give a discussion of our results in Sec. VIII. Comparison is made with other calculations [result of Levinger and Rustgi,[10] and partial-wave expansion calculations of Brown *et al.*,[7] Cornille and Chapdelaine,[8] and Johnson and Lin (private communication)]. The interference with Delbrück scattering is also considered. Finally, the paper contains two appendices of a mathematical nature.

II. HIGH-ENERGY LIMIT OF MATRIX ELEMENT

We consider the initial state of the process to consist of a K-shell electron of magnetic quantum number $m_1=\pm\frac{1}{2}$ and a photon of momentum $\vec{\kappa}_1$ and polarization vector $\vec{s}_1$. After the photon has undergone the elastic scattering, we consider the K-shell electron to be in a state of magnetic quantum number $m_2=\pm\frac{1}{2}$, the photon having momentum $\vec{\kappa}_2$ ($\kappa_1=\kappa_2=\kappa$) and polarization vector $\vec{s}_2$. We thus allow for the possibility of electron spin flip in the process ($m_1\neq m_2$). This can occur only for a hydrogenlike atom. We shall allow for this more general case because of its theoretical interest.

The relativistic matrix element of Waller can be written as[21,22]

$$\mathfrak{M}_{m_2m_1}=-m\left(\sum_n \frac{\langle 0m_2|\vec{\alpha}\cdot\vec{s}_2 e^{-i\vec{\kappa}_2\cdot\vec{r}}|n\rangle\langle n|\vec{\alpha}\cdot\vec{s}_1 e^{i\vec{\kappa}_1\cdot\vec{r}}|0m_1\rangle}{E_n-(E_0+\kappa+i\epsilon)} + \sum_n \frac{\langle 0m_2|\vec{\alpha}\cdot\vec{s}_1 e^{i\vec{\kappa}_1\cdot\vec{r}}|n\rangle\langle n|\vec{\alpha}\cdot\vec{s}_2 e^{-i\vec{\kappa}_2\cdot\vec{r}}|0m_1\rangle}{E_n-(E_0-\kappa-i\epsilon)}\right). \tag{2}$$

Here, the initial- and final-state vectors of the electron are denoted by $|0m_1\rangle$ and $|0m_2\rangle$, respectively, and the summation is to be carried out over the complete set of relativistic (positive and negative) energy eigenstates $|n\rangle$. The infinitesimal positive quantities ϵ in the denominators prevent the occurrence of singularities for $\kappa>m-E_0$ in the case of the first term, and for $\kappa>m+E_0$ in the case of the second term.

The K-shell contribution to Rayleigh scattering is given in the independent-electron approximation by the coherent sum

$$\mathfrak{M}_K=\sum_{m=\pm 1/2}\mathfrak{M}_{mm}. \tag{3}$$

The matrix-element equation (2) can be expressed in terms of the Green's-function matrix for the Dirac Hamiltonian of the electron. This has the following eigenspinor expansion:

$$G(\vec{r}_2,\vec{r}_1;\Omega)=\sum_n \frac{u_n(\vec{r}_2)\,u_n^\dagger(\vec{r}_1)}{E_n-\Omega}, \tag{4}$$

where the sum is extended over all positive and negative energy eigenspinors. Taking this into account Eq. (2) becomes

$$\mathfrak{M}_{m_2m_1}=\mathfrak{M}^{(1)}_{m_2m_1}+\mathfrak{M}^{(2)}_{m_2m_1}, \tag{5}$$

where

$$\mathfrak{M}^{(1)}_{m_2m_1}=-m\int\int u^\dagger_{m_2}(\vec{r}_2)e^{-i\vec{\kappa}_2\cdot\vec{r}_2}(\vec{\alpha}\cdot\vec{s}_2)G(\vec{r}_2,\vec{r}_1;\Omega_1)\times(\vec{\alpha}\cdot\vec{s}_1)e^{i\vec{\kappa}_1\cdot\vec{r}_1}u_{m_1}(\vec{r}_1)\,d\vec{r}_1\,d\vec{r}_2, \tag{6}$$

$$\mathfrak{M}^{(2)}_{m_2m_1}=-m\int\int u^\dagger_{m_2}(\vec{r}_2)e^{i\vec{\kappa}_1\cdot\vec{r}_2}(\vec{\alpha}\cdot\vec{s}_1)G(\vec{r}_2,\vec{r}_1;\Omega_2)\times(\vec{\alpha}\cdot\vec{s}_2)e^{-i\vec{\kappa}_2\cdot\vec{r}_1}u_{m_1}(\vec{r}_1)\,d\vec{r}_1 d\vec{r}_2,$$

and

$$\Omega_1=E_0+\kappa+i\epsilon,\quad \Omega_2=E_0-\kappa-i\epsilon. \tag{7}$$

Note that $\mathfrak{M}^{(2)}_{m_2m_1}$ can be obtained from $\mathfrak{M}^{(1)}_{m_2m_1}$ by interchanging $\vec{s}_1$ and $\vec{s}_2$, $\vec{\kappa}_1$ and $-\vec{\kappa}_2$, and by replacing Ω_1 with Ω_2.

The Green's function G associated with the linear Dirac equation can be expressed in terms of the Green's function G_I for the iterated (second-order) Dirac equation as follows[23]:

$$G(\vec{r}_2,\vec{r}_1\Omega)=-(\vec{\alpha}\cdot\vec{p}_2+m\beta+a/r_2+\Omega)\beta G_I(\vec{r}_2,\vec{r}_1;\Omega)\beta, \tag{8}$$

with $a=\alpha Z$. G_I satisfies the integral equation

$$G_I(\vec{r}_2,\vec{r}_1;\Omega)=G_0(\vec{r}_2,\vec{r}_1;\Omega)-\int G_0(\vec{r}_2,\vec{r}_3;\Omega)W(\vec{r}_3)G_I(\vec{r}_3,\vec{r}_1;\Omega)\,d\vec{r}_3, \tag{9}$$

where G_0 is the nonrelativistic Green's function with modified parameters and $W(\vec{r})$ is the operator

$$W(\vec{r})=ia\vec{\alpha}\cdot\vec{r}/r^3+a^2/r^2. \tag{10}$$

By iterating Eq. (9) one can express G_I in the form of an expansion in ascending powers of a.

We prefer to write the matrix elements (6) as integrals in momentum space. In fact we need consider only the case of $\mathfrak{M}^{(1)}_{m_2m_1}$, because $\mathfrak{M}^{(2)}_{m_2m_1}$ can be derived simply from it. By introducing the Fourier transforms of the quantities involved and by using Eq. (8), we get

$$\mathfrak{M}^{(1)}_{m_2m_1} = m\int\int u^{\dagger}_{m_2}(\vec{p}_1-\vec{\kappa}_2)(\vec{\alpha}\cdot\vec{s}_2)[\vec{\alpha}\cdot\vec{p}_2+m\beta+v(\vec{p}_2)+\Omega_1] \times\beta G_I(\vec{p}_2,\vec{p}_1;\Omega_1)\beta(\vec{\alpha}\cdot\vec{s}_1) \times u_{m_1}(\vec{p}_1-\vec{\kappa}_1)\,d\vec{p}_1\,d\vec{p}_2\,. \tag{11}$$

Here $v(\vec{p}_2)$ is the integral operator in momentum space of the potential

$$v(\vec{p}_2)G(\vec{p}_2,\vec{p}_1)=\int V(\vec{p}_2-\vec{q})G(\vec{q},\vec{p}_1)\,d\vec{q}\,, \tag{12}$$

with $V(\vec{p})$ the Fourier transform of a/r. Equation (11) can be further written

$$\mathfrak{M}^{(1)}_{m_2m_1}= m\int\int u^{\dagger}_{m_2}(\vec{p}_2+\vec{\Delta})(\vec{\alpha}\cdot\vec{s}_2) \times[\vec{\alpha}\cdot(\vec{p}_2+\vec{\kappa}_1)+m\beta+v(\vec{p}_2+\vec{\kappa}_1)+E_0+\kappa] \times\beta G_I(\vec{p}_2+\vec{\kappa}_1,\vec{p}_1+\vec{\kappa}_1,\Omega_1)\beta(\vec{\alpha}\cdot\vec{s}_1) \times u_{m_1}(\vec{p}_1)\,d\vec{p}_1\,d\vec{p}_2\,, \tag{13}$$

where we have introduced the photon momentum transfer $\vec{\Delta}=\vec{\kappa}_1-\vec{\kappa}_2$.

$G_I(\vec{p}_2,\vec{p}_1,\Omega)$ appearing in Eq. (13) can be expressed by the Fourier transform of Eq. (9),

$$G_I(\vec{p}_2,\vec{p}_1;\Omega)=G_0(\vec{p}_2,\vec{p}_1;\Omega) -\int\int G_0(\vec{p}_2,\vec{p}_3;\Omega)W(\vec{q}) \times G_I(\vec{p}_3-\vec{q},\vec{p}_1;\Omega)\,d\vec{p}_3\,d\vec{q}\,. \tag{14}$$

Here $W(\vec{q})$ is the Fourier transform of $W(\vec{r})$ of Eq. (10).

We want to obtain the high-energy limit of Eq. (13). As in Eq. (1), we denote

$$\overline{\mathfrak{M}}^{(i)}_{m_2m_1}=\lim\mathfrak{M}^{(i)}_{m_2m_1},\quad i=1,2\,. \tag{15}$$

Here and in the following the symbol "lim" will mean $\kappa\to\infty$, with Δ being kept constant. Then, to lowest order in $1/\kappa$, we have $\vec{\kappa}_1=\vec{\kappa}_2=\vec{\kappa}=\kappa\vec{\nu}$, where $\vec{\nu}$ is the unit vector of $\vec{\kappa}$; also, $\vec{\nu}\vec{\Delta}=0$, $\vec{\nu}\vec{s}_1=\vec{\nu}\vec{s}_2=0$.

To begin, let us consider the limit

$$\lim\kappa G_I(\vec{p}_2+\vec{\kappa},\vec{p}_1+\vec{\kappa};E_0+\kappa+i\epsilon)\,, \tag{16}$$

where $\vec{p}_1$ and $\vec{p}_2$ belong to a finite domain. We want to prove that it is equal to the function

$$g^{(1)}(\vec{p}_2,\vec{p}_1)\equiv\lim\kappa G_0(\vec{p}_2+\vec{\kappa},\vec{p}_1+\vec{\kappa};E_0+\kappa+i\epsilon)\,. \tag{17}$$

For this purpose we shall use the iteration expansion of Eq. (14).

First we show that the limit defined by Eq. (17) exists. This can be done by using the Schwinger integral representation for G_0, given by[18,24]

$$G_0(\vec{p}_2,\vec{p}_1,\Omega)=-\frac{1}{4\pi^2}X^3\int_0^1\rho^{-\tau}\frac{d}{d\rho}\left(\frac{1-\rho^2}{\rho}\frac{1}{[X^2(\vec{p}_1-\vec{p}_2)^2+(p_1^2+X^2)(p_2^2+X^2)(1-\rho)^2/4\rho]^2}\right)d\rho\,, \tag{18}$$

where, in the present case,

$$X^2=m^2-\Omega^2\,,\quad \tau=a\Omega/X\,,\quad \mathrm{Re}X>0\,. \tag{19}$$

If we replace Ω in Eq. (19) with Ω_1 of Eq. (7), we get in the high-energy limit

$$X_1^2+\kappa^2\simeq-2E_0\kappa-i\epsilon\,,\quad X_1\simeq-i\kappa\,,\quad \tau_1\simeq ia\,. \tag{20}$$

Hence by introducing Eq. (18) into Eq. (17), we find[25]

$$g^{(1)}(\vec{p}_2,\vec{p}_1)=-\frac{i}{4\pi^2}\int_0^1\rho^{-ia}\frac{d}{d\rho}\left(\frac{1-\rho^2}{\rho}\frac{1}{[(\vec{p}_1-\vec{p}_2)^2-(E_0-\vec{p}_1\vec{\nu}+i\epsilon)(E_0-\vec{p}_2\vec{\nu}+i\epsilon)(1-\rho)^2/\rho]^2}\right)d\rho\,. \tag{21}$$

This shows that $g^{(1)}$ is indeed well defined.

Next, we consider the contribution to the limit (16) of the second term of the iteration expansion of G_I, Eq. (14):

$$T(\vec{p}_2,\vec{p}_1)=-\int\int G_0(\vec{p}_2\vec{p}_3;\Omega_1)W(\vec{q}) \times G_0(\vec{p}_3-\vec{q},\vec{p}_1;\Omega_1)\,d\vec{p}_3\,d\vec{q}\,. \tag{22}$$

From here it follows that we can write

$$T(\vec{p}_2+\vec{\kappa},\vec{p}_1+\vec{\kappa})=-\int\int G_0(\vec{p}_2+\vec{\kappa},\vec{p}_3+\vec{\kappa};\Omega_1)W(\vec{q}) \times G_0(\vec{p}_3+\vec{\kappa}-\vec{q},\vec{p}_1+\vec{\kappa};\Omega_1) \times d\vec{p}_3\,d\vec{q}\,. \tag{23}$$

Therefore in the limit $\kappa\to\infty$ and with $\vec{p}_1,\vec{p}_2$ bounded,

by taking into account Eq. (17) we get the dominant behavior[26]

$$\kappa T(\vec{p}_2+\vec{\kappa},\vec{p}_1+\vec{\kappa}) \simeq -\frac{1}{\kappa}\int\int g^{(1)}(\vec{p}_2,\vec{p}_3)W(\vec{q}) \times g^{(1)}(\vec{p}_3-\vec{q},\vec{p}_1)\,d\vec{p}_3\,d\vec{q}\,. \tag{24}$$

This term is of order $1/\kappa$ and is vanishing with respect to the term given by Eq. (17), which is finite. Similarly, one can show that the nth term of the iteration expansion of G_I, Eq. (14), yields a contribution of order $1/\kappa^{n-1}$ to the limit Eq. (16). Hence, as stated,

$$\lim \kappa G_I(\vec{p}_2+\vec{\kappa},\vec{p}_1+\vec{\kappa};E_0+\kappa+i\epsilon) = g^{(1)}(\vec{p}_2,\vec{p}_1)\,. \tag{25}$$

This equation is exact in the atomic charge Z.[27]

Let us now consider the limit (15) of the matrix element (13). By noting that the spinors $u_m(p)$ are rapidly decreasing functions of p,[28] it follows that only an essentially finite domain of momentum space will contribute to the integral of Eq. (13). In Eqs. (15) and (13) we can invert the integral with the limit and use Eq. (25).[26] Also, the term $\vec{\alpha}\vec{p}_2 + m\beta + E_0$ in the square bracket of Eq. (13) should be neglected.[29] Moreover, from Eqs. (12) and (25) we find that for large values of κ

$$v(\vec{p}_2+\vec{\kappa})G_I(\vec{p}_2+\vec{\kappa},\vec{p}_1+\vec{\kappa};\Omega_1) = \int V(\vec{p}_2-\vec{q})G_I(\vec{q}+\vec{\kappa},\vec{p}_1+\vec{\kappa})\,d\vec{q} \simeq (1/\kappa)v(\vec{p}_2)g^{(1)}(\vec{p}_2,\vec{p}_1)\,, \tag{26}$$

which shows that the term is vanishing and should be dropped.

We are finally left with

$$\overline{\mathfrak{M}}^{(1)}_{m_2m_1} = m\int\int u^{\dagger}_{m_2}(\vec{p}_2+\vec{\Delta})(\vec{\alpha}\cdot\vec{s}_2)(1+\vec{\alpha}\cdot\vec{\nu})(\vec{\alpha}\cdot\vec{s}_1) \times g^{(1)}(\vec{p}_2,\vec{p}_1)u_m(\vec{p}_1)\,d\vec{p}_1\,d\vec{p}_2\,. \tag{27}$$

The case of $\mathfrak{M}^{(2)}_{m_2m_1}$ can be handled similarly. We now define

$$g^{(2)}(\vec{p}_2,\vec{p}_1) = \lim \kappa G_0(\vec{p}_2-\vec{\kappa},\vec{p}_1-\vec{\kappa};E_0-\kappa-i\epsilon)\,. \tag{28}$$

The expression of $g^{(2)}$ can be obtained from Eqs. (18) and (19) by replacing Ω with Ω_2, given by Eq. (7). However, instead of Eq. (20), we now use

$$X_2^2+\kappa^2 \simeq +2E_0\kappa - i\epsilon\,,\quad X_2 \simeq -i\kappa\,,\quad \tau_2 \simeq -ia\,. \tag{29}$$

We thus find

$$g^{(2)}(\vec{p}_2,\vec{p}_1) = -[g^{(1)}(\vec{p}_2,\vec{p}_1)]^*\,, \tag{30}$$

with $g^{(1)}$ defined by Eq. (21). Finally, we get

$$\overline{\mathfrak{M}}^{(2)}_{m_2m_1} = -m\int\int u^{\dagger}_{m_2}(\vec{p}_2+\vec{\Delta})(\vec{\alpha}\cdot\vec{s}_1)(1+\vec{\alpha}\cdot\vec{\nu})(\vec{\alpha}\cdot\vec{s}_2)g^{(2)}(\vec{p}_2,\vec{p}_1)u_{m_1}(\vec{p}_1)\,d\vec{p}_1\,d\vec{p}_2\,. \tag{31}$$

We will now show that the high-energy limit we derived for the Waller matrix element, Eqs. (5), (15), (27), and (31), agrees with the result of Goldberger and Low. To this end we need the expression of $\overline{\mathfrak{M}}^{(1)}_{m_2m_1}$, Eq. (27), as a configuration-space integral,

$$\overline{\mathfrak{M}}^{(1)}_{m_2m_1} = m\int\int u^{\dagger}_{m_2}(\vec{r}_2)e^{i\vec{\Delta}\cdot\vec{r}_2}(\vec{\alpha}\cdot\vec{s}_2)(1+\vec{\alpha}\cdot\vec{\nu})(\vec{\alpha}\cdot\vec{s}_1)\left(\lim_{\kappa\to\infty}\kappa e^{-i\vec{\kappa}\cdot\vec{r}_2}G_0(\vec{r}_2,\vec{r}_1;\Omega_1)e^{i\vec{\kappa}\cdot\vec{r}_1}\right)u_{m_1}(\vec{r}_1)\,d\vec{r}_1\,d\vec{r}_2\,.$$

This can also be written

$$\overline{\mathfrak{M}}^{(1)}_{m_2m_1} = m\int u^{\dagger}_{m_2}(\vec{r})e^{i\vec{\Delta}\cdot\vec{r}}(\vec{\alpha}\cdot\vec{s}_2)(1+\vec{\alpha}\cdot\vec{\nu})(\vec{\alpha}\cdot\vec{s}_1)(\lim \kappa e^{-i\vec{\kappa}\cdot\vec{r}}[H_0^{(1)}-X_1^2]^{-1}e^{i\vec{\kappa}\cdot\vec{r}})u_{m_1}(\vec{r})\,d\vec{r}\,, \tag{32}$$

where

$$H_0 = -\vec{P}^2 + 2a\Omega/r\,, \tag{33}$$

and $[H_0-X^2]^{-1}$ is the operator form of the Green's function G_0.[23]

We then apply the operator equality[30]

$$e^{i\vec{\kappa}\cdot\vec{r}}[H(\vec{P}+\vec{\kappa})-\omega]^{-1}e^{-i\vec{\kappa}\cdot\vec{r}} = [H(\vec{P})-\omega]^{-1}\,, \tag{34}$$

where $H(\vec{P})$ is a Hamiltonian and $H(\vec{P}+\vec{\kappa})$ is the result when the momentum operator $\vec{P}$ is replaced by $\vec{P}+\vec{\kappa}$; ω is a constant which does not belong to the spectrum of H.

By setting in Eq. (34) $H=H_0^{(1)}$ and $\omega=X_1^2$, we get

$$e^{-i\vec{\kappa}\cdot\vec{r}}[H_0^{(1)}-X_1^2]^{-1}e^{i\vec{\kappa}\cdot\vec{r}} = [-(\vec{P}+\vec{\kappa})^2+2a\Omega_1/r-X_1^2]^{-1}\,.$$

Inserting this into Eq. (33) and taking the high-energy limit, it follows [see also Eqs. (7) and (20)] that

$$\overline{\mathfrak{M}}^{(1)}_{m_2m_1} = m\int u^{\dagger}_{m_2}(\vec{r})e^{i\vec{\Delta}\cdot\vec{r}}(\vec{\alpha}\cdot\vec{s}_2)\frac{1+\vec{\alpha}\cdot\vec{\nu}}{2}(\vec{\alpha}\cdot\vec{s}_1) \times\left(E_0+\frac{a}{r}-\vec{P}\cdot\vec{\nu}+i\epsilon\right)^{-1}u_{m_1}(\vec{r})\,d\vec{r}\,. \tag{35}$$

This represents the first term of the result of Goldberger and Low, Ref. 12, Eq. (2.7). By proceeding similarly with $\overline{\mathfrak{M}}^{(2)}_{m_2m_1}$ one obtains also

their second term, and thereby the desired equivalence is established.[31]

III. INVARIANT AMPLITUDES

In the following we analyze the high-energy limit of the Waller matrix element in terms of rotation-invariant amplitudes and derive their analytical form.

The exact K-shell eigenspinors in momentum space have the following form:

$$u_m(\vec{p}) = [a(p) + \tfrac{1}{2} b(p) \vec{\alpha} \cdot \vec{p}\,] \chi_m . \tag{36}$$

Here χ_m is the constant spinor of a free particle at rest. For $m = \frac{1}{2}$, χ_m is $(1, 0, 0, 0)$, and for $m = -\frac{1}{2}$, χ_m is $(0, 1, 0, 0)$. The functions $a(p)$ and $b(p)$ can be expressed in terms of the integral representations[32-34]

$$a(p) = \frac{N}{\Gamma(1-\gamma)} \int_0^\infty x^{-\gamma} \frac{\lambda(1+x)}{[p^2 + \lambda^2(1+x)^2]^2}\, dx , \tag{37}$$

$$b(p) = \frac{2N}{\Gamma(1-\gamma)} \frac{a}{1+\gamma} \int_0^\infty x^{-\gamma} \frac{1}{[p^2 + \lambda^2(1+x)^2]^2}\, dx , \tag{38}$$

where N is the normalization constant

$$N = (2^{\gamma+1/2}/\pi)\lambda^{3/2}[(1+\gamma)/\Gamma(2\gamma+1)]^{1/2} . \tag{39}$$

We have denoted here

$$a = \alpha Z , \quad \lambda = am , \quad \gamma = (1 - a^2)^{1/2} . \tag{40}$$

The energy eigenvalue of the ground state is

$$E_0 = m\gamma . \tag{41}$$

By inserting Eq. (36) into Eq. (27) we get

$$\overline{\mathfrak{M}}^{(1)}_{m_2 m_1} = \chi^\dagger_{m_2} \Big(A(\vec{\alpha}\cdot\vec{s}_2)(\vec{\alpha}\cdot\vec{s}_1) + \tfrac{1}{2}(\vec{\alpha}\cdot\vec{s}_2)(\vec{\alpha}\cdot\vec{\nu})(\vec{\alpha}\cdot\vec{s}_1)(\vec{\alpha}\cdot\vec{B}_1) + \tfrac{1}{2}(\vec{\alpha}\cdot\vec{B}_2)(\vec{\alpha}\cdot\vec{s}_2)(\vec{\alpha}\cdot\vec{\nu})(\vec{\alpha}\cdot\vec{s}_1) + \tfrac{1}{4}\sum_{i,j} \alpha_j (\vec{\alpha}\cdot\vec{s}_2)(\vec{\alpha}\cdot\vec{s}_1)\alpha_i C_{ij} \Big) \chi_{m_1} , \tag{42}$$

where

$$A = m \iint a(\vec{p}_2 + \vec{\Delta}) a(\vec{p}_1) g^{(1)}(\vec{p}_2, \vec{p}_1)\, d\vec{p}_1\, d\vec{p}_2 , \tag{43}$$

$$\vec{B}_1 = m \iint \vec{p}_1\, a(\vec{p}_2 + \vec{\Delta}) b(\vec{p}_1) g^{(1)}(\vec{p}_2, \vec{p}_1)\, d\vec{p}_1\, d\vec{p}_2 , \tag{44}$$

$$\vec{B}_2 = m \iint (\vec{p}_2 + \vec{\Delta}) b(\vec{p}_2 + \vec{\Delta}) a(\vec{p}_1) g^{(1)}(\vec{p}_2, \vec{p}_1)\, d\vec{p}_1\, d\vec{p}_2 , \tag{45}$$

$$C_{ij} = m \iint p_{1i}(p_{2j} + \Delta_j) b(\vec{p}_2 + \vec{\Delta}) b(\vec{p}_1) \times g^{(1)}(\vec{p}_2, \vec{p}_1)\, d\vec{p}_1\, d\vec{p}_2 . \tag{46}$$

The last three integrals must have the following forms:

$$\vec{B}_1 = B'\vec{\nu} + B''\vec{\delta} , \tag{47}$$

$$\vec{B}_2 = B'\vec{\nu} - B''\vec{\delta} , \tag{48}$$

$$C_{ij} = C\delta_{ij} + D\nu_i\nu_j + E(\nu_i\delta_j - \nu_j\delta_i) + F\delta_i\delta_j . \tag{49}$$

Here the δ_i are the components of the unit vector of the momentum transfer, $\vec{\delta} = \vec{\Delta}/\Delta$, whereas δ_{ij} is the Kronecker symbol. The special form of Eqs. (47)–(49) can be established by changing the integration variables according to $\vec{p}_1 \to \vec{p}_2 - \vec{\Delta}$, $\vec{p}_2 \to \vec{p}_1 - \vec{\Delta}$, and noting that from Eq. (21) we have

$$g^{(1)}(\vec{p}_1 - \vec{\Delta}, \vec{p}_2 - \vec{\Delta}) = g^{(1)}(\vec{p}_2, \vec{p}_1) ,$$

since $\vec{\nu}\vec{\Delta} = 0$. Therefore

$$\vec{B}_1(-\vec{\Delta}) = \vec{B}_2(\vec{\Delta}), \quad C_{ij}(-\vec{\Delta}) = C_{ji}(\vec{\Delta}) ,$$

from which Eqs. (47)–(49) readily follow.

After carrying out the matrix algebra implied by Eq. (42) and taking into account that $\vec{s}_1 \cdot \vec{\nu} = \vec{s}_2 \cdot \vec{\nu} = 0$, $\vec{\nu} \cdot \vec{\delta} = 0$, we find that $\overline{\mathfrak{M}}^{(1)}$ can be cast in the general form

$$\overline{\mathfrak{M}}^{(i)}_{m_2 m_1} = \chi^\dagger_{m_2} \mathfrak{U}^{(i)} \chi_{m_1} , \quad i = 1, 2 , \tag{50}$$

where $\mathfrak{U}^{(i)}$ is the matrix

$$\mathfrak{U}^{(i)} = H^{(i)}(\vec{s}_2 \cdot \vec{s}_1) + iK^{(i)}\vec{\sigma} \cdot (\vec{s}_2 \times \vec{s}_1) + iL^{(i)}(\vec{s}_2 \cdot \vec{s}_1)\vec{\sigma} \cdot (\vec{\nu} \times \vec{\delta}) . \tag{51}$$

In the present case $i = 1$ and we get

$$H^{(1)} = A - B' + \tfrac{3}{4}C + \tfrac{1}{4}D + \tfrac{1}{4}F , \tag{52}$$

$$K^{(1)} = A - B' - \tfrac{1}{4}C + \tfrac{1}{4}D - \tfrac{1}{4}F , \tag{53}$$

$$L^{(1)} = -(B'' + \tfrac{1}{2}E) . \tag{54}$$

The case of $\overline{\mathfrak{M}}^{(2)}_{m_2 m_1}$ immediately reduces to that of $\overline{\mathfrak{M}}^{(1)}_{m_2 m_1}$, if one considers Eqs. (28), (30), and (31). We find that Eqs. (50) and (51) remain valid also for $i = 2$, and that

$$H^{(2)} = H^{(1)*}, \quad K^{(2)} = -K^{(1)*}, \quad L^{(2)} = L^{(1)*} \tag{55}$$

By summing the two contributions for $i = 1$ and $i = 2$, we get

$$\overline{\mathfrak{M}}_{m_2 m_1} = \chi^\dagger_{m_2} \mathfrak{U} \chi_{m_1} , \tag{56}$$

with

$$\mathfrak{A} = H(\vec{s}_2 \cdot \vec{s}_1) + K\vec{\sigma} \cdot (\vec{s}_2 \times \vec{s}_1) + iL(\vec{s}_2 \cdot \vec{s}_1)\vec{\sigma} \cdot (\vec{\nu} \times \vec{\delta}), \tag{57}$$

where H, K, L are given by

$$H = 2\,\mathrm{Re}H^{(1)}, \quad K = -2\,\mathrm{Im}K^{(1)}, \quad L = 2\,\mathrm{Re}\,L^{(1)}. \tag{58}$$

We have thus achieved the decomposition of the matrix element Eq. (56) in terms of three invariant amplitudes. Note that all of them are real.

From Eqs. (56) and (57) it is apparent that for $m_1 = m_2$ all amplitudes contribute, whereas in the case of electron spin flip, $m_1 \neq m_2$, only K and L contribute.

The K-shell contribution to Rayleigh scattering is given by [see Eq. (3)]

$$\overline{\mathfrak{M}}_K = 2H(\vec{s}_1 \cdot \vec{s}_2). \tag{59}$$

H is therefore the physically important amplitude and we shall be concerned mainly with it in what follows.

IV. ANALYTIC EXPRESSIONS FOR THE AMPLITUDES

We are now faced with the calculation of the amplitudes (58) and (52)–(54). Taking into account Eqs. (43)–(46), (21), (37), and (38), these are expressed in terms of three parametric integrals, followed by momentum-space integrals. Many of these can be carried out analytically, but not all. We shall try to pursue the analytic calculation as far as possible and only in the end apply numerical methods.

For all of Eqs. (43)–(46) the first step will be to change the order of integrations: perform first the integrations in momentum space and leave the parametric integrals to the end. Further, instead of using the integral representation Eq. (21) for $\mathcal{G}^{(1)}$, we prefer to consider it defined by Eq. (17) in terms of G_0, and use the integral representation Eq. (18) for the latter. This is because we have already calculated elsewhere the relevant momentum-space integrals in terms of G_0 (for finite $\vec{\kappa}$).[3]

Let us consider the case of A, Eq. (43). By inserting here the integral representations Eqs. (18) and (37) we can write

$$A = \lim\left(-\frac{m}{4\pi^2}\right)\left(\frac{N}{\Gamma(1-\gamma)}\right)^2 \kappa X_1^3 \int_0^1 d\rho\, \rho^{-\tau_1} \int_0^\infty dx \int_0^\infty dy\, (xy)^{-\gamma} \frac{d}{d\rho}\left(\frac{1-\rho^2}{\rho} U\right), \tag{60}$$

where

$$U = \iint \frac{\lambda(1+y)\lambda(1+x)\, d\vec{p}_1\, d\vec{p}_2}{[(\vec{p}_2 - \vec{\kappa}_2)^2 + \lambda^2(1+y)^2]^2 [X_1^2(\vec{p}_2 - \vec{p}_1)^2 + \alpha(p_1^2 + X_1^2)(p_2^2 + X_1^2)]^2 [(\vec{p}_1 - \vec{\kappa}_1)^2 + \lambda^2(1+x)^2]^2}. \tag{61}$$

We have abbreviated

$$\alpha = (1-\rho)^2/4\rho. \tag{62}$$

The momentum-space integral equation (61) can be expressed in terms of

$$J = \iint \frac{d\vec{p}_1\, d\vec{p}_2}{[(\vec{p}_2 - \vec{\kappa}_2)^2 + \mu^2][X^2(\vec{p}_2 - \vec{p}_1)^2 + \alpha(p_1^2 + X^2)(p_2^2 + X^2)]^2 [(\vec{p}_1 - \vec{\kappa}_1)^2 + \lambda^2]}. \tag{63}$$

Indeed, we can write[25]

$$A = -\frac{m}{4\pi^2}\left(\frac{N}{\Gamma(1-\gamma)}\right)^2 \int_0^1 d\rho\, \rho^{-ia} \int_0^\infty \int_0^\infty dx\, dy\, (xy)^{-\gamma} \left\{ \lim \kappa X_1^3 \frac{1}{4} \frac{\partial^2}{\partial\lambda\, \partial\mu} \frac{d}{d\rho} \frac{1-\rho^2}{\rho} J \right\}, \tag{64}$$

where it is understood that after performing the derivatives contained in the curly brackets of Eq. (64), λ and μ should be replaced by $\lambda(1+x)$ and $\lambda(1+y)$, respectively.

The integral (63) was encountered and calculated in a previous work. We have calculated there [Ref. 3, Eq. (23)] also the derivative

$$\frac{d}{d\rho}\left(\frac{1-\rho^2}{\rho} J\right) = \frac{16\pi^4}{X^2} \frac{1}{c}, \tag{65}$$

with

$$c = \{[(X+\lambda)^2 + \kappa_1^2] - \rho[(X-\lambda)^2 + \kappa_1^2]\}\{[(X+\mu)^2 + \kappa_2^2] - \rho[(X-\mu)^2 + \kappa_2^2]\} + 4\rho X^2 \Delta^2 + 4\rho X^2 (\lambda - \mu)^2. \tag{66}$$

From this it follows that

$$\frac{\partial^2}{\partial\lambda\partial\mu}\frac{d}{d\rho}\left(\frac{1-\rho^2}{\rho}J\right)=64\pi^4\left(\frac{(X+\lambda)(X+\mu)+2\rho(2X^2-\lambda\mu)+(X-\lambda)(X-\mu)\rho^2}{c^2X^2}-8\rho\Delta^2\frac{[X+\lambda+\rho(X-\lambda)][X+\mu+\rho(X-\mu)]}{c^3}\right.$$
$$\left.-8\rho(1-\rho)(\lambda-\mu)\frac{\kappa_2^2[X+\lambda+\rho(X-\lambda)]-\kappa_1^2[X+\mu+\rho(X-\mu)]}{c^3}\right). \tag{67}$$

In the high-energy limit, by taking $X=X_1$ and using Eq. (20), we get

$$\lim\kappa X_1^3\frac{1}{4}\frac{\partial^2}{\partial\lambda\,\partial\mu}\frac{d}{d\rho}\left(\frac{1-\rho^2}{\rho}J\right)=i\pi^4\left(\frac{1+4\rho+\rho^2}{\bar{c}'^2}+2\rho\frac{\Delta^2(1+\rho)^2-(\lambda-\mu)^2(1-\rho)^2}{\bar{c}'^3}\right), \tag{68}$$

where

$$\bar{c}'=\lim(1/4\kappa^2)c=[E_0+i\lambda-\rho(E_0-i\lambda)][E_0+i\mu-\rho(E_0-i\mu)]-\rho[\Delta^2+(\lambda-\mu)^2]. \tag{69}$$

In Eqs. (68) and (69) we now have to replace λ and μ by $\lambda(1+x)$ and $\lambda(1+y)$. When we insert the result into Eq. (64), we get

$$A=-\frac{i\pi^2}{4}m\left(\frac{N}{\Gamma(1-\gamma)}\right)^2\int_0^1 d\rho\,\rho^{-ia}\int_0^\infty\int_0^\infty dx\,dy\,(xy)^{-\gamma}\left(\frac{1+4\rho+\rho^2}{\bar{c}^2}+2\rho\frac{\Delta^2(1+\rho)^2-\lambda^2(x-y)^2(1-\rho)^2}{\bar{c}^3}\right). \tag{70}$$

$\bar{c}$ is obtained by making the aforementioned substitution in Eq. (69), which gives

$$\bar{c}=c_1c_2-\rho\Delta^2;\quad c_1=[E_0+i\lambda-\rho(E_0-i\lambda)]+i\lambda x+i\lambda\rho y,\quad c_2=[E_0+i\lambda-\rho(E_0-i\lambda)]+i\lambda\rho x+i\lambda y. \tag{71}$$

We shall consider next the case of $\vec{B}_1$, Eq. (44). By using the integral representations (18), (37), and (38) we can write

$$\vec{B}_1=\lim\left(-\frac{m}{4\pi^2}\right)\frac{2a}{1+\gamma}\left(\frac{N}{\Gamma(1-\gamma)}\right)^2\kappa X_1^3\int_0^1 d\rho\,\rho^{-ia}\int_0^\infty\int_0^\infty dx\,dy\,(xy)^{-\gamma}\frac{d}{d\rho}\left(\frac{1-\rho^2}{\rho}\vec{V}_1\right), \tag{72}$$

where

$$\vec{V}_1=\iint\frac{\lambda(1+y)(\vec{p}_1-\vec{\kappa}_1)\,d\vec{p}_1\,d\vec{p}_2}{[(\vec{p}_2-\vec{\kappa}_2)^2+\lambda^2(1+y)^2][X_1^2(\vec{p}_2-\vec{p}_1)^2+\alpha(p_1^2+X_1^2)(p_2^2+X_1^2)]^2[(\vec{p}_1-\vec{\kappa}_1)^2+\lambda^2(1+x)^2]^2}. \tag{73}$$

This also can be reduced to the evaluation of the integral J, Eq. (63),[25]

$$\vec{B}_1=-\frac{m}{4\pi^2}\frac{2a}{1+\gamma}\left(\frac{N}{\Gamma(1-\gamma)}\right)^2\int_0^1 d\rho\,\rho^{-ia}\int_0^\infty\int_0^\infty dx\,dy\,(xy)^{-\gamma}\left[-\lim\kappa X_1^3\frac{1}{4}\frac{\partial^2}{\partial\mu\,\partial\vec{\kappa}_1}\frac{d}{d\rho}\left(\frac{1-\rho^2}{\rho}J\right)\right]. \tag{74}$$

Again, after having taken the derivatives in the large brackets of Eq. (74), one should replace λ and μ by $\lambda(1+x)$ and $\lambda(1+y)$.

From Eq. (65) we find

$$\frac{\partial^2}{\partial\mu\,\partial\vec{\kappa}_1}\frac{d}{d\rho}\left(\frac{1-\rho^2}{\rho}J\right)=64\pi^4\left[\left(\frac{(1-\rho)[X+\mu+\rho(X-\mu)]}{c^2X^2}-\frac{8\rho(1-\rho)(\lambda-\mu)\{(X+\mu)^2+\kappa_2^2-\rho[(X-\mu)^2+\kappa_2^2]\}}{c^3}\right.\right.$$
$$\left.-\frac{8\rho(1-\rho)[X+\mu+\rho(X-\mu)][\Delta^2+(\lambda-\mu)^2]}{c^3}\right)\vec{\kappa}_1$$
$$\left.+8\rho\frac{\{[(X+\mu)+\rho(X-\mu)][(X+\lambda)^2+\kappa_1^2-\rho[(X-\lambda)^2+\kappa_1^2]]-4\rho X^2(\lambda-\mu)\}}{c^3}\vec{\Delta}\right]. \tag{75}$$

With $X=X_1$, we get in high-energy limit

$$-\lim\kappa X_1^3\frac{1}{4}\frac{\partial^2}{\partial\mu\,\partial\vec{\kappa}_1}\frac{d}{d\rho}\left(\frac{1-\rho^2}{\rho}J\right)=+16\pi^4\left[\left(\frac{(1-\rho^2)}{16\bar{c}'^2}-\frac{i\rho(1-\rho)(\lambda-\mu)[E_0+i\mu-\rho(E_0-i\mu)]}{4\bar{c}'^3}\right.\right.$$
$$\left.+\frac{(\lambda-\mu)^2\rho(1-\rho^2)}{8\bar{c}'^3}+\frac{\Delta^2\rho(1-\rho^2)}{8\bar{c}'^3}\right)\nu_i$$
$$\left.+\frac{\rho(1+\rho)[E_0+i\lambda-\rho(E_0-i\lambda)]-2i\rho(\lambda-\mu)}{4\bar{c}'^3}\Delta_i\right]. \tag{76}$$

We next have to make the required substitutions for λ and μ and insert into Eq. (74). By taking into account the decomposition Eq. (47) we find[35]

$$B' = -\frac{\pi^2}{4} m \frac{2a}{1+\gamma}\left(\frac{N}{\Gamma(1-\gamma)}\right)^2 \int_0^1 d\rho\, \rho^{-ia}(1-\rho^2) \int_0^\infty\int_0^\infty dx\, dy\, (xy)^{-\gamma}\left(\frac{1}{\bar{c}^2} + 2\Delta^2\frac{\rho}{\bar{c}^3}\right), \tag{77}$$

$$B'' = -\pi^2 m \Delta \frac{2a}{1+\gamma}\left(\frac{N}{\Gamma(1-\gamma)}\right)^2 \int_0^1 d\rho\, \rho^{-ia+1}(1+\rho) \int_0^\infty\int_0^\infty dx\, dy\, (xy)^{-\gamma}[E_0 + i\lambda - \rho(E_0 - i\lambda) + i\lambda x(1+\rho)]\frac{1}{\bar{c}^3}. \tag{78}$$

Finally, we consider the integral C_{ij} of Eq. (46). Proceeding as before, we get

$$C_{ij} = \lim\left(-\frac{m}{4\pi^2}\right)\left(\frac{2a}{1+\gamma}\right)^2\left(\frac{N}{\Gamma(1-\gamma)}\right)^2 \kappa X_1^3 \int_0^1 d\rho\, \rho^{-ia} \int_0^\infty\int_0^\infty dx\, dy\, (xy)^{-\gamma} \frac{d}{d\rho}\left(\frac{1-\rho^2}{\rho} W_{ij}\right), \tag{79}$$

where

$$W_{ij} = \iint \frac{(p_{1i} - \kappa_{1i})(p_{2j} - \kappa_{2j})\, d\vec{p}_1\, d\vec{p}_2}{[(\vec{p}_2 - \vec{\kappa}_2)^2 + \lambda^2(1+y)^2]^2[X_1^2(\vec{p}_2 - \vec{p}_1)^2 + \alpha(p_1^2 + X_1^2)(p_2^2 + X_1^2)]^2[(\vec{p}_1 - \vec{\kappa}_1)^2 + \lambda^2(1+x)^2]^2}. \tag{80}$$

Also, this can be expressed in terms of Eq. (63),[25]

$$C_{ij} = -\frac{m}{4\pi^2}\left(\frac{2a}{1+\gamma}\right)^2\left(\frac{N}{\Gamma(1-\gamma)}\right)^2 \int_0^1 d\rho\, \rho^{-ia} \int_0^\infty\int_0^\infty dx\, dy\, (xy)^{-\gamma}\left[\lim\kappa X_1^3 \frac{1}{4} \frac{\partial^2}{\partial\kappa_{1i}\,\partial\kappa_{2j}} \frac{d}{d\rho}\left(\frac{1-\rho^2}{\rho} J\right)\right], \tag{81}$$

with the usual substitutions for λ and μ in the square brackets.

From Eq. (65) we find

$$\begin{aligned}\frac{\partial^2}{\partial\kappa_{1i}\,\partial\kappa_{2j}} \frac{d}{d\rho}\left(\frac{1-\rho^2}{\rho} J\right) = 64\pi^4\Bigg[&\frac{2\rho}{c^2}\delta_{ij} - \frac{8\rho(1-\rho)\{(X+\mu)^2 + \kappa_2^2 - \rho[(X-\mu)^2 + \kappa_2^2]\} + 32\rho^2X^2}{c^3}\kappa_{1i}\kappa_{1j} \\ &+ \Bigg(\frac{(1-\rho)^2}{c^2X^2} - \frac{8\rho(1-\rho)^2[\Delta^2 + (\lambda-\mu)^2]}{c^3} + \frac{8\rho(1-\rho)\{(X+\lambda)^2 + \kappa_1^2 - \rho[(X-\lambda)^2 + \kappa_1^2]\}}{c^3} \\ &+ \frac{8\rho(1-\rho)\{(X+\mu)^2 + \kappa_2^2 - \rho[(X-\mu)^2 + \kappa_2^2]\}}{c^3} + \frac{32\rho^2X^2}{c^3}\Bigg)\kappa_{1i}\kappa_{2j} \\ &+ \frac{32\rho^2X^2}{c^3}\kappa_{2i}\kappa_{1j} - \frac{8\rho(1-\rho)\{(X+\lambda)^2 + \kappa_1^2 - \rho[(X-\lambda)^2 + \kappa_1^2]\} + 32\rho^2X^2}{c^3}\kappa_{2i}\kappa_{2j}\Bigg].\end{aligned} \tag{82}$$

With $X = X_1$, and in the high-energy limit, Eq. (82) yields

$$\begin{aligned}\lim\kappa X_1^3 \frac{1}{4} \frac{\partial^2}{\partial\kappa_{1i}\,\partial\kappa_{2j}} \frac{d}{d\rho}\left(\frac{1-\rho^2}{\rho} J\right) = 2\pi^4\Bigg[&\frac{i\rho}{\bar{c}'^2}\delta_{ij} - i\left(\frac{(1-\rho)^2}{2\bar{c}'^2} + \frac{\rho(1-\rho)^2[\Delta^2 + (\lambda-\mu)^2]}{\bar{c}'^3}\right)\nu_i\nu_j + \frac{4i\rho^2}{\bar{c}'^3}\Delta_i\Delta_j \\ &+ 2i\frac{\rho(1-\rho)[E_0 + i\mu - \rho(E_0 - i\mu)]}{\bar{c}'^3}\nu_i\Delta_j \\ &- 2i\frac{\rho(1-\rho)[E_0 + i\lambda - \rho(E_0 - i\lambda)]}{\bar{c}'^3}\nu_j\Delta_i\Bigg].\end{aligned} \tag{83}$$

We next perform the substitutions for λ and μ, insert into Eq. (81), and decompose the integral according to Eq. (49). This yields

$$C = -\frac{i\pi^2}{2} m\left(\frac{2a}{1+\gamma}\right)^2\left(\frac{N}{\Gamma(1-\gamma)}\right)^2 \times \int_0^1 d\rho\, \rho^{1-ia} \int_0^\infty\int_0^\infty dx\, dy\, (xy)^{-\gamma}(\bar{c})^{-2}, \tag{84}$$

$$D = \frac{i\pi^2}{4} m\left(\frac{2a}{1+\gamma}\right)^2\left(\frac{N}{\Gamma(1-\gamma)}\right)^2 \times \int_0^1 d\rho\, \rho^{-ia}(1-\rho)^2 \times \int_0^\infty\int_0^\infty dx\, dy\, (xy)^{-\gamma}\left(\frac{1}{\bar{c}^2} + \frac{2\rho[\Delta^2 + \lambda^2(x-y)^2]}{\bar{c}^3}\right), \tag{85}$$

$$E = -i\pi^2 m\Delta\left(\frac{2a}{1+\gamma}\right)^2\left(\frac{N}{\Gamma(1-\gamma)}\right)^2 \int_0^1 d\rho\, \rho^{1-ia}(1-\rho) \times \int_0^\infty\int_0^\infty dx\, dy\, (xy)^{-\gamma} \times \frac{E_0 + i\lambda(1+y) - \rho[E_0 - i\lambda(1+y)]}{\bar{c}^3}, \tag{86}$$

$$F = -2i\pi^2 m\Delta^2\left(\frac{2a}{1+\gamma}\right)^2\left(\frac{N}{\Gamma(1-\gamma)}\right)^2 \times \int_0^1 d\rho\, \rho^{2-ia} \int_0^\infty\int_0^\infty dx\, dy\, (xy)^{-\gamma}(\bar{c})^{-3}. \tag{87}$$

We have so far carried out the momentum-space integrations contained in the integrals (52)–(54) and (43)–(49). We are left with threefold integrals over the parameters ρ, x, y. The integration can

be carried out analytically one step further, thereby reducing the threefold integrals to one-dimensional integrals over hypergeometric functions of two variables of Appell's type F_1.[36]

Indeed, let us first define

$$I_1(\rho)=\frac{1}{[\Gamma(1-\gamma)]^2}\int_0^\infty\int_0^\infty (xy)^{-\gamma}(\bar{c})^{-2}\,dx\,dy\,, \tag{88}$$

$$I_2(\rho)=\frac{1}{[\Gamma(1-\gamma)]^2}\int_0^\infty\int_0^\infty (xy)^{-\gamma}(\bar{c})^{-3}\,dx\,dy\,, \tag{89}$$

$$I_3(\rho)=\frac{1}{[\Gamma(1-\gamma)]^2}\int_0^\infty\int_0^\infty x(xy)^{-\gamma}(\bar{c})^{-3}\,dx\,dy\,, \tag{90}$$

$$I_4(\rho)=\frac{1}{2[\Gamma(1-\gamma)]^2}\int_0^\infty\int_0^\infty (x-y)^2(xy)^{-\gamma}(\bar{c})^{-3}\,dx\,dy\,. \tag{91}$$

We can then write Eqs. (70), (77), (78), and (84)–(87) as follows:

$$A=-\frac{i\pi^2}{4}N^2 m\int_0^1 \rho^{-ia}[(1+4\rho+\rho^2)I_1(\rho)+4\rho(1-\rho)^2(i\lambda)^2 I_4(\rho)+2\rho(1+\rho)^2 I_2(\rho)\Delta^2]\,d\rho\,, \tag{92}$$

$$B'=-\frac{\pi^2}{4}\frac{2a}{1+\gamma}N^2 m\int_0^1 \rho^{-ia}[(1-\rho^2)I_1(\rho)+2\rho(1-\rho^2)I_2(\rho)\Delta^2]\,d\rho\,, \tag{93}$$

$$B''=-\pi^2\frac{2a}{1+\gamma}N^2 m\Delta\int_0^1 \rho^{1-ia}(1+\rho)\{[E_0+i\lambda-\rho(E_0-i\lambda)]I_2(\rho)+i\lambda(1+\rho)I_3(\rho)\}\,d\rho\,, \tag{94}$$

$$C=-\frac{i\pi^2}{2}\left(\frac{2a}{1+\gamma}\right)^2 N^2 m\int_0^1 \rho^{1-ia}I_1(\rho)\,d\rho\,, \tag{95}$$

$$D=+\frac{i\pi^2}{4}\left(\frac{2a}{1+\gamma}\right)^2 N^2 m\int_0^1 \rho^{-ia}[(1-\rho)^2 I_1(\rho)-4\rho(1-\rho)^2(i\lambda)^2 I_4(\rho)+2\rho(1-\rho)^2 I_2(\rho)\Delta^2]\,d\rho\,, \tag{96}$$

$$E=-i\pi^2\left(\frac{2a}{1+\gamma}\right)^2 N^2 m\Delta\int_0^1 \rho^{1-ia}(1-\rho)\{[E_0+i\lambda-\rho(E_0-i\lambda)]I_2(\rho)+i\lambda(1+\rho)I_3(\rho)\}\,d\rho\,, \tag{97}$$

$$F=-2i\pi^2\left(\frac{2a}{1+\gamma}\right)^2 N^2 m\Delta^2\int_0^1 \rho^{2-ia}I_2(\rho)\,d\rho\,. \tag{98}$$

Equations (52) and (92)–(98) then give

$$\begin{aligned}H^{(1)}=-\tfrac{1}{4}i\pi^2 N^2 m\int_0^1 \rho^{-ia}\Big\{&\Big[(1+4\rho+\rho^2)+i\frac{2a}{1+\gamma}(1-\rho^2)-\Big(\frac{a}{1+\gamma}\Big)^2(1-8\rho+\rho^2)\Big]I_1(\rho)\\&+2\rho\Delta^2\Big[(1+\rho)^2+i\frac{2a}{1+\gamma}(1-\rho^2)-\Big(\frac{a}{1+\gamma}\Big)^2(1-6\rho+\rho^2)\Big]I_2(\rho)\\&+4(ia)^2\rho(1-\rho^2)\Big[1+\Big(\frac{a}{1+\gamma}\Big)^2\Big]I_4(\rho)\Big\}\,d\rho\,.\end{aligned} \tag{99}$$

Similar expressions can be found for $K^{(1)}$ and $L^{(1)}$.

The integrals (88)–(91) are calculated in Appendix A. From Eqs. (A25)–(A27) we find

$$I_1(\rho)=[\Gamma(2+2\gamma)/6(i\lambda)^{2-2\gamma}][\tfrac{1}{2}(1+\rho)]^{2\gamma-2}\{[E_0+i\lambda-\rho(E_0-i\lambda)]^2-\rho\Delta^2\}^{-1-\gamma}F_1(1-\gamma;1+\gamma,\tfrac{1}{2};\tfrac{5}{2};z_1,z_2)\,, \tag{100}$$

$$I_2(\rho)=[\Gamma(4+2\gamma)/120(i\lambda)^{2-2\gamma}][\tfrac{1}{2}(1+\rho)]^{2\gamma-2}\{[(E_0+i\lambda)-\rho(E_0-i\lambda)]^2-\rho\Delta^2\}^{-2-\gamma}F_1(1-\gamma;2+\gamma,\tfrac{1}{2};\tfrac{7}{2};z_1,z_2)\,, \tag{101}$$

$$\begin{aligned}I_3(\rho)=&[(1-\gamma)\Gamma(3+2\gamma)/120(i\lambda)^{3-2\gamma}][\tfrac{1}{2}(1+\rho)]^{2\gamma-3}[E_0+i\lambda-\rho(E_0-i\lambda)]\\&\times\{[(E_0+i\lambda)-\rho(E_0-i\lambda)]^2-\rho\Delta^2\}^{-2-\gamma}F_1(2-\gamma;2+\gamma,\tfrac{1}{2};\tfrac{7}{2};z_1,z_2)\,,\end{aligned} \tag{102}$$

$$I_4(\rho)=[(1-\gamma)\Gamma(2+2\gamma)/120(i\lambda)^{4-2\gamma}][\tfrac{1}{2}(1+\rho)]^{2\gamma-4}\{[E_0+i\lambda-\rho(E_0-i\lambda)]^2-\rho\Delta^2\}^{-1-\gamma}F_1(2-\gamma;1+\gamma,\tfrac{3}{2};\tfrac{7}{2};z_1,z_2)\,, \tag{103}$$

with

$$z_1=\rho\Delta^2/\{\rho\Delta^2-[E_0+i\lambda-\rho(E_0-i\lambda)]^2\}\,,$$
$$z_2=[(1-\rho)/(1+\rho)]^2\,. \tag{104}$$

Equations (92)–(104) represent our ultimate analytic result for the amplitudes of Eqs. (58) and (52)–(54), which are expressed as integrals over Appell functions F_1. Note that $H^{(1)}$, Eq. (99), contains three distinct such functions. From here on we must continue by numerical computation. However, before doing so, in the following two sections we shall discuss various aspects of our analytic results.

V. FORWARD SCATTERING

In the case of forward scattering the matrix $\mathfrak{U}$ occurring in the matrix element $\mathfrak{M}_{m_2m_1}$, Eqs. (56)–(57), reduces to

$$\mathfrak{U}_0=H_0(\vec{s}_1\cdot\vec{s}_2)+K_0\vec{\sigma}\cdot(\vec{s}_2\times\vec{s}_1)\,, \tag{105}$$

where subscript zero refers to $\Delta=0$. From Eqs. (54), (94), and (97) we have $L_0=0$.

In this section we first write formulas for H_0 and K_0. Next we discuss the connection of our results with the high-energy limit of the cross sections for photoeffect and pair production. Finally, we give the first terms of the expansions of H_0 and K_0 in powers of a.

The expressions of H for forward scattering are simpler than in the general case. This is due to the fact that the variable z_1 of Eq. (104) vanishes. Therefore by using the general formula

$$F_1(a\,;\,b_1,b_2\,;\,c\,;\,0,z_2)={}_2F_1(a,b_2,c\,;\,z_2)\,,$$

the Appell functions F_1 in Eqs. (100)–(103) reduce to Gauss functions ${}_2F_1$. We then insert these into Eqs. (92)–(96).

For Eq. (92) we find

$$A_0=\frac{2f}{3}\int_0^1\rho^{-ia}(1+\rho)^{2\gamma-2}(1-\rho\zeta)^{-2-2\gamma}\times\{(1+4\rho+\rho^2)\,{}_2F_1(1-\gamma,\tfrac{1}{2},\tfrac{5}{2}\,;\,z_2)+\tfrac{4}{5}(1-\gamma)\rho z_2\,{}_2F_1(2-\gamma,\tfrac{3}{2},\tfrac{7}{2}\,;\,z_2)\}\,d\rho\,, \tag{106}$$

with

$$\zeta=(E_0-i\lambda)/(E_0+i\lambda)=(\gamma-ia)/(\gamma+ia)\,, \tag{107}$$

and [see Eq. (39)]

$$f=-\tfrac{1}{2}i(1+\gamma)(1+2\gamma)a^{2\gamma+1}e^{i\pi(\gamma-1)}(\gamma+ia)^{-2\gamma-2}\,. \tag{108}$$

The two Gauss functions occurring in the curly brackets of Eq. (106) can be reduced to a single one by some transformations.[37] We find

$$A_0=f\int_0^1\rho^{-ia}(1+\rho)^{2\gamma}(1-\rho\zeta)^{-2\gamma-2}\,{}_2F_1(-\gamma,\tfrac{1}{2},\tfrac{3}{2}\,;\,z_2)\,d\rho\,. \tag{109}$$

We apply a similar procedure to transform D_0 of Eq. (96), obtaining

$$D_0=-4f\left(\frac{a}{1+\gamma}\right)^2\int_0^1\rho^{-ia}(1+\rho)^{2\gamma}(1-\rho\zeta)^{-2-2\gamma}\left(\frac{4}{3}\frac{1+\rho+\rho^2}{(1+\rho)^2}\,{}_2F_1(1-\gamma,\tfrac{1}{2},\tfrac{5}{2}\,;\,z_2)-{}_2F_1(-\gamma,\tfrac{1}{2},\tfrac{3}{2}\,;z_2)\right)d\rho\,. \tag{110}$$

We now insert Eqs. (109) and (110), and Eqs. (93) and (95) (taken for $\Delta=0$), into Eq. (52). The result is

$$H_0^{(1)}=f\int_0^1\rho^{-ia}(1+\rho)^{2\gamma}(1-\rho\zeta)^{-2-2\gamma}\left\{{}_2F_1(-\gamma,\tfrac{1}{2},\tfrac{3}{2}\,;z_2)+\frac{4i}{3}\frac{a}{1+\gamma}z_2^{1/2}\,{}_2F_1(1-\gamma,\tfrac{1}{2},\tfrac{5}{2}\,;z_2)+\left(\frac{a}{1+\gamma}\right)^2[F(-\gamma,\tfrac{1}{2},\tfrac{3}{2}\,;z_2)-\tfrac{4}{3}z_2\,{}_2F_1(1-\gamma,\tfrac{1}{2},\tfrac{5}{2}\,;z_2)]\right\}d\rho\,. \tag{111}$$

H_0 appearing in Eq. (105) follows from Eq. (58).

Equations (52) and (53) give

$$K_0^{(1)}=H_0^{(1)}-C_0\,, \tag{112}$$

since $F_0=0$. K_0 is then obtained from Eq. (58).

Next, we want to relate the result of Eq. (111) to the high-energy limit of the total cross section for the photoeffect from the K shell. Let us denote by $\mathfrak{M}_0^{(i)}$ the matrix elements defined by Eqs. (2) and (5), taken for forward scattering ($\vec{\kappa}_1=\vec{\kappa}_2=\vec{\kappa}$), with no change of photon polarization ($\vec{s}_1=\vec{s}_2=\vec{s}$) or electron-spin projection ($m_1=m_2=m$). From Eq. (2) it follows that[38]

$$\mathrm{Im}\mathfrak{M}_0^{(1)}=-(\kappa/4\pi r_0)\sigma^{pe}\,,\quad \mathrm{Im}\mathfrak{M}_0^{(2)}=+(\kappa/4\pi r_0)\sigma^{pp}\,. \tag{113}$$

Here σ^{pe} is the total cross section for the photoeffect involving the photon $\vec{\kappa}$, $\vec{s}$, and a bound K-shell electron in the magnetic state m; σ^{pp} is the total cross section for pair production by the same photon, with the electron left in the state m of the K shell and the positron flying away; $r_0=e^2/m$ is the classical electron radius.

In the high-energy limit we get from Eqs. (50)–(52) and (55)

$$\mathrm{Im}\bar{\mathfrak{M}}_0^{(1)} = \mathrm{Im}H_0^{(1)}\,, \quad \mathrm{Im}\bar{\mathfrak{M}}_0^{(2)} = -\,\mathrm{Im}H_0^{(1)}\,. \tag{114}$$

We thus find the limiting behavior[39]

$$\bar{\sigma}_K^{pe} = \bar{\sigma}_K^{pp} = -\,(8\pi r_0/\kappa)\,\mathrm{Im}H_0^{(1)}\,, \tag{115}$$

where $\bar{\sigma}_K^{pe}$, $\bar{\sigma}_K^{pp}$ are now cross sections for the whole K shell. Equation (115) shows that the two cross sections become equal at high energies, which is a point of interest in connection with the application of dispersion relations to our problem.[40]

Equations (115) and (111) give a formula for calculating the high-energy limit of the photoeffect cross section. This limit was studied by Hall[19] and subsequently in more detail by Pratt,[20] who derived several equivalent expressions for $\bar{\sigma}_K^{pe}$ in terms of double integrals. Let us now show that our result agrees with his.

We first consider $H_0^{(1)*}$, the complex conjugate of Eq. (111), in which we change the integration variable according to $\rho' = 1/\rho$. We thus find

$$\mathrm{Im}\,H_0^{(1)} = (1/2i)(H_0^{(1)} - H_0^{(1)*}) = \frac{1}{2i}\int_0^\infty \rho^{-ia}(1+\rho)^{2\gamma}(1-\rho\xi)^{-2-2\gamma}\{\cdots\}d\rho\,, \tag{116}$$

which contains the same curly brackets $\{\cdots\}$ as Eq. (111). In Eq. (116) we replace the Gauss functions ${}_2F_1$ by their standard integral representation,[41] written in terms of a variable t. By then changing the variables of integration ρ and t according to $\rho = (1+x)/(1-x)$ and $t = y^2$, we obtain Eqs. (47) and (48) of Ref. 20.[42]

Finally, we give the first terms of the series expansions of H_0 and K_0, Eqs. (105) and (58), in powers of a. Although, in principle, the derivation is rather straightforward, in practice it turns out to be extremely tedious. In the case of H_0 the expansion was carried out to order a^4 (included), whereas in the case of K_0 only to order a (included). The calculations are given in Appendix B.

Equations (B14) and (B20) yield

$$H_0 = 1 - \tfrac{1}{2}a^2 + \tfrac{13}{24}a^4 + O(a^5)\,, \tag{117}$$

$$K_0 = \tfrac{1}{3}a^5 a^{2\gamma-2}[\exp(-2a\cos^{-1}a)][(1 - \tfrac{4}{5}\pi a + O(a^2)]\,. \tag{118}$$

In Eq. (118) we have kept some corrective terms of order a^2 which are numerically important.

These quantities were calculated also in Ref. 12, Eq. (3.4), to about the same order in a. However, the results obtained there differ from ours in that the coefficient of a^4 in Eq. (117) is found to be $\frac{5}{24}$ and the one of the leading term in a^5 of Eq. (118) is found to be $\frac{11}{48}$ (instead of $\frac{1}{3}$).[31,43] The numerical computation we have carried out for H_0 (see Sec. VII) favors at low a our results, Eqs. (117) and (118).[44,45]

VI. FORM FACTOR APPROXIMATION AND BEHAVIOR FOR LARGE Δ

As noted by Goldberger and Low, for small a the high-energy-limit matrix element Eqs. (3), (5), (15), and (35) [or, equivalently, (27)] reduces to the form-factor approximation. Let us analyze in some detail the small-a limit of our analytic result, Eqs. (59), (58), and (99)–(104), for all Δ.

Equation (99) shows that $H^{(1)}$ can be considered to be a function of the two variables a and Δ/λ. Consequently, from Eq. (58), the same applies to H:

$$H = H(\Delta/\lambda, a)\,.$$

We are interested in deriving the lowest-order terms of the expansion of H in powers of a, for fixed Δ/λ. We write the expansion as

$$H\left(\frac{\Delta}{\lambda}, a\right) = F_0\left(\frac{\Delta}{\lambda}\right) + aF_1\left(\frac{\Delta}{\lambda}\right) + a^2F_2\left(\frac{\Delta}{\lambda}\right) + \cdots\,. \tag{119}$$

This is only apparently an expansion in a, since the variable Δ/λ also depends on a. Nevertheless, for sufficiently small values of Δ/λ, the expansion is useful, since it successive terms decrease. This can be inferred from the fact that for $\Delta = 0$, $H(0, a)$ reduces to H_0 given in Eq. (117), and, by comparison, the values of $F_0(0)$ are found to be of order 1 while, on the other hand, $a < 1$. For large values of Δ/λ the expansion becomes meaningless, as we shall see in the following.

In order to obtain $F_0(\Delta/\lambda)$ we can neglect all quantities of order a^2 in Eq. (99). Thus γ can be replaced by 1 and the Appell functions F_1 which are needed in Eqs. (100)–(102) reduce to 1. One can also replace ρ^{-ia} by 1 in the integrand of Eq. (99) (lowest-order Born approximation of the Green's function, free-particle intermediate states) and neglect some terms of order a, to get[46]

$$H^{(1)} = -2ia^3\int_0^1\left(\frac{1+4\rho+\rho^2}{\{[1+ia-\rho(1-ia)]^2-\rho(\Delta/m)^2\}^2} + 2\left(\frac{\Delta}{m}\right)^2\frac{\rho(1+\rho)^2}{\{[1+ia-\rho(1-ia)]^2-\rho(\Delta/m)^2\}^3}\right)d\rho\,.$$

After integrating and dropping some more terms of order a, we find

$$F_{NR} \equiv F_0(\Delta/\lambda) = [1 + (\Delta/2\lambda)^2]^{-2}\,. \tag{120}$$

In order to obtain $F_1(\Delta/\lambda)$ of Eq. (119) we can again set $\gamma = 1$ in Eq. (99), but we must now retain all terms of order a (including those contributed by ρ^{-ia}). The derivation is somewhat involved and

we shall omit it. The result is $F_1(\Delta/\lambda)=0$.

F_{NR} of Eq. (120) represents the *nonrelativistic* form factor.[47] Since the corrective terms in Eq. (119) are of order a^2, it is expected to approximate quite well $H(\Delta/\lambda, a)$ for very-low-Z atoms, like hydrogen, and not too large Δ/λ. However, as stated before, the form-factor approximation is not uniform with respect to Δ/λ. For large values of this variable, the approximation breaks down, whatever the value of Z, and the asymptotic behavior of $H(\Delta/\lambda, a)$ with respect to Δ is different from the one given by Eq. (120).

In order to find the correct asymptotic behavior we have to start from the integral of Eq. (99). For $\Delta\to\infty$ it is only the neighborhood of $\rho=0$ which contributes. This is because for any $\rho\neq 0$ the factors in curly brackets, $\{\cdots\}$, in Eqs. (100)–(103) tend to zero, whereas for $\rho=0$ they remain finite. (The functions F_1 stay finite for all ρ and Δ). Hence it is convenient to divide the integration interval $(0,1)$ into two parts by a point r/Δ, where r is some constant. On the interval $(0, r/\Delta)$ we can neglect powers of ρ with respect to 1. Therefore the variable z_2 becomes equal to 1 and the functions F_1 reduce to ${}_2F_1$.[48] However, the term $\rho\Delta^2$ appearing in the curly brackets of Eqs. (100)–(103) must be kept, since it can be large. We next change the integration variable to $u=\rho\Delta^2/r$ and consequently the integration interval becomes $(0,\Delta)$. Besides, the integrand becomes independent of Δ, while the power Δ^{-2+2ia} factors out. Since the integral can be extended to the interval $(0,\infty)$ without perturbing the dominant asymptotic behavior, the latter will be given by Δ^{-2+2ia}.[49] Thus we get the following contribution to $H^{(1)}$:

$$\bar{H}^{(1)}\simeq -h(a)(\Delta/m)^{-2+2ia}\ . \tag{121}$$

The coefficient $h(a)$ can be calculated exactly; we give only the result

$$\begin{aligned} h(a)=+\,&2^{2\gamma-1}e^{-\pi a}e^{-i\pi(1-\gamma)}a^{2\gamma+2}[(1+\gamma)\Gamma(1+2\gamma)]^{-1}\\ &\times\Gamma(1-ia)(1+\gamma+ia)^2[\Gamma(\gamma+ia)]^2\\ &\times\{\Gamma(1+ia)(\gamma+ia)^{2\gamma+2ia}\}^{-1}\ . \end{aligned} \tag{122}$$

To lowest order in a we find

$$\bar{H}^{(1)}\simeq -2a^4(\Delta/m)^{-2}\ . \tag{123}$$

Let us denote by $\hat{H}^{(1)}$ the contribution of the interval $(r/\Delta, 1)$. On this interval we can neglect finite quantities with respect to $\rho\Delta^2$. Thus the variable z_1 reduces to 1 and again the F_1 functions become ${}_2F_1$.[48] By expanding the integrand in powers of ρ one finds upon integrating that the contribution of the dominant term in Δ is of order

$$\hat{H}^{(1)}\simeq O(\Delta^{-2-\gamma+ia}\ln\Delta)\ . \tag{124}$$

By comparing Eqs. (121) and (124) we see that for $a<1$ it is $\bar{H}^{(1)}$ which dominates for $\Delta\to\infty$ and therefore

$$H=2\,\mathrm{Re}H^{(1)}\simeq 2\,\mathrm{Re}\bar{H}^{(1)}\ . \tag{125}$$

This shows that asymtotically H is negative, in disagreement with F_{NR} Eq. (120), and besides, it decreases more slowly than the latter. However, it should be kept in mind that Eq. (125) is valid only for extremely large Δ. Note also that, to lowest order in a, the asymptotic term Eqs. (125), (123) would belong to the term $a^2F_2(\Delta,\lambda)$ of Eq. (119).

In order to understand what happens for moderately large values of Δ ($\Delta\gg\lambda$) we note that F_{NR} and the asymptotic term Eq. (123) are both of order a^4. In order that F_{NR} be still a good approximation to H, Δ should be such that Eq. (120) is large with respect to Eq. (125), that is,

$$(\Delta^2/4\lambda^2)^{-2}\gg 4a^4(\Delta/m)^{-2}\quad\text{or}\quad(\Delta^2/m^2)\ll 4\ .$$

In terms of the variable Δ/λ, this becomes $(\Delta/\lambda)^2\ll O(a^{-2})$. This shows that the smaller the Z, the larger the interval of agreement between F_{NR} and H, when both are represented in terms of Δ/λ (as we shall do later; see Table I).

For reasonably large values of Δ, F_{NR} and the asymptotic term of Eqs. (121) and (125), both contained in H [consider Eq. (119)], are not very different in magnitude. Besides, they are of opposite sign. Therefore it is expected that the amplitude H will pass from positive values (which are below those predicted by F_{NR}) to negative values, tending to zero afterwards).[50,51]

VII. NUMERICAL METHODS

The evaluation of the basic integral $H^{(1)}$, Eqs. (99)–(104), was done in two stages: first, the computation of the Appell functions F_1 contained in the integrand, and then the integration over ρ.

The most accurate and fastest way of computing the functions F_1 is to sum their series expansions. When possible, we have adopted this approach. The expansion[52] of $F_1(z_1, z_2)$ converges for $|z_1|<1$ and $|z_2|<1$. In our case we indeed have $0<z_2<1$, but z_1 can be smaller or larger than 1 in modulus depending on ρ, Δ^2, and a. When the modulus $|z_1|$ became larger than 0.8, we have switched over to the computation of the F_1 functions from their integral representation.[53]

In what concerns the integration over ρ in Eq. (99), one problem here is the fact that $\rho^{-ia}=\cos(a\ln\rho)-i\sin(a\ln\rho)$ in the integrand has an infinite number of oscillations for $\rho\to 0$. Consequently, we have isolated an interval $(0,\epsilon)$ in which we have approximated the rest of the integrand by a polynomial, the integration then being performed analytically. On the interval $(\epsilon, 1)$ the integration was carried out by Simpson's rule.[54]

TABLE I. K-shell Rayleigh amplitude H for various Z and $\Delta/\alpha Zm$ [see Eq. (59)].

$\frac{\Delta}{\alpha Zm}$ \ Z	1[a]	6	13	29	42	50	60	73	80	82	92	α^{-1}
0.0	1.0000	0.9990	0.9955	0.9785	0.9566	0.9401	0.9167	0.8820	0.8614	0.8552	0.8221	0.3822
0.1	0.9950	0.9941	0.9906	0.9737	0.9519	0.9355	0.9122	0.8778	0.8573	0.8512	0.8183	0.3813
0.2	0.9803	0.9794	0.9759	0.9593	0.9379	0.9219	0.8991	0.8654	0.8454	0.8394	0.8072	0.3788
0.3	0.9565	0.9556	0.9522	0.9361	0.9154	0.8998	0.8778	0.8453	0.8260	0.8202	0.7893	0.3746
0.4	0.9246	0.9237	0.9205	0.9050	0.8851	0.8703	0.8492	0.8183	0.8000	0.7945	0.7651	0.3689
0.5	0.8858	0.8850	0.8819	0.8672	0.8484	0.8343	0.7996	0.7854	0.7682	0.7631	0.7355	0.3618
0.6	0.8417	0.8409	0.8380	0.8241	0.8065	0.7933	0.7748	0.7478	0.7319	0.7271	0.7017	0.3535
0.7	0.7936	0.7929	0.7902	0.7772	0.7608	0.7486	0.7315	0.7067	0.6920	0.6877	0.6645	0.3441
0.8	0.7432	0.7425	0.7400	0.7279	0.7127	0.7014	0.6858	0.6632	0.6499	0.6460	0.6251	0.3339
0.9	0.6916	0.6909	0.6886	0.6774	0.6634	0.6531	0.6389	0.6184	0.6066	0.6030	0.5844	0.3229
1.0	0.6400	0.6394	0.6372	0.6269	0.6141	0.6047	0.5918	0.5735	0.5629	0.5598	0.5433	0.3114
1.1	0.5894	0.5889	0.5869	0.5774	0.5656	0.5571	0.5455	0.5291	0.5198	0.5170	0.5027	0.2996
1.2	0.5407	0.5401	0.5383	0.5295	0.5188	0.5111	0.5006	0.4860	0.4778	0.4754	0.4630	0.2876
1.3	0.4942	0.4973	0.4920	0.4839	0.4741	0.4671	0.4576	0.4447	0.4375	0.4354	0.4248	0.2755
1.4	0.4504	0.4500	0.4484	0.4409	0.4319	0.4255	0.4170	0.4055	0.3993	0.3975	0.3884	0.2635
1.5	0.4096	0.4092	0.4077	0.4007	0.3925	0.3866	0.3790	0.3688	0.3633	0.3617	0.3541	0.2516
1.6	0.3718	0.3714	0.3700	0.3635	0.3559	0.3505	0.3436	0.3345	0.3298	0.3284	0.3220	0.2400
1.7	0.3370	0.3367	0.3354	0.3293	0.3222	0.3173	0.3109	0.3028	0.2987	0.2976	0.2922	0.2287
1.8	0.3052	0.3049	0.3037	0.2980	0.2913	0.2868	0.2810	0.2737	0.2701	0.2691	0.2647	0.2177
1.9	0.2763	0.2760	0.2748	0.2694	0.2632	0.2589	0.2536	0.2470	0.2439	0.2430	0.2395	0.2072
2	0.2500	0.2497	0.2486	0.2434	0.2376	0.2336	0.2286	0.2227	0.2199	0.2192	0.2164	0.1970
2.1	0.2262	0.2259	0.2248	0.2200	0.2144	0.2107	0.2060	0.2006	0.1982	0.1976	0.1953	0.1873
2.25	0.1948	0.1945	0.1935	0.1889	0.1838	0.1803	0.1761	0.1713	0.1693	0.1688	0.1673	0.1736
2.4	0.1680	0.1677	0.1667	0.1624	0.1576	0.1543	0.1505	0.1462	0.1445	0.1441	0.1431	0.1608
2.5	0.1523	0.1520	0.1511	0.1469	0.1422	0.1392	0.1355	0.1315	0.1300	0.1296	0.1289	0.1529
2.6	0.1382	0.1379	0.1370	0.1330	0.1285	0.1255	0.1220	0.1182	0.1169	0.1166	0.1162	0.1453
2.75	0.1197	0.1194	0.1186	0.1147	0.1104	0.1076	0.1043	0.1008	0.0997	0.0994	0.0993	0.1348
2.9	0.1039	0.1037	0.1028	0.0991	0.0950	0.0923	0.0892	0.0860	0.0850	0.0848	0.0849	0.1251
3.00	0.0947	0.0944	0.0936	0.0900	0.0860	0.0834	0.0804	0.0774	0.0764	0.0763	0.0765	0.1191
3.5	0.0606	0.0604	0.0597	0.0565	0.0529	0.0507	0.0481	0.0457	0.0451	0.0450	0.0455	0.0940
4	0.0400	0.0398	0.0392	0.0363	0.0332	0.0312	0.0290	0.0271	0.0267	0.0266	0.0273	0.0753
4.5	0.0272	0.0270	0.0265	0.0233	0.0212	0.0195	0.0176	0.0160	0.0157	0.0157	0.0164	0.0614
5	0.0190	0.0189	0.0184	0.0161	0.0137	0.0122	0.0106	0.0094	0.0092	0.0092	0.0099	0.0510
5.5	0.0136	0.0135	0.0130	0.0110	0.0089	0.0076	0.0063	0.0053	0.0052	0.0052	0.0059	0.0430
6	0.0100	0.0099	0.0095	0.0077	0.0058	0.0047	0.0036	0.0028	0.0028	0.0028	0.0035	0.0369
7	0.0057	0.0056	0.0053	0.0038	0.0024	0.0016	0.000 83	0.000 38	0.000 45	0.000 51	0.0012	0.0283
8	0.0035	0.0034	0.0031	0.0019	0.000 83	0.000 24	-0.000 29	-0.000 51	-0.000 37	-0.000 30	0.000 34	0.0228
9	0.0022	0.0021	0.0019	0.000 97	0.000 10	-0.000 34	-0.000 70	-0.000 76	-0.000 58	-0.000 50	0.000 12	0.0190
10	0.0015	0.0014	0.0012	0.000 45	-0.000 24	-0.000 57	-0.000 81	-0.000 76	-0.000 55	-0.000 47	0.000 12	0.0163
12	0.000 73	0.000 69	0.000 55	-0.000 00	-0.000 44	-0.000 63	-0.000 72	-0.000 56	-0.000 33	-0.000 23	0.000 32	0.0128
15	0.000 31	0.000 28	0.000 18	-0.000 16	-0.000 41	-0.000 49	-0.000 47	-0.000 24	-0.000 023	0.000 052	0.000 50	0.0096

[a] The values given for $Z=1$ are those of F_{NR} Eq. (120).

For $\Delta/\lambda<7$, the estimated relative error on $H^{(1)}$ is less than 10^{-5}. For $\Delta/\lambda>7$, when the computation becomes more difficult and time consuming, the relative error is about 10^{-3}.

$H^{(1)}$ was computed for values of Z such that $6 \leq Z \leq 137$. For small Z, such as $Z<6$, one is faced with severe problems in the ρ integration.[55] However, already for $Z=6$ the Δ/λ dependence of H, obtained from Eq. (58), nearly coincides for $\Delta/\lambda<15$ (the range we consider) with F_{NR} of Eq. (120).

Our results for H, Eq. (58), are contained in Table I. We have considered only values of $\Delta/\lambda < 15$, because beyond this H becomes very small and Rayleigh scattering is negligible with respect to the other competing scattering processes. The values of Z chosen are distributed rather uniformly from 1 to 92. For the reason explained above, for $Z=1$ we give the values of F_{NR}, Eq. (120).

Table I also contains results for $Z=\alpha^{-1}$ ($a=1$). This case represents the high-Z limit of any relativistic quantum-mechanical calculation based on the Dirac equation for a point nucleus. Although nonphysical, we have considered it because it sheds additional insight into our problem. Besides, in this case the general analytic expressions obtained for $H^{(1)}$ become simpler[56] and have been calculated numerically independently of the main computation. We thus had the possibility of checking the latter.

The computation of $K_0^{(1)}$, given by Eqs. (112) and (111), proceeds along the same lines as for $H^{(1)}$ but is simpler because instead of Appell functions, in the integrand we now have Gauss functions and, besides, $\Delta=0$. The results for K_0 of Eq. (58) are presented in Table II. For $Z=1$ we encountered difficulties in the computation, similar to the ones described above for $H^{(1)}$, so that we have used instead the expansion Eq. (118), which is quite accurate in this case.[45]

Besides the checks on the analytical part of the calculations we have mentioned already, we have also made some checks on the computation. These refer mainly to the case $\Delta=0$ which, however, is a special case of the general program for $\Delta\neq 0$. For example, we checked the Z dependence of $\mathrm{Im}H_0^{(1)}$ by using Eq. (115) and the existing results for σ_K^{pe}; the agreement was excellent.[57] Also, we compared the numerical results for H_0, K_0 at small a with the values obtained from the series expansions (117) and (118) and found good agreement.[44,45] For $\Delta\neq 0$, a qualitative check is offered by the dependence of H on Δ/λ at values of Z which are either small or very large (nearly equal to α^{-1}). In the former case H nearly coincides with F_{NR}, whereas in the latter (e.g., for $Z=136$) we found that its behavior becomes close to the one derived for $Z=\alpha^{-1}$ by independent calculation.

VIII. DISCUSSION

The Rayleigh-matrix element at high energies for a single K-shell electron is given by Eqs. (56)–(58) in terms of the three amplitudes H, K, and L. The result for the full K shell is expressed in terms of H only [see Eq. (59)]. We now want to comment on the numerical values obtained for the latter in Table I.

Let us first study the Δ/λ dependence of H, for fixed Z. For $Z=1$, H practically coincides with the nonrelativistic form factor F_{NR} given by Eq. (120),[58] which is a monotonically decreasing function of Δ/λ. For not too large values of Z (e.g., $Z\leq 50$) and moderate Δ/λ (e.g., $\Delta/\lambda\leq 3$), H is not very different from F_{NR}. The agreement is better for small Z and Δ/λ. In fact, the reason we have chosen to represent H as a function of Δ/λ [rather than of Δ/m or $\Delta/Z^2\mathcal{R}$] is to illustrate that F_{NR} is the underlying structure of H and to display the Z dependence of the corrections. For large Δ/λ and Z these are quite important.

H remains monotonically decreasing with respect to Δ/λ over the entire range $0\leq\Delta/\lambda\leq 15$, for all $Z\leq 29$. However, for $42\leq Z\leq 82$, H has a negative minimum, which occurs for Δ/λ larger than 8. For $Z=92$ the minimum still exists, but the corresponding value of H is then positive and small. In the limiting case of $Z=\alpha^{-1}$ the minimum has disappeared and H is again monotonically decreasing. This behavior agrees with the qualitative description given in Sec. VI, based on the interference of F_{NR} with the term in H which is dominant for asymptotic Δ/λ.

Let us also consider the variation of H when Δ/λ is kept fixed and Z is increased. For forward scattering ($\Delta=0$), H decreases monotonically with Z, starting from a value $H=1$. The decrease is rather slow for $Z\leq 92$, but then as Z approaches

TABLE II. Electron spin-flip amplitude K_0 for forward scattering and various Z [see Eqs. (56) and (105)].

Z	K_0	Z	K_0
1[a]	0.662×10^{-11}	60	-0.109×10^{-3}
3	0.148×10^{-8}	73	-0.118×10^{-2}
6	0.422×10^{-7}	80	-0.242×10^{-2}
13	0.149×10^{-5}	82	-0.290×10^{-2}
29	0.388×10^{-4}	92	-0.636×10^{-2}
42	0.105×10^{-3}	120	-0.328×10^{-1}
50	0.100×10^{-3}	a^{-1}	-0.808×10^{-1}

[a] Evaluated with Eq. (118).

the critical value α^{-1}, H becomes appreciably smaller than 1.

The decrease of H with Z is maintained for all Δ/λ up to 2.1. However, beginning with $\Delta/\lambda = 2.25$ a minimum appears in our table for $Z = 92$. The minimum shifts to smaller Z as Δ/λ increases. For $\Delta/\lambda \geq 8$, the minimum of H is negative but becomes shallower. The *minimum minimorum* in our table occurs for $Z = 60$ and $\Delta/\lambda = 10$.

The preceding discussion referred to the dependence of H on Z for a fixed value of the variable Δ/λ, which depends itself on Z. It can be easily seen that one does not obtain in general a monotonic variation of H with Z, even if Δ is taken to be the other variable.

The electron spin-flip amplitudes of Eqs. (56)–(58) were considered by us only in the case of forward scattering. The dependence of K_0 on Z is given in Table II, while $L_0 = 0$. K_0 has a remarkable behavior. It starts by being very small and positive for small values of Z such as $Z = 1$. It then increases many orders of magnitude when Z grows up to 50. Between $Z = 50$ and $Z = 60$ it drops to zero and becomes negative. The values for $Z = 50$ and $Z = 60$ are nearly the same in absolute value. Beyond $Z = 60$, K_0 becomes more and more negative until it attains its lowest value for $Z = \alpha^{-1}$. We cannot offer a physical explanation of why the probability of electron spin flip should vanish when the strength of the Coulomb field varies from $Z = 50$ to $Z = 60$.

We shall now compare our results with previous ones. Levinger and Rustgi[10] have computed the high-energy-limit matrix element $\overline{\mathfrak{M}}(\Delta; Z)$ for lead and forward scattering from the Kramers-Kronig dispersion relation (the generalized Thomas-Reiche-Kuhn sum rule).[13,12] The result they found was $\frac{1}{2}\overline{\mathfrak{M}}_K(0; 82) = 0.86$. As noted in Sec. I this cast doubt on the suggestion in Ref. 13, according to which $\frac{1}{2}\overline{\mathfrak{M}}_K(0; Z)$ should have been equal to 1. The good agreement with our value 0.8552 given in Table I is somewhat fortuitous, because some of the input data they used have been meanwhile recognized as incorrect.[59]

Our result for $\overline{\mathfrak{M}}_K(\Delta; Z)$ was derived in the limit of high photon energies (mathematically, $\kappa \to \infty$) and finite momentum transfers Δ. A natural question is to what extent the result can be applied to the case of finite κ. In order to discuss this question we need to digress regarding the general form of the Rayleigh matrix element $\mathfrak{M}_K$.

Rotation invariance requires that in the case of a closed shell or an atom with closed shells, the matrix element should have the following form:

$$\mathfrak{M} = M(\vec{s}_1 \cdot \vec{s}_2^{\,*}) + N(\vec{s}_1 \cdot \vec{\nu}_2)(\vec{s}_2^{\,*} \cdot \vec{\nu}_1) , \tag{126}$$

where $\vec{\nu}_1$ and $\vec{\nu}_2$ are the unit vectors of $\vec{\kappa}_1$ and $\vec{\kappa}_2$.[60] The amplitudes M and N are in general complex.

When the matrix element for one electron is defined as in Eq. (2), the differential cross section for polarized initial and final photons is

$$d\sigma_p = r_0^2 |\mathfrak{M}|^2 \, d\Omega , \tag{127}$$

with $r_0 = e^2/m$. By averaging over the initial and summing over the final photon polarizations, we get

$$\begin{aligned} d\sigma &= \tfrac{1}{2} r_0^2 [\, |\mathfrak{M}_\perp|^2 + |\mathfrak{M}_\parallel|^2] \, d\Omega \\ &= r_0^2 [\, |\mathfrak{M}_{NSF}|^2 + |\mathfrak{M}_{SF}|^2] \, d\Omega , \end{aligned} \tag{128}$$

where[61]

$$\mathfrak{M}_\perp = M, \quad \mathfrak{M}_\parallel = M\cos\theta - N\sin^2\theta ; \tag{129}$$

$$\begin{aligned} \mathfrak{M}_{SF} &= \tfrac{1}{2}[M(\cos\theta - 1) - N\sin^2\theta] , \\ \mathfrak{M}_{NSF} &= \tfrac{1}{2}[M(\cos\theta + 1) - N\sin^2\theta] . \end{aligned} \tag{130}$$

Therefore the process can be described either by the amplitudes (129), or (130). The existing numerical calculations have used both alternatives.

In the high-energy limit and for finite momentum transfers Δ it is only the first term of Eq. (126) which survives [see Eq. (59)].[62] If we want to apply Eq. (59) at lower photon energies and finite scattering angles, this suggests the following approximate forms for the amplitudes Eq. (129) for the K shell:

$$\tilde{\mathfrak{M}}_\perp = 2H, \quad \tilde{\mathfrak{M}}_\parallel = 2H\cos\theta . \tag{131}$$

We can now discuss the conditions under which Eq. (131) should be a valid approximation to Eq. (129). A significant fact is that the *high-energy-limit* result H reduces for small a and sufficiently small Δ/λ to F_{NR}, Eq. (120), which is a *nonrelativistic* approximation (see Sec. VI). The assumptions made in nonrelativistic theory in deriving the form factor are that orders a^2 and κ/m can be neglected, but that κ is large enough with respect to the *binding energy* of the electron $I(Z)$ [in the relativistic Coulomb case we have $I(Z) = m(1 - \gamma)$]. Combining this with our results one can expect that Eq. (131) will give a uniform approximation in κ from nonrelativistic energies $\kappa \gg I$ up to extremely high relativistic energies, at least for small a and Δ/λ not too large.[63] Moreover, the statement seems to be true also for large values of a. It is hard to prove this directly by analyzing the magnitude of the terms neglected in deriving Eq. (59), because of their multitude and complexity. However, a qualitative analysis we have made indicates that this is the case. We also find that in order to have the agreement of the matrix elements $\mathfrak{M}_K$ and $\overline{\mathfrak{M}}_K$, Δ should be limited by a condition of the form $\Delta^2/\lambda^2 \ll O(\kappa/m)$.

However, it is more conclusive to compare our numerical results with the existing computations at higher energies (Brown and Mayers,[7] Cornille and Chapdelaine,[8] Johnson and Lin[64]).[65] The comparison is presented in Table III.

For $Z=13$ and $\kappa=0.08m=17.7I$, $\frac{1}{2}\mathrm{Re}\mathfrak{M}_\perp$ is very well approximated by $\frac{1}{2}\tilde{\mathfrak{M}}_\perp$ and F_{NR}. By comparing $\frac{1}{2}\tilde{\mathfrak{M}}_\parallel$ with $\frac{1}{2}\mathrm{Re}\mathfrak{M}_\parallel$ one sees that the difference is small, except about 90°, but there it is irrelevant since $\frac{1}{2}\mathrm{Re}\mathfrak{M}_\parallel$ is small anyway with respect to $\frac{1}{2}\mathrm{Re}\mathfrak{M}_\perp$. The imaginary parts of $\frac{1}{2}\mathfrak{M}_\perp$ and $\frac{1}{2}\mathfrak{M}_\parallel$ are small and give a totally negligible contribution when entered in the cross section, Eq. (128). Thus in the present case, the exact amplitudes $\mathfrak{M}_\perp$ and $\mathfrak{M}_\parallel$ are well approximated by using $\tilde{\mathfrak{M}}_\perp$ and $\tilde{\mathfrak{M}}_\parallel$ with H or F_{NR} in Eq. (131). This was to be expected from what has been said before, since both a (=0.095) and κ/m (=0.08) are small.

For $Z=50$ and $\kappa=m=14.5I$, the agreement depends on θ (or Δ). For $\theta\leq 40°$, $\frac{1}{2}\mathrm{Re}\mathfrak{M}_\perp$ agrees well with $\frac{1}{2}\tilde{\mathfrak{M}}_\perp$, better than with F_{NR}; the effect of the imaginary parts of $\frac{1}{2}\mathfrak{M}_\perp$ and $\frac{1}{2}\mathfrak{M}_\parallel$ is negligible. For $\theta>40°$ the agreement with $\frac{1}{2}\tilde{\mathfrak{M}}_\perp$ rapidly deteriorates and the difference becomes orders of magnitude, whereas F_{NR} remains closer to $\frac{1}{2}\mathrm{Re}\mathfrak{M}_\perp$. Because of this we have omitted the results for larger angles. However, in these cases the matrix element is relatively small.

In the case of $Z=80$, $\kappa=2.56m=13.6I$ and $\kappa=5.12m=27.2I$, for smaller angles $\frac{1}{2}\tilde{\mathfrak{M}}_\perp$ is a better approximation to $\frac{1}{2}\mathrm{Re}\mathfrak{M}_\perp$ than F_{NR}. But then, beginning with about 60°, both are in error by orders of magnitude.

Thus Table III indicates that for $\kappa\gg I$ our high-energy approximation is good when the matrix element is not too small, even if κ is not large with respect to m. The case of $Z=80$ also seems to indicate that the agreement extends to larger

TABLE III. Comparison of $\tilde{\mathfrak{M}}_\perp$, and $\tilde{\mathfrak{M}}_\parallel$, Eq. (131), with the results of other calculations for $\mathfrak{M}_\perp$, $\mathfrak{M}_\parallel$ at higher energies.

θ (deg.)	$\frac{1}{2}\mathrm{Re}\mathfrak{M}_\perp$	$\frac{1}{2}\mathrm{Im}\mathfrak{M}_\perp$	$\frac{1}{2}\tilde{\mathfrak{M}}_\perp$	F_{NR}	$\frac{1}{2}\mathrm{Re}\mathfrak{M}_\parallel$	$\frac{1}{2}\mathrm{Im}\mathfrak{M}_\parallel$	$\frac{1}{2}\tilde{\mathfrak{M}}_\parallel$
			$Z=13$,	$\kappa=0.08m$[a]			
0	1.0055	−0.0042	0.9955	1.0000	1.0055	−0.0042	0.9955
30	0.9165	−0.0041	0.9071	0.9111	0.7925	−0.0034	0.7856
60	0.7260	−0.0039	0.7178	0.7209	0.3603	−0.0016	0.3589
90	0.5495	−0.0037	0.5418	0.5442	−0.0032	0.0005	0.0000
120	0.4306	−0.0035	0.4234	0.4253	−0.2174	0.0021	−0.2117
150	0.3667	−0.0033	0.3596	0.3614	−0.3180	0.0030	−0.3114
180	0.3469	−0.0033	0.3398	0.3415	−0.3469	0.0033	−0.3398
			$Z=50$,	$\kappa=m$[a]			
0	0.9495	−0.0099	0.9401	1.0000	0.9495	−0.0099	0.9401
10	0.8520	−0.0095	0.8428	0.8950	0.8375	−0.0089	0.8300
20	0.6355	−0.0085	0.6280	0.6648	0.5910	−0.0064	0.5901
30	0.4242	−0.0072	0.4180	0.4426	0.3571	−0.0035	0.3620
40	0.2721	−0.0059	0.2657	0.2833	0.1952	−0.0010	0.2035
60	0.1162	−0.0039	0.1086	0.1207	0.0431	0.0014	0.0543
90	0.0447	−0.0023	0.0352	0.0442	−0.0115	0.0016	0.0000
			$Z=80$,	$\kappa=2.56\,m$[b]			
0	0.8494	−0.0237	0.8614	1.0000	0.8494	−0.0237	0.8614
15	0.4963	−0.0182	0.5008	0.5674	0.4730	−0.0138	0.4837
30	0.1640	−0.0106	0.1658	0.1910	0.1265	−0.0027	0.1436
45	0.0541	−0.0077	0.0524	0.0687	0.0199	−0.0011	0.0371
60	0.0213	−0.0060	0.0178	0.0297	−0.0080	−0.0010	0.0089
			$Z=80$,	$\kappa=5.12\,m$[c]			
0	0.8655	−0.0123	0.8614	1.0000	0.8655	−0.0123	0.8614
10	0.3546	−0.0079	0.3534	0.3984	0.3465	−0.0068	0.3480
20	0.0736	−0.0051	0.0728	0.0908	0.0650	−0.0018	0.0684
30	0.0162	−0.0044	0.0151	0.0264	0.0099	−0.0014	0.0131
40	0.0040	−0.0039	0.0028	0.0100	−0.0008	−0.0013	0.0021

[a] $\mathfrak{M}_\perp$ and $\mathfrak{M}_\parallel$ taken from W. R. Johnson and Chien-ping Lin (private communication).
[b] $\mathfrak{M}_\perp$ and $\mathfrak{M}_\parallel$ taken from G. E. Brown and D. F. Mayers [(Ref. 7(d)].
[c] $\mathfrak{M}_\perp$ and $\mathfrak{M}_\parallel$ taken from H. Cornille and M. Chapdelaine (Ref. 8).

values of Δ/λ as κ increases.

We finally consider the interference of Rayleigh scattering by the K-shell electrons and Delbrück scattering, in the energy range where the Rayleigh matrix element can be approximated by our Eq. (131). The results for the Delbrück matrix element will be taken from the recent work of Papatzacos and Mork,[16(a)] where it is denoted by $(\alpha Z)^2 a$. Since the two matrix elements have to be added coherently in the total matrix element for the elastic scattering by the atom, it is important to know their relative phases. The most natural way of finding these is by referring to the S matrix, from which both matrix elements are derived.[66] By doing so, we find that the two matrix elements have been defined in a way such that, in order to be consistent, they have to be subtracted so as to get their contribution to the atomic matrix element, $\mathfrak{M} = (\alpha Z)^2 a - \mathfrak{M}_K$.[67] In the present case the atomic differential cross section is also given by Eqs. (127)–(130).[68,69]

The amplitudes (131) corresponding to the matrix element a were calculated in Ref. 16(a) for several photon energies, in the lowest-order Born approximation (a is then Z independent). We consider here the angular distribution for $\kappa = 10.83$ MeV $= 21.18m$ and $Z = 73$. This is presented in Table IV, together with our results for Eq. (131). Since $\kappa = 138I$, we expect our approximation, Eq. (131), to be quite good.

From Table IV we see that for Ta and angles up to 5°, Rayleigh scattering plays a significant role in the interference with Delbrück scattering. However, beyond 5° the Rayleigh amplitudes tend to become negligibly small.

The Delbrück amplitudes are proportional to Z^2 (in the Born approximation) and have a rather slow decrease with the angle,[70] whereas the K-shell Rayleigh amplitudes decrease rapidly with the angle at high energies, decreasing faster with smaller Z. Taking this into account, it is Rayleigh scattering which is predominant for $\kappa = 10.83$ MeV and $Z < 73$ at small angles, whereas for $Z > 73$ even at small angles it will give a minor contribution (see, however, Ref. 69). As the photon energy increases, the Rayleigh effect will play a role only for decreasing Z and scattering angles.

ACKNOWLEDGMENTS

The authors are grateful to Dr. R. H. Pratt for stimulating discussions and for his hospitality at the University of Pittsburgh. They would also like to thank Dr. W. R. Johnson and Dr. Chienping Lin for communicating their unpublished results. One of us (M.G.) has also enjoyed instructive discussions with Dr. B. Nagel.

APPENDIX A: EVALUATION OF SOME INTEGRALS

We shall now show that all integrals $I(\rho)$, Eqs. (88)–(91), can be evaluated analytically and that they can be derived from the following one:

$$\mathcal{I}_n = \int_0^\infty \int_0^\infty \frac{(xy)^{-\gamma}}{[(1+sx+ty)(1+tx+sy)+u]^n}\, dx\, dy \ . \tag{A1}$$

Here γ, s, t, and u are complex numbers and n is a positive integer. In order that the integral be meaningful these quantities must satisfy certain conditions.

To find them, we change the variables x and y in Eq. (A1) to polar coordinates. It is then easily seen that in order that the integral be convergent near the origin we must have $\mathrm{Re}\gamma < 1$. For it to be convergent at infinity one must have $1 - n < \mathrm{Re}\gamma$. Combining these conditions we get

$$1 - n < \mathrm{Re}\gamma < 1 \ . \tag{A2}$$

However, in order that the integral be meaningful, its parameters s, t, and u also need to be such that the denominator of the integrand does not vanish. This is certainly true if s, t, and u are positive numbers. We shall consider this case first.

TABLE IV. Comparison of Rayleigh K-shell amplitudes, Eq. (131), with Delbrück amplitudes of Papatzacos and Mork [Ref. 16(a)] for $Z = 73$ and $\kappa = 10.83$ MeV.

θ (deg.)	$(\alpha Z)^2\,\mathrm{Re}a_\perp$	$(\alpha Z)^2\,\mathrm{Im}a_\perp$	$\tilde{\mathfrak{M}}_\perp$	$(\alpha Z)^2\,\mathrm{Re}a_\parallel$	$(\alpha Z)^2\,\mathrm{Im}a_\parallel$	$\tilde{\mathfrak{M}}_\parallel$
0.01	1.856	1.850	1.764	1.856	1.850	1.764
0.5	1.342	1.711	1.667	1.416	1.751	1.667
1	1.007	1.461	1.418	1.101	1.555	1.418
1.5	0.769	1.229	1.110	0.863	1.356	1.110
2	0.610	1.033	0.820	0.698	1.183	0.819
3	0.429	0.735	0.409	0.508	0.908	0.408
5	0.227	0.394	0.094	0.289	0.568	0.094
10	0.074	0.116	0.001	0.116	0.241	0.001

The denominator of the integral can be expanded in powers of $[u/(1+sx+ty)(1+tx+sy)]$. The series is (uniformly) convergent if $0 \le u < 1$, whenever $s, t \ge 0$ and $x, y \ge 0$; in the following we shall consider u to be restricted by this condition. We therefore can interchange the order of the summation and integration to find

$$g_n = \sum_{p=0}^{\infty} \frac{(n)_p (-u)^p}{(1)_p} g_n^p , \tag{A3}$$

with

$$g_n^p = \int_0^\infty \int_0^\infty (xy)^{-\gamma} (c_1 c_2)^{-n-p} \, dx \, dy , \tag{A4}$$

$$c_1 = 1 + sx + ty, \quad c_2 = 1 + tx + sy . \tag{A5}$$

In order to calculate the integral Eq. (A4) we start from

$$\frac{1}{c_1 c_2} = \int_0^1 \frac{dz}{[c_1 z + c_2 (1-z)]^2} . \tag{A6}$$

By differentiating this with respect to c_1 and c_2 we get

$$\frac{1}{(c_1 c_2)^m} = \frac{1}{[\Gamma(m)]^2} \frac{\partial^{2m-2}}{\partial c_1^{m-1} \partial c_2^{m-1}} \frac{1}{c_1 c_2}$$

$$= \frac{\Gamma(2m)}{[\Gamma(m)]^2} \int_0^1 \frac{z^{m-1}(1-z)^{m-1}}{[c_1 z + c_2 (1-z)]^{2m}} \, dz . \tag{A7}$$

We then insert Eq. (A7) into Eq. (A4) and interchange the order of integrations:

$$g_n^p = \frac{\Gamma(2n+2p)}{[\Gamma(n+p)]^2} \int_0^1 dz \, z^{n+p-1} (1-z)^{n+p-1}$$

$$\times \int_0^\infty \int_0^\infty dx \, dy \, \frac{(xy)^{-\gamma}}{[c_1 z + c_2 (1-z)]^{2n+2p}} . \tag{A8}$$

The double integral over x and y now has a denominator which is linear in these variables [see Eq. (A5)]. According to a known formula,[71] we find

$$\int_0^\infty \int_0^\infty dx \, dy \, (xy)^{-\gamma} [c_1 z + c_2 (1-z)]^{-2n-2p}$$

$$= \frac{[\Gamma(1-\gamma)]^2 \, \Gamma(2n+2p+2\gamma-2)}{\Gamma(2n+2p)}$$

$$\times [(s-t)z + t]^{\gamma-1} [s - (s-t)z]^{\gamma-1} . \tag{A9}$$

Therefore

$$g_n^p = [\Gamma(1-\gamma)]^2 \frac{\Gamma(2n+2p+2\gamma-2)}{[\Gamma(n+p)]^2}$$

$$\times \int_0^1 [z(1-z)]^{n+p-1} [(s-t)^2 z(1-z) + st]^{\gamma-1} dz .$$

We decompose this integral into two parts, one corresponding to the interval $(0, \frac{1}{2})$, the other to $(\frac{1}{2}, 1)$, and make in each of them the change of variable $\zeta = 4z(1-z)$. This gives

$$g_n^p = [\Gamma(1-\gamma)]^2 \frac{\Gamma(2n+2p+2\gamma-2)}{[\Gamma(n+p)]^2} \, 2^{1-2n-2p}$$

$$\times \int_0^1 \zeta^{n+p-1} (1-\zeta)^{-1/2} \left(st + (s-t)^2 \frac{\zeta}{4} \right)^{\gamma-1} d\zeta .$$

By using now the standard integral representation of a Gauss function ${}_2F_1$,[72] we find

$$g_n^p = [\Gamma(1-\gamma)]^2 \frac{\Gamma(\frac{1}{2})\Gamma(2n+2p+2\gamma-2)}{\Gamma(n+p)\Gamma(n+p+\frac{1}{2})} \, 2^{1-2n-2p} (st)^{\gamma-1}$$

$$\times {}_2F_1(n+p, \, 1-\gamma, \, n+p+\tfrac{1}{2}; \, -(s-t)^2/4st) . \tag{A10}$$

The coefficient in Eq. (A10) can be written as follows:

$$\frac{\Gamma(\frac{1}{2})\Gamma(2n+2p+2\gamma-2)}{\Gamma(n+p)\Gamma(n+p+\frac{1}{2})}$$

$$= 2^{2n+2p-1} \frac{\Gamma(2n+2\gamma-2)}{\Gamma(2n)} \frac{(n+\gamma-1)_p (n+\gamma-\frac{1}{2})_p}{(n+\frac{1}{2})_p (n)_p} ,$$

where $(a)_p = \Gamma(a+p)/\Gamma(a)$.[73] Putting this back into Eq. (A10) and using one of the transformation formulas for ${}_2F_1$ we get

$$g_n^p = [\Gamma(1-\gamma)]^2 \frac{\Gamma(2n+2\gamma-2)}{\Gamma(2n)}$$

$$\times \frac{(n+\gamma-1)_p (n+\gamma-\frac{1}{2})_p}{(n)_p (n+\frac{1}{2})_p} \left(\frac{s+t}{2} \right)^{2\gamma-1} (st)^{-1/2}$$

$$\times {}_2F_1\left(\tfrac{1}{2}, n+p+\gamma-\tfrac{1}{2}, \; n+p+\tfrac{1}{2}; \; -\frac{(s-t)^2}{4st} \right). \tag{A11}$$

We now insert Eq. (A11) into Eq. (A3) to obtain

$$g_n = [\Gamma(1-\gamma)]^2 \frac{\Gamma(2n+2\gamma-2)}{\Gamma(2n)} \left(\frac{s+t}{2} \right)^{2\gamma-1} (st)^{-1/2}$$

$$\times \sum_{p=0}^{\infty} \frac{(n+\gamma-1)_p (n+\gamma-\frac{1}{2})_p}{(n+\frac{1}{2})_p (1)_p} (-u)^p$$

$$\times {}_2F_1\left(\tfrac{1}{2}, n+p+\gamma-\tfrac{1}{2}, n+p+\tfrac{1}{2}; \; -\frac{(s-t)^2}{4st} \right). \tag{A12}$$

If $0 \le u < 1$, which we have already assumed, the series occurring here can be summed to give a hypergeometric function of two variables, of Appell's type, $F_1(a; b_1, b_2; c; z_1 z_2)$.[52] Equation (A12) thus becomes

$$\mathcal{J}_n=[\Gamma(1-\gamma)]^2[\Gamma(2n+2\gamma-2)/\Gamma(2n)]\,(st)^{-1/2}[\tfrac{1}{2}(s+t)]^{2\gamma-1}\,F_1(n+\gamma-\tfrac{1}{2};n+\gamma-1,\tfrac{1}{2};n+\tfrac{1}{2};-u,-(s-t)^2/4st)\ . \tag{A13}$$

This can be written in an alternative way, by using a transformation property of the F_1 function[74]:

$$\mathcal{J}_n=[\Gamma(1-\gamma)]^2[\Gamma(2n+2\gamma-2)/\Gamma(2n)]\,[\tfrac{1}{2}(s+t)]^{2\gamma-2}(1+u)^{1-\gamma-n}F_1(1-\gamma;n+\gamma-1,\tfrac{1}{2};n+\tfrac{1}{2};u/(u+1),[(s-t)/(s+t)]^2)\ . \tag{A14}$$

We have derived Eqs. (A13) and (A14) under the assumption that $s,t\geq 0$ and $0\leq u<1$. These restrictions can be relaxed by looking at the analyticity properties of the integral Eq. (A1) with respect to its parameters s, t, and u. This is simpler to do after first performing in Eq. (A1) a change of integration variables of the form $sx\rightarrow x$, $ty\rightarrow y$, which yields

$$\mathcal{J}_n=(st)^{\gamma-1}\int_0^\infty\int_0^\infty (xy)^{-\gamma}[(1+x+y)(1+vx+v^{-1}y)+u]^{-n}\,dx\,dy\ ,$$

where $v=t/s$. The preceding integral is analytic with respect to its parameters v,u in the vicinity of any point v_0,u_0 for which the denominator of the integrand does not vanish. The domain of analyticity can be obtained by analytic continuation. We shall not attempt to study the general case here. Instead we shall consider what happens when v_0 is fixed and positive. It is then easily seen that the domain of analyticity of the integral with respect to u coincides with the complex plane of this variable, cut along the real negative axis from $-\infty$ to -1.

On the other hand, by dividing the right-hand side of Eq. (A14) by $(st)^{\gamma-1}$, we get a function which depends only on $v=t/s$ and which is analytic with respect to u. Since the equality (A14) holds for $v>0$ and $0\leq u<1$, by analytic continuation with respect to u it will remain true for any u in the complex plane cut along $(-\infty,-1)$.[75]

The integral Eq. (A1) can be expressed in an alternative manner. By changing the variables according to

$$x\rightarrow x(1+u)^{1/2},\quad y\rightarrow y(1+u)^{1/2}\ , \tag{A15}$$

we can write

$$\mathcal{J}_n=(1+u)^{1-\gamma-n}\,\mathcal{J}_n\ , \tag{A16}$$

$$\mathcal{J}_n=\int_0^\infty\int_0^\infty\frac{(xy)^{-\gamma}}{[1+p(x+y)+qxy+r(x-y)^2]^n}\,dx\,dy\ . \tag{A17}$$

The connection between s,t,u and p,q,r is given by

$$p=(s+t)(1+u)^{-1/2},\quad q=(s+t)^2,\quad r=st\ . \tag{A18}$$

From Eqs. (A14)–(A17) we get

$$\mathcal{J}_n=[\Gamma(1-\gamma)]^2[\Gamma(2n+2\gamma-2)/\Gamma(2n)]\,(\tfrac{1}{4}q)^{\gamma-1}\times F_1(1-\gamma;n+\gamma-1,\tfrac{1}{2};n+\tfrac{1}{2};1-(p^2/q),1-(4r/q))\ . \tag{A19}$$

Let us consider also the integrals

$$\mathcal{K}_n=\int_0^\infty\int_0^\infty\frac{x(xy)^{-\gamma}}{[1+p(x+y)+qxy+r(x-y)^2]^n}\,dx\,dy\ , \tag{A20}$$

$$\mathcal{L}_n=\frac{1}{2}\int_0^\infty\int_0^\infty\frac{(x-y)^2(xy)^{-\gamma}}{[1+p(x+y)+qxy+r(x-y)^2]^n}\,dx\,dy\ . \tag{A21}$$

These can be expressed in terms of $\mathcal{J}_n$ by

$$\mathcal{K}_n=-\frac{1}{2(n-1)}\frac{\partial}{\partial p}\mathcal{J}_{n-1},\quad \mathcal{L}_n=-\frac{1}{2(n-1)}\frac{\partial}{\partial r}\mathcal{J}_{n-1}\ . \tag{A22}$$

For $\mathcal{J}_{n-1}$ occurring here we shall use the expression Eq. (A19). The parameters p and r, with respect to which we have to take the derivatives, are contained only in the variables of the function F_1 in Eq. (A19). By using the appropriate differentiation formula for F_1,[76] we find

$$\mathcal{K}_n=(1-\gamma)[\Gamma(1-\gamma)]^2[\Gamma(2n+2\gamma-3)/2\Gamma(2n)]\,p(\tfrac{1}{4}q)^{\gamma-2}\times F_1(2-\gamma;n+\gamma-1,\tfrac{1}{2};n+\tfrac{1}{2};1-(p^2/q),1-(4r/q))\ , \tag{A23}$$

$$\mathcal{L}_n=(1-\gamma)[\Gamma(1-\gamma)]^2[\Gamma(2n+2\gamma-4)/\Gamma(2n)]\,(\tfrac{1}{4}q)^{\gamma-2}\ .\times F_1(2-\gamma;n+\gamma-2,\tfrac{3}{2};n+\tfrac{1}{2};1-(p^2/q),1-(4r/q))\ . \tag{A24}$$

We can now apply the general formulas Eqs. (A14), (A23), and (A24) to calculate our integrals $I(\rho)$ of Sec. IV. From Eqs. (88) and (89) we have

$$I_1(\rho)=[\Gamma(1-\gamma)]^{-2}[(E_0+i\lambda)-\rho(E_0-i\lambda)]^{-4}\mathcal{J}_2\ ,$$
$$I_2(\rho)=[\Gamma(1-\gamma)]^{-2}[(E_0+i\lambda)-\rho(E_0-i\lambda)]^{-6}\mathcal{J}_3\ , \tag{A25}$$

with $\mathcal{J}_n$ given by Eq. (A1) and

$$s=ia/[(\gamma+ia)-\rho(\gamma-ia)]\ ,$$
$$t=ia\rho/[(\gamma+ia)-\rho(\gamma-ia)]\ , \tag{A26}$$
$$u=-\frac{\rho(\Delta/m)^2}{[\gamma+ia-\rho(\gamma-ia)]^2}\ .$$

Taking into account Eq. (A14) we get Eqs. (100) and (101).[75]

Further, if in Eqs. (90) and (91) we change the integration variables according to Eq. (A15), we can write

$$I_3(\rho)=[\Gamma(1-\gamma)]^{-2}[E_0+i\lambda-\rho(E_0-i\lambda)]^{-6}\times(1+u)^{-\gamma-3/2}\mathcal{K}_3\ ,$$

$$I_4(\rho)=[\Gamma(1-\gamma)]^{-2}[E_0+i\lambda-\rho(E_0-i\lambda)]^{-6}\times(1+u)^{-\gamma-1}\mathcal{L}_3\ , \tag{A27}$$

where $\mathcal{K}_3$, $\mathcal{L}_3$ are the integrals Eqs. (A20) and (A21) with p, q, r calculated from Eq. (A18) in terms of Eq. (A26).[77] By inserting in Eq. (A27) the results Eqs. (A23) and (A24) we find Eqs. (102) and (103).

APPENDIX B: EXPANSIONS IN a FOR FORWARD SCATTERING

In the following we derive the first terms of the expansions in powers of a of H_0 and K_0, defined in Eqs. (105) and (58).

We consider first the case of H_0, whose expansion we want to determine to order a^4 (included). We illustrate our procedure on A_0, Eq. (109), which is the first term of $H_0^{(1)}$ [see Eqs. (58) and (52)].

Since for $0\le\rho\le1$, one has $0\le z_2\le1$ [see Eq. (104)]; we can expand the Gauss function ${}_2F_1$ in the integrand of Eq. (109) in a power series in z_2. This series is uniformly convergent on the interval $0\le\rho\le1$, so that one can interchange the order of the summation and integration. Therefore the result can be expressed as

$$A_0=+f\Gamma(1-ia)\sum_{n=0}^{\infty}\frac{(\frac12)_n(-\gamma)_n}{(\frac32)_n(1)_n}\frac{\Gamma(2n+1)}{\Gamma(2n+2-ia)}F_1(1-ia;\,2n-2\gamma,\,2+2\gamma;\,2n+2-ia;\,-1,\zeta)\ , \tag{B1}$$

with ζ and f given by Eqs. (107) and (108).

We next try to expand the function F_1 in powers of a.[78] For this purpose we use the formula of analytic continuation,[79]

$$\begin{aligned}F_1(a;b_1,b_2;c;x,y)=&\frac{\Gamma(c)\Gamma(a+b_2-c)}{\Gamma(a)\Gamma(b_2)}(1-x)^{b_2-1}(1-y)^{c-a-b_2}(y-x)^{1-b_1-b_2}y^{-c+b_1+b_2}\\&\times F_1\left(1-b_2;1-a,c-b_1-b_2;c-a-b_2+1;\frac{1-y}{1-x},\frac{x(1-y)}{y(1-x)}\right)\\&+\frac{\Gamma(c)\Gamma(c-a-b_2)}{\Gamma(c-a)\Gamma(c-b_2)}(1-x)^{-b_1}y^{-b_2}G_2\left(b_1,b_2;b_2-c+1,c-a-b_2;\frac{x}{1-x},\frac{1-y}{y}\right).\end{aligned} \tag{B2}$$

By applying this formula to the function F_1 in Eq. (B1) and using some of the properties of the Γ functions, A_0 can be decomposed according to

$$A_0=A_0'+A_0''\ , \tag{B3}$$

where

$$A_0'=\frac{1+\gamma}{4\gamma}\zeta^{ia}\sum_{n=0}^{\infty}\frac{(\frac12)_n(\frac12)_n}{(\frac32)_n(-\gamma+\frac12)_n}\left(i\frac{a}{\gamma}\right)^{2n}F_1\left(-1-2\gamma;ia,-ia;2n-2\gamma;\frac{ia}{\gamma+ia},-\frac{ia}{\gamma-ia}\right), \tag{B4}$$

$$A_0''=+f\Gamma(1-ia)2^{2\gamma}\zeta^{-2\gamma-2}\sum_{n=0}^{\infty}\frac{(\frac12)_n(-\gamma)_n}{(\frac32)_n(1)_n}\frac{\Gamma(2n-2\gamma-1)}{\Gamma(2n-ia-2\gamma)}2^{-2n}\times G_2\left(2n-2\gamma,2+2\gamma;2\gamma+ia+1-2n,2n-2\gamma-1;-\tfrac12,\frac{2ia}{\gamma-ia}\right). \tag{B5}$$

The first quantity A_0' is real. The successive terms of its series, Eq. (B4), contain ascending powers of a^2. Furthermore, the functions F_1 in the coefficients can be expanded in double series in their variables, both of order a.[80] Because of the power $(ia/\gamma)^{2n}$, when n increases less terms have to be retained in the expansion of the corresponding F_1 in order to get A_0' to the desired order a^4. However, it should be kept in mind that also the parameters of F_1 depend on a, through $\gamma=(1-a^2)^{1/2}$, and have to be expanded. Proceeding thus, we get, to the order of magnitude needed,

$$\begin{aligned}&n=0:\ F_1=1-5a^2+\tfrac{19}{6}a^4+O(a^6)\ ,\\&n=1:\ F_1=7-\tfrac{17}{2}a^2+O(a^4)\ ,\\&n=2:\ F_1=1+O(a^2)\ .\end{aligned} \tag{B6}$$

Inserting these in Eq. (B4) and expanding also the

other quantities involved we get

$$A_0' = \tfrac{1}{2} - \tfrac{5}{24}a^2 + \tfrac{341}{480}a^4 + O(a^6) \ . \qquad \text{(B7)}$$

The calculation of A_0'', Eq. (B5), is more involved. This is due to the fact that only one of the variables of G_2 is of order a while the other one is independent of a. We start from the series expansion[81]

$$G_2(a_1, a_2; b_1, b_2; x, y) = \sum_{m=0}^{\infty} \frac{(a_1)_m (b_2)_m}{(1-b_1)_m (1)_m} (-x)^m \times {}_2F_1(a_2, b_1 - m, 1 - b_2 - m; -y) \ . \qquad \text{(B8)}$$

By using this for the function G_2 in Eq. (B5), with $x = 2ia/(\gamma - ia)$ and $y = -\frac{1}{2}$, and interchanging the order of summations, we get

$$A_0'' = -f\,\Gamma(1-ia)2^{ia-1} \times \frac{\sin\pi(2\gamma+ia)}{\sin(2\pi\gamma)} \frac{\Gamma(1+2\gamma+ia)}{\Gamma(2+2\gamma)} \xi^{-2\gamma-2} \times \sum_{m=0}^{\infty} \frac{(1+2\gamma+ia)_m}{(1)_m} \left(-\frac{2ia}{\gamma - ia}\right)^m T_m \ , \qquad \text{(B9)}$$

where

$$T_m = \sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n (-\gamma)_n (-\gamma - \frac{1}{2} - \frac{1}{2}m)_n (-\gamma - \frac{1}{2}m)_n}{(\frac{3}{2})_n (1)_n (-\gamma - \frac{1}{2}ia - \frac{1}{2}m)_n (-\gamma - \frac{1}{2}ia + \frac{1}{2} - \frac{1}{2}m)_n} \times {}_2F_1(1 - ia, -ia - m, 2n - 2\gamma - ia - m; \tfrac{1}{2}) \ . \qquad \text{(B10)}$$

The terms of the series in Eq. (B9) contain ascending powers of a. However, the series of coefficients T_m do not have this property. Therefore no systematic simple procedure can be applied in order to get the expansion of A_0''. Nevertheless, we now show that it is possible to extract all terms to order a^4 (included).

The coefficient in front of the sum of Eq. (B9) is of order a^2. Therefore we need take into account only the first terms ($m = 0, 1, 2$) of the series in Eq. (B9). On the other hand, for each of these values of m only a finite number of terms need be considered in the corresponding T_m. This is because $(-\gamma)_n$, for $n \geq 2$, contains the factor $(1-\gamma)$, which is of order a^2. However, we also have factors like $[(-\gamma - \frac{1}{2}ia - p)_n]^{-1}$ in the coefficients ($p \geq 0$ is an integer); these become of order a^{-1} for $n \geq p+2$. By taking such factors into account one concludes that for $m = 0, 2$ one needs two terms ($n = 0, 1$) in the expansions Eq. (B10), whereas for $m = 1$ one needs three ($n = 0, 1, 2$). One is then faced with the expansion in a of functions such as

$${}_2F_1(1 - ia, -p - ia, q - 2\gamma - ia; \tfrac{1}{2}) \ ,$$

with $p = 0, 1, 2$ and $q = 0, \pm 1, \pm 2$. The difficulty here lies in the fact that for $a \to 0$ we have $c \equiv q - 2\gamma - ia \simeq q - 2$, a nonpositive integer, and therefore ${}_2F_1$ becomes singular. However, this can be avoided by applying a number of times the equations among contiguous functions ${}_2F_1$, so as to raise the parameter c to become positive for $a \to 0$. After having done this we expand the functions ${}_2F_1$ in power series of $\frac{1}{2}$. Because of the structure of their parameters only a few terms need be kept. We thus get the expansions of T_0, T_1, T_2 which, inserted in Eq. (B9), give finally

$$A_0'' = -\tfrac{8}{15}a^4 + O(a^5) \ . \qquad \text{(B11)}$$

Hence from Eqs. (B3), (B7), and (B11)

$$A_0 = \tfrac{1}{2} - \tfrac{5}{24}a^2 + \tfrac{17}{96}a^4 + O(a^5) \ . \qquad \text{(B12)}$$

Now, A_0 is only the first term of $H_0^{(1)}$ [see Eqs. (53), (52), and (109)–(111)]. Proceeding similarly for the other terms, we find

$$B_0' = \tfrac{1}{6}a^2 + \tfrac{1}{12}a^4 + O(a^5), \quad C_0 = \tfrac{1}{6}a^2 + \tfrac{1}{8}a^4 + O(a^5), \quad D_0 = \tfrac{1}{3}a^4 + O(a^5) \ . \qquad \text{(B13)}$$

By inserting Eqs. (B12) and (B13) into Eq. (52) we find

$$H_0^{(1)} = \tfrac{1}{2}[1 - \tfrac{1}{2}a^2 + \tfrac{13}{24}a^4 + O(a^5)] \ . \qquad \text{(B14)}$$

Finally, from Eq. (58) we get Eq. (117).

To the order considered in Eq. (B14), $H_0^{(1)}$ is real. Indeed, its imaginary part starts with order a^5, as can be seen from Eq. (115) by noting tnat $\bar{\sigma}_K^{pe}$ is of order $\alpha(\alpha Z)^5$. The first terms of the expansion in a of $\bar{\sigma}_K^{pe}$ were obtained in Ref. 20. By combining this with our Eq. (115) we find

$$\mathrm{Im}H_0^{(1)} = -\tfrac{1}{2}a^5 a^{2\gamma-2} \exp(-2a\cos^{-1}a) \times [1 - \tfrac{4}{15}\pi a + O(a^2)] . \qquad \text{(B15)}$$

We have included here only some terms of order a^2, which are numerically important.

We next consider the amplitude K_0, which is obtained from the imaginary part of $K_0^{(1)}$ [see Eq. (58)]. We want to use Eq. (112), since we already know Eq. (B15). However, we still have to find $\mathrm{Im}C_0$.

From Eq. (95) we have

$$C_0 = \frac{4f}{3}\left(\frac{2a}{1+\gamma}\right)^2 \int_0^1 \rho^{-ia+1}(1+\rho)^{2\gamma-2}(1-\rho\xi)^{-2\gamma-2} \times {}_2F_1(\tfrac{1}{2}, 1-\gamma, \tfrac{5}{2}; z_2)\, d\rho \ , \qquad \text{(B16)}$$

with f defined in Eq. (108). Let us take the com-

plex conjugate of this and then change the integration variable according to $\rho' = 1/\rho$. We end up with the expression in Eq. (B16) with a minus sign in front and the integration interval changed to $(1, \infty)$. Therefore

$$\mathrm{Im}C_0 = \frac{1}{2i}(C_0 - C_0^*) = \frac{2}{3i} f\left(\frac{2a}{1+\gamma}\right)^2 \int_0^\infty \rho^{1-ia}(1+\rho)^{2\gamma-2}(1-\rho\zeta)^{-2\gamma-2} \times {}_2F_1(\tfrac{1}{2}, 1-\gamma, \tfrac{5}{2}; z_2)\, d\rho , \tag{B17}$$

with the same ${}_2F_1$ function as in Eq. (B16). The factor in front of the integral is of order a^5, which is also the lowest order in Eq. (B15). On the other hand, the integral is of order 1. Therefore by setting $\gamma = 1$ in the integrand of Eq. (B17) we neglect terms of order a^7. By doing so the ${}_2F_1$ function reduces to 1, and the integral Eq. (B17) can be expressed in terms of a beta function,[82]

$$\mathrm{Im}C_0 = (8/3i)(a/(1+\gamma))^2 f \times [(-\zeta)^{ia-2} B(2-ia, 2+ia) + O(a^2)] . \tag{B18}$$

This can be simplified by taking into account some properties of the Γ functions[83] and the definition of f, Eq. (108):

$$\mathrm{Im}C_0 = -\tfrac{1}{3} a^{2\gamma+3} e^{-\pi a} \zeta^{ia} \frac{\pi a}{sh\pi a}[1 + O(a^2)] = -\tfrac{1}{3} a^{2\gamma+3} \exp(-2a\cos^{-1}a)[1 + O(a^2)] . \tag{B19}$$

In conjunction with Eq. (B15) this gives

$$\mathrm{Im}K_0^{(1)} = -\tfrac{1}{6} a^5 a^{2\gamma-2}[\exp(-2a\cos^{-1}a)] \times [1 - \tfrac{4}{5}\pi a + O(a^2)] . \tag{B20}$$

We can now combine Eqs. (58) and (B20) to get Eq. (118).

---

*Work supported in part by a joint grant of the National Science Foundation and the National Council for Science and Technology of Romania.

[1]I. Waller, Z. Phys. 58, 75 (1929).
[2]M. Gavrila, Phys. Rev. 163, 147 (1967), the case of the dipole approximation.
[3]M. Gavrila and A. Costescu, Phys. Rev. A 2, 1752 (1970); 4, 1688 (1971); retardation included.
[4]W. Franz, Z. Phys. 95, 652 (1935); 98, 314 (1936).
[5]J. S. Levinger, Phys. Rev. 87, 656 (1952).
[6]G. E. Brown and J. B. Woodward, Proc. Phys. Soc. Lond. A 65, 977 (1952). See also V. G. Gorshkov, A. I. Mikhailov, V. S. Polikanov, and S. G. Sherman, Phys. Lett. 30A, 455 (1969).
[7](a) G. E. Brown, R. E. Peierls, and J. B. Woodward, Proc. R. Soc. Lond. A 227, 51 (1954); (b) S. Brenner, G. E. Brown, and J. B. Woodward, *ibid.* 227, 59 (1954); (c) G. E. Brown and D. F. Mayers, *ibid.* 234, 387 (1955); and (d) G. E. Brown and D. F. Mayers, *ibid.* 242, 89 (1957).
[8]H. Cornille and M. Chapdelaine, Nuovo Cimento 14, 1386 (1959).
[9]Chien-ping Lin, Kwok-tsang Cheng, and W. R. Johnson, Phys. Rev. A 11, 1946 (1975). This calculation includes electron correlation effects beyond the Hartree-Fock approximation for the case of He.
[10]J. S. Levinger and M. L. Rustgi, Phys. Rev. 103, 439 (1956).
[11]W. R. Johnson and F. D. Feiock, Phys. Rev. 168, 22 (1968); D. T. Cromer and D. Liberman, J. Chem. Phys. 53, 1891 (1970).
[12]M. L. Goldberger and F. Low, Phys. Rev. 176, 1778 (1968).
[13]M. Gell-Mann, M. L. Goldberger, and W. Thirring, Phys. Rev. 95, 1612 (1954).
[14]This is the result given by the Klein-Nishina amplitude for forward scattering at all energies (differential cross section equal to r_0^2, with $r_0 = e^2/m$). For an atomic electron the result is still true in nonrelativistic theory (in the dipole approximation or with retardation included); see Refs. 2 and 3.
[15]See T. Erber, Ann. Phys. (N.Y.) 6, 319 (1959).
[16](a) P. Papatzacos and K. Mork, Phys. Rev. D 12, 206 (1975); (b) a review of the theoretical and experimental results on Delbrück scattering by the same authors is given in Phys. Rep. 21, 81 (1975).
[17]L. Hostler, J. Math. Phys. 5, 591 (1964), Secs. II and III.
[18]J. Schwinger, J. Math. Phys. 5, 1606 (1964), Eq. (3′). Equivalent results were obtained by the authors quoted in Ref. 9 of Ref. 2.
[19]H. Hall, Rev. Mod. Phys. 8, 358 (1936).
[20]R. H. Pratt, Phys. Rev. 117, 1017 (1960).
[21]We are using natural units ($\hbar = c = 1$).
[22]A. J. Akhiezer and V. B. Berestetskii, *Quantum Electrodynamics* (Interscience, New York, 1965). See Eqs. (35.8) and (35.8′). The connection of our matrix element to the corresponding S matrix element is $S_{21}^R = -2\pi i\,\delta(\kappa_1 - \kappa_2)(2\pi r_0/\kappa)\mathfrak{M}_{21}$.
[23]See Ref 17, Sec. III. Our G is related to Hostler's K by $G = -K\beta$, but our G_I and G_0 coincide with his [see Ref. 17, Eqs. (3.9) and (2.5)].
[24]By comparing Eq. (2.5) of Ref. 17, which defines G_0, with the first equation of Ref. 18, which defines the nonrelativistic Green's function G_{NR}, we find $G_0 = G_{NR}/2m$. (However, the parameters of G_{NR} have to be modified so as to take into account relativity.) Further, by considering Eq. (3′) of Ref. 18, one gets the integral representation of our Eqs. (18) and (19). This is subject to the condition $\mathrm{Re}\tau < 1$, which is satisfied in our case [see Eqs. (20) and (29)].
[25]We have inverted here the integration with the limit, which implies that we consider the integral to be a continuous function of the variable $1/\kappa$, when this is

in the vicinity of zero. The statement is true because the integrand is a continuous function of the integration variable and of $1/\kappa$, when κ is large. See E. Whittaker and G. Watson, *A Course of Modern Analysis* (Cambridge U.P., Cambridge, 1927), Chap. IV.

[26]See Ref. 25. Besides, the integral is uniformly convergent.

[27]In order to derive Eqs. (25) and (17), G_0 must take into account the whole Coulomb potential a/r. If G_0 does not contain its contribution (or some part of it), Eq. (14) will still hold with W modified to include the term $2a\,\Omega_1/r$ (or part of it). However, Eq. (24) will no longer be true because this term will give a finite contribution in the limit $\kappa \to \infty$.

[28]To lowest order (in the Pauli approximation) they contain the factor $(p^2+a^2m^2)^{-2}$.

[29]From this it would seem that $\kappa \to \infty$ means physically that $\kappa \gg m$. This is not so in the case of the total K-shell matrix element $\mathfrak{M}_K$, given by Eq. (3). Indeed, by retaining in Eq. (13) only that part of G_I which gives the leading contribution for small a at high energies [i.e., $(2m\kappa)^{-1}\delta(\vec{p}_2-\vec{p}_1)$], the terms $m\beta+E_0$ contained in $\mathfrak{M}_{mm}^{(1)}$ cancel with the corresponding ones of $\mathfrak{M}_{mm}^{(2)}$ when one sums over the magnetic quantum number m [see Eqs. (30) and (31)]. Thus the terms neglected are not of order m/κ but rather of order a^2m/κ.

[30]Since $H(\vec{P})$ is a polynomial in the components of $\vec{P}$, if we denote its eigenvalues and eigenfunctions by E_n and $u_n(\vec{r})$, the eigenvalues and eigenfunctions of $H(\vec{P}+\vec{\kappa})$ will be E_n and $u_n(\vec{r})e^{-i\vec{\kappa}\cdot\vec{r}}$. The equality Eq. (34) then follows by applying both sides to an arbitrary function $\phi(\vec{r})=\sum_n c_n u_n(\vec{r})$ and taking into account the way that the resolvent $[H(\vec{P})-\omega]^{-1}$ acts on it.

[31]Our matrix element (2) is related to theirs by $f^{1/2}=-r_0\mathfrak{M}$ [see Ref. 12, Eq. (2.7)].

[32]Equations (36)–(38) are obtained by taking the Fourier transforms of the K-shell spinors in coordinate space and by using an integral representation for the power $r^{\gamma-1}$ they contain. This is taken from Ref. 33, Eq. (5), p. 1, with $z=1-\gamma$ and $s=r$. The procedure was used previously by Boyer, Ref. 34.

[33]A. Erdélyi, W. Magnus, F. Oberhettinger, and F. G. Tricomi, *Higher Transcendental Functions* (McGraw-Hill, New York, 1953), Vol. I. This will be referred to in the following by HTF.

[34]R. H. Boyer, Phys. Rev. 117, 475 (1960).

[35]Some terms cancel because of symmetry properties of the integrand with respect to x and y.

[36]The most comprehensive treatment of hypergeometric functions of several variables is found in P. Appell and J. Kampé de Fériet, *Fonctions Hypergéometriques et Hypersphériques* (Gauthier-Villars, Paris, 1926).

[37]We first apply the last equation on p. 21 of Ref. 36, taken for $y=0$ and $\alpha \to 1-\gamma$, $\beta \to \frac{1}{2}$, and $\gamma \to \frac{5}{2}$. Therefore the terms in the curly brackets in Eq. (106) become $(1-\rho)^2\,{}_2F_1(1-\gamma,\frac{1}{2},\frac{5}{2};z_2)+6\rho\,{}_2F_1(1-\gamma,\frac{1}{2},\frac{3}{2};z_2)$. Further, by using Ref. 33, Eq. (43), p. 104, with $a=\frac{1}{2}$, $b=1-\gamma$, and $c=\frac{3}{2}$, we get Eq. (109).

[38]Equations (113) can be proved by using in Eq. (2) the equality $\mathrm{Im}(E_n-E_0\mp\kappa\mp i\epsilon)^{-1}=\pm\pi\delta(E_n-E_0\mp\kappa)$. By adding Eqs. (113) one obtains the optical theorem for Rayleigh scattering by a bound electron [see Ref. 15, Eq. (12.1)].

[39]The right-hand sides in Eqs. (114) are independent of the magnetic state m. Combining with Eq. (113) it follows that for the K shell the high-energy cross sections $\bar{\sigma}^{pe}$ and $\bar{\sigma}^{pp}$ are themselves independent of m and $\bar{\sigma}_K=2\bar{\sigma}$. (This can be proved directly and is valid for any energy κ.)

[40]Indeed, the difference $\sigma_K^{pe}-\sigma_K^{pp}$ will therefore behave like $1/\kappa^2$ at high energies, which ensures the convergence of the integral in the dispersion relation (1.1a) of Ref. 15, or in the second equation on p. 1778 of Ref. 12.

[41]HTF (Ref. 33), p. 114, Eq. (1).

[42]The factor $(-2\xi+3)$ contained in Eq. (47) of Ref. 20 should read $-(2\xi+3)$.

[43]No indication is given in Ref. 12 as to how the authors derived their Eq. (3.4).

[44]For H_0 the numerical computation gives $H_0=0.995\,54$ for $Z=13$ and $H_0=0.978\,51$ for $Z=29$. The corresponding values given by Eq. (117) are 0.995 54 and 0.978 69, whereas the ones given by Eq. (3.4) of Ref. 12 are 0.995 51 and 0.978 03.

[45]For K_0 the computation gives $K_0=0.1482\times10^{-8}$ for $Z=3$ and 0.4224×10^{-7} for $Z=6$. The corresponding values given by Eq. (118) are 0.1483×10^{-8} and 0.4202×10^{-7}, whereas the ones given by Eq. (3.4) of Ref. 12 are 0.2304×10^{-8} and 0.7376×10^{-8}.

[46]We cannot neglect the λ in the denominators of the integrand because the integral would become singular.

[47]See Ref. 3, Eq. (11), where it was denoted by $\mathcal{O}$.

[48]See Ref. 36, p. 22, Eq. (23).

[49]Our procedure cannot be used in the limiting case $a=1$ ($\gamma=0$) because the integral would become divergent.

[50]This conclusion does not apply to the case of $a=1$ (see Ref. 49).

[51]In fact, for extremely large Δ (which is not of physical interest because then $\mathfrak{M}_K$ becomes too small in comparison with the matrix elements of the other competing processes) Eqs. (125) and (121) predict oscillations in sign.

[52]Reference 36, p. 15, Eq. (15).

[53]HTF (Ref. 33), p. 231, Eq. (5). The condition $\mathrm{Re}c > \mathrm{Re}a > 0$ is fulfilled in our cases.

[54]The number of points used for the integration and their distribution were chosen so as to take into account the peculiar behavior of the integrand.

[55]These are caused by the factor in curly brackets in Eqs. (100)–(103), which becomes singular on the real axis for $Z \to 0$, whatever the value of Δ. (However, $H^{(1)}$ stays finite because of the factor N^2 in front of the integral, which is proportional to Z^3.)

[56]Setting $a=1$, $\gamma=0$ in Eqs. (99)–(104), one obtains functions F_1 which reduce to combinations of rational and logarithmic functions.

[57]The values of σ_K^{pe} were obtained from R. H. Pratt, A. Ron, and H. K. Tseng, Rev. Mod. Phys. 45, 273 (1973), Table 6.1, and private communication. For $29 \le Z \le 135$ the agreement was complete to four significant figures. For $Z=13$ and $Z=6$ the difference was of three units in the fourth significant figure. (The accuracy on $\mathrm{Re}H_0^{(1)}$ is, however, much higher.)

[58]This is true for all the values of Δ/λ in Table I. For larger Δ/λ deviations will occur (see Sec. VI).

[59]The "Hall factor," equal to $1/2.2=0.454$ for lead, was used to correct the cross sections at high energies for Coulomb effects, instead of using the correct value, 0.223 (see Ref. 57, Table 6.1). Our attention was drawn to this by Dr. R. H. Pratt.

[60]We allow here the vectors $\vec{s}_1$ and $\vec{s}_2$ to be complex in order to be able to account for circular polarization.

[61]The amplitudes $\mathfrak{M}_\perp$ and $\mathfrak{M}_\parallel$ represent the values of the matrix element when the photon polarizations are linear and both perpendicular or both parallel to the scattering plane. The amplitudes $\mathfrak{M}_{SF}$ and $\mathfrak{M}_{NSF}$ correspond to the cases when the photons are circularly polarized and there is a change in the sense of polarization (a photon spin flip) or there is none (no photon spin flip).

[62]Equation (126) can be rewritten as $\mathfrak{M}=M(\vec{s}_1\cdot\vec{s}_2^*)-(N\Delta^2/\kappa^2)(\vec{s}_1\cdot\vec{\delta})(\vec{s}_2^*\cdot\vec{\delta})$, with $\vec{\delta}=\vec{\Delta}/\Delta$. By comparing this with Eq. (59) we find $\lim(N\Delta^2/\kappa^2)=0$. Note that this does not imply $\lim N=0$, as might be erroneously inferred from the comparison of Eqs. (59) and (126).

[63]For example, see Ref. 29.

[64]The results of Johnson and Lin in Table III (private communication) were calculated for a purely Coulomb field and no electron correlation, in contrast to those of Ref. 9.

[65]We have transformed the results of Brown and Mayers and Cornille and Chapdelaine, which were given in terms of $\mathfrak{M}_{NSF}$ and $\mathfrak{M}_{SF}$, into $\mathfrak{M}_\perp$ and $\mathfrak{M}_\parallel$. The results of Johnson and Lin were given directly in terms of the latter.

[66]An alternative method of fixing the relative phases via optical theorms and Kramers-Kronig dispersion relations has also been used.

[67]The matrix element $(\alpha Z)^2a$ of Papatzacos and Mork, [see also Ref. 16(b), Sec. 3.4], is related to the corresponding S-matrix element by the equation $S_{21}^{(D)}=2\pi i\delta(\kappa_1-\kappa_2)(2\pi r_0/\kappa)(\alpha Z)^2a$. By comparing this with $S_{21}^{(R)}$ given in Ref. 22 we see that indeed $\mathfrak{M}_K$ and $(\alpha Z)^2a$ have to be added with opposite signs.

[68]Because of spherical symmetry the matrix element a also has the general form of Eq. (126).

[69]The contribution of the higher shells to Rayleigh scattering is rather small if the momentum transfer is not very small [see M. Schumacher, F. Smend, and I. Borchert, Nucl. Phys. A 206, 531 (1973), Figs. 3–5]. (However, right at $\Delta=0$ the total Rayleigh matrix element is roughly equal to Z, whereas $\mathfrak{M}_K\simeq 2$).

[70]However, Coulomb corrections to Delbrück scattering in the Born approximation probably become large at large Δ, as is indicated by the high-energy result of *H*. Cheng and T. T. Wu, Phys. Rev. D 5, 3077 (1972).

[71]I. S. Gradshteyn and I. M. Ryzhik, *Tables of Integrals, Series and Products* (Academic, New York, 1967), p. 636, Eq. (4.638.2).

[72]HTF (Ref. 33), p. 114, Eq. (1).

[73]We have used also HTF (Ref. 33), p. 5, Eq. (15).

[74]HTF (Ref. 33), p. 239, Eq. (1).

[75]In the case we are interested in, $v=t/x$ is indeed positive and real [see Eq. (A26)], so that we can apply Eq. (A14).

[76]Reference 36, p. 19, Eq. (19).

[77]Because Eq. (A19) is valid for the values of p, q, and r derived from Eqs. (A26) and (A18), Eqs. (A23) and (A24) will also be valid.

[78]A_0 cannot be expanded in *power series* in a because of $\ln a$ contributions stemming from quantities such as ζ^{ia}.

[79]We have combined Eq. (15) of P. O. M. Olsson, J. Math. Phys. 5, 420 (1964), with HTF (Ref. 33), p. 240, Eq. (4). The first reference contains the definition of the functions $G_2(a_1,a_2;b_1,b_2;x,y)$. The principal branches of the complex powers appearing in Eq. (B2) should be taken.

[80]The double series of the F_1 functions converge if the modulus of their variables is smaller than 1, which happens for all physical a, since $a<1$.

[81]See M. Gavrila, Phys. Rev. A 6, 1360 (1972), Eq. (22).

[82]HTF (Ref. 33), p. 10, Eq. (12).

[83]HTF (Ref. 33), p. 9, Eq. (5), and p. 4, Eq. (8).