Nonlinear theory of parametric instabilities in plasmas

D. F. DuBois and B. Bezzerides

Theoretical Division, Los Alamos Scientific Laboratory, University of California, Los Alamos, New Mexico 87544

(Received 29 April 1976)

The weak plasma turbulence theory recently formulated by one of us is applied to parametrically unstable plasmas. A dispersion relation for normal modes of the turbulent plasma is derived. The condition of marginal stability of these modes provides a nonlinear equation describing the quasistationary turbulent spectrum. Two classes of nonlinearities arise: (i) Diagonal corrections, similar to the renormalized frequencies and damping rates of usual weak turbulence theory but which must be computed using the frequencies of the coupled parametric modes, and (ii) coupling saturation corrections which are renormalized mode coupling coefficients related to spectral cross coupling coefficients such as $\langle E(\vec{k},\omega)E(\vec{k}-\vec{k}_0,\omega-\omega_0)\rangle$. The coupling saturation theory is applied to the electron-electron decay instability ($\omega \sim 2\omega_{pe}$) and strongly inhibits the conversion of pump radiation into plasma waves.

In this paper we apply the space-time formulation of plasma turbulence theory recently formulated by one of us[1] (hereafter referred to as I) to the nonlinear theory of parametric instabilities in plasmas.[2] We adopt a model in which the driving pump field $\vec{E}_0\cos(\omega_0 t-\vec{k}_0\cdot\vec{r})$ which modulates the properties of the plasma is the only coherent field in the problem. The parametrically excited waves are excited from thermal fluctuations and are taken to be incoherent. Such problems have been treated in the past[3-10] by essentially straightforward application of the usual weak-turbulence techniques[11-13] involving the random-phase approximation (RPA). In the present paper we take into account the phase correlations between Fourier modes intrinsic to a parametrically driven system and show that new effects arise. The most important of these we call coupling saturation which results in a modification of the mode-coupling coefficients by the turbulence generated by the instability. In the cases which we have investigated, the turbulent modification of the mode-coupling coefficients *reduces* the parametric growth rate causing a saturation of the instability.

The formulation of plasma turbulence theory developed in I is based on the concept of the response function of the turbulent plasma to an infinitesimal, localized external perturbation. This infinitesimal response function describes the normal modes of the turbulent system. The theory is developed as an expansion in powers of the assumed small ratio of the incoherent wave energy to the particle kinetic energy. It is also assumed that while the wave fluctuations are weakly correlated, strong coherent fields may also be present which make the system arbitrarily nonstationary in time and inhomogeneous in space. The equations were therefore developed in physical space and time rather than in Fourier space. This formulation was shown to be equivalent to the well-known theory of Kadomstev in the case of a quasistationary, infinite homogeneous plasma.

The major consequence of the imposed coherent fields is to invalidate the RPA, i.e., modes with different values of $\vec{k}$ and ω are now correlated. In the present paper, the only imposed field considered will be the sinusoidal pump field $\vec{E}_0\cos(\omega_0 t-\vec{k}_0\cdot\vec{r})$ which drives the parametric instability. The parameters of the pump field will be taken to be given constants; i.e., we neglect pump depletion.

In Sec. I, we examine, in a general way, the consequences of the symmetry breaking by the pump field which couples together modes whose wave vectors and frequencies differ by integral multiples of k_0 and ω_0, respectively. We do this by computing the infinitesimal response $d^+(1,1')=4\pi\delta\langle U(1)\rangle/\delta\rho_{ex}(1')$ of the average electrostatic potential $\langle U(1)\rangle$ at the space-time point $(1)\equiv(\vec{r}_1,t_1)$ to an infinitesimal test charge perturbation $\delta\rho_{ex}(1')$ at $(1')\equiv(\vec{r}_1',t_1')$. This infinitesimal-response function yields a dispersion relation for the normal modes of the *pump-driven turbulent* plasma. This dispersion relation is formally identical to the familiar linear dispersion relation for parametric instabilities except that the parameters—e.g., mode frequencies, damping rates, and mode-coupling coefficients—become prescribed functionals of the turbulent wave intensity. These results do not depend on any particular approximation of weak-turbulence theory.

In Sec. II, we compute the nonlinear contributions to the polarization response $q^+(1,2)$. These contributions produce dominant terms which describe induced scattering from ions and resonant decay of Langmuir waves into Langmuir waves plus ion waves in the weak-turbulence theory of stationary, homogeneous, infinite plasmas. It is the only nonlinearity which arises to first order in the wave intensity if particle effects are neglected—

i.e., in a fluid theory. In the pump-modulated problem, however, this nonlinearity produces a new effect in addition to the familiar damping and frequency shifts associated with induced scattering. This new effect, which we call coupling saturation, produces corrections to the mode-coupling coefficients, which are porportional to the cross-correlation products $\langle E_{\vec{k},\omega}E_{\vec{k}-n\vec{k}_0,\omega-n\omega_0}\rangle$. We emphasize that coupling saturation occurs in the same order of the weak-turbulence expansion as the familiar induced-scattering nonlinearity, and is a result of the breakdown of the RPA because of pump-induced correlations between modes.

In Sec. III, we examine the case of parametric-decay instabilities in which only two coupled modes are resonantly excited by the pump. A steady-state wave-fluctuation spectrum is obtained by imposing the condition of marginal stability on the potentially unstable modes of the pumped, turbulent system.

In Sec. IV, we apply the theory to the electron-electron decay (or two-plasmon decay) instability where $\omega_0 \gtrsim 2\omega_{pe}$. The saturation of this instability has been considered by Pustovalov, Silin, and Tikhonchuk (PST)[9] and others,[10] but not taking into account the coupling-saturation effect. We show that this is, in fact, the dominant saturation effect which greatly reduces the conversion of pump radiation into plasmons over that previously computed.

We briefly discuss the application of this theory to other parametric decay instabilities in Sec. V. We conclude that such a weak-turbulence expansion is inapplicable to modulational-type instabilities such as the oscillating-two-stream instability because of intrinsic divergences in the theory which arise in this case.

I. INFINITESIMAL RESPONSE FUNCTION

The nonlinear dispersion relation of the parametric system can be found from the infinitesimal response which obeys the general space-time equation

$$\nabla_1^2 d^+(1,1') = \delta^4(1-1') + \int d2\, q^+(1,2) d^+(2,1'). \tag{1.1}$$

The intense pump wave which drives the parametric instabilities modulates the particle trajectories in a coherent way. In this paper we will consider mostly traveling pump waves of the form

$$\vec{E}_0 \cos(\omega_0 t - \vec{k}_0 \cdot \vec{r}). \tag{1.2}$$

This coherent modulation leads naturally to the following Fourier series-integral representation for any quantity depending on two space-time points:

$$A(1;2) = A(\vec{r}_1, t_1; \vec{r}_2, t_2)$$
$$= \sum_{l=-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \int \frac{d^3k}{(2\pi)^3} \exp\{-i[\omega(t_1-t_2) - \vec{k}\cdot(\vec{r}_1-\vec{r}_2)]\} \exp\{-i\tfrac{1}{2}l[\omega_0(t_1+t_2) - \vec{k}_0\cdot(\vec{r}_1+\vec{r}_2)]\} A_l(\vec{k},\omega). \tag{1.3}$$

This representation automatically satisfies the space-time translational symmetry imposed by the pump

$$A(\vec{r}_1, t_1; \vec{r}_2, t_2) = A\left(\vec{r}_1 + \frac{2\pi n \vec{k}_0}{|\vec{k}_0|^2}, t_1 + \frac{2\pi n}{\omega_0}; \vec{r}_2 + \frac{2\pi n \vec{k}_0}{|\vec{k}_0|^2}, t_2 + \frac{2\pi n}{\omega_0}\right). \tag{1.4}$$

This periodicity imposed by the coherent pump is a fundamental change in the space-time translational properties of the plasma. We assume that in the absence of the pump the plasma is homogeneous in space and stationary in time. If $A(1,2)$ is a real quantity [such as $d^+(1,2)$ or $q^+(1,2)$], it is clear that

$$A_l(\vec{k},\omega) = A^*_{-l}(-\vec{k}, -\omega^*).$$

If we substitute the transformation (1.3) for each two point function in (1.1), it is readily seen that the following coupled set of equations is obtained for the coefficients:

$$-k^2 d_l^+\left(\vec{k} + \frac{l}{2}\vec{k}_0, \omega + \frac{l}{2}\omega_0\right) = \delta_{l,0} + \sum_{l'} q_{l'}^+\left(\vec{k} + \frac{l'}{2}\vec{k}_0, \omega + \frac{l'}{2}\omega_0\right) \times d^+_{l-l'}\left(\vec{k} + \frac{l+l'}{2}\vec{k}_0, \omega + \frac{l+l'}{2}\omega_0\right). \tag{1.5}$$

This set of equations can usually be truncated to a set of two or three coupled equations depending on how many modes of the system can simultaneously be on resonance for a given choice of $\vec{k}_0, \omega_0$. These arguments are familiar in the linearized theory of parametric instabilities and, since they are essentially unchanged in the present theory, we will not discuss them in detail. It is sufficient for our present purpose to consider the set of three coupled equations: (for simplicity we will often write the arguments $(\vec{k},\omega)$ of Fourier transforms simply as ω, suppressing the $\vec{k}$ dependence where this cannot cause confusion.)

$$
\begin{aligned}
&-k^2 d_0^+(\omega) = 1 + q_0^+(\omega) d_0^+(\omega) + q_{-1}^+(\omega - \tfrac{1}{2}\omega_0) d_1^+(\omega - \tfrac{1}{2}\omega_0) + q_1^+(\omega + \tfrac{1}{2}\omega_0) d_{-1}^+(\omega + \tfrac{1}{2}\omega_0),\\
&-(\vec{k} - \vec{k}_0)^2 d_1^+(\omega - \tfrac{1}{2}\omega_0) = q_0^+(\omega - \omega_0) d_1^+(\omega - \tfrac{1}{2}\omega_0) + q_1^+(\omega - \tfrac{1}{2}\omega_0) d_0^+(\omega),\\
&-(\vec{k} + \vec{k}_0)^2 d_{-1}^+(\omega + \tfrac{1}{2}\omega_0) = q_0^+(\omega + \omega_0) d_{-1}^+(\omega + \tfrac{1}{2}\omega_0) + q_{-1}^+(\omega + \tfrac{1}{2}\omega_0) d_0^+(\omega).
\end{aligned} \tag{1.6}
$$

The second equation is obtained from (1.5) with $l=1$ and shifting $\vec{k}, \omega$ to $\vec{k}-\vec{k}_0$, $\omega-\omega_0$, the third equation by setting $l=-1$ and shifting $\vec{k}, \omega$ to $\vec{k}+\vec{k}_0$, $\omega+\omega_0$. We can write these equations in matrix form which makes clearer the connection to previous work:

$$
\begin{bmatrix}
(\vec{k}+\vec{k}_0)^2 \bar{\epsilon}_{\omega+\omega_0} & q_{-1}^+(\omega+\tfrac{1}{2}\omega_0) & 0\\
q_1^+(\omega+\tfrac{1}{2}\omega_0) & k^2 \bar{\epsilon}_\omega & q_{-1}^+(\omega-\tfrac{1}{2}\omega_0)\\
0 & q_1^+(\omega-\tfrac{1}{2}\omega_0) & (\vec{k}-\vec{k}_0)^2 \bar{\epsilon}_{\omega-\omega_0}
\end{bmatrix}
\begin{bmatrix}
d_{-1}^+(\omega+\tfrac{1}{2}\omega_0)\\
d_0^+(\omega)\\
d_1^+(\omega-\tfrac{1}{2}\omega_0)
\end{bmatrix}
=
\begin{bmatrix}
0\\ -1\\ 0
\end{bmatrix} \tag{1.7}
$$

where $\bar{\epsilon}_{\vec{k}\omega} = 1 + q_0(\omega)/k^2$. The same arguments which applied in linear theory for truncating this at three modes $\omega, \omega \pm \omega_0$ apply here. Note that now the q_n's contain not only the renormalization effects of the pump but also nonlinear corrections arising from the turbulence level. We will calculate some of these terms in Sec. II.

The solution for the Fourier coefficients of the infinitesimal response follows directly from (1.6). Note that each component $(d_0^+, d_{\pm 1}^+)$ is inversely proportional to the determinant of the matrix, where the determinant is

$$
\begin{aligned}
D(\vec{k}, \omega) = &(\vec{k}+\vec{k}_0)^2 (\vec{k})^2 (\vec{k}-\vec{k}_0)^2 \bar{\epsilon}_{\omega+\omega_0} \bar{\epsilon}_\omega \bar{\epsilon}_{\omega-\omega_0}\\
&- (\vec{k}+\vec{k}_0)^2 \bar{\epsilon}_{\omega+\omega_0} q_1^+(\omega - \tfrac{1}{2}\omega_0) q_{-1}^+(\omega - \tfrac{1}{2}\omega_0)\\
&- (\vec{k}-\vec{k}_0)^2 \bar{\epsilon}_{\omega-\omega_0} q_1^+(\omega + \tfrac{1}{2}\omega_0) q_{-1}^+(\omega + \tfrac{1}{2}\omega_0).
\end{aligned} \tag{1.8}
$$

The condition $D(\vec{k}, \omega) = 0$ constitutes the *nonlinear dispersion relation* of the parametric system. The complex roots $\omega(\vec{k})_{\rm NL} - i\gamma(\vec{k})_{\rm NL}$ obtained from this represent the infinitesimal excitations of the turbulent system. The condition $\gamma(\vec{k})_{\rm NL} = 0$ for marginal stability can provide a condition determining the possible steady-state or saturated-wave spectrum of the turbulent plasma.

In this paper we will consider only the steady-state excitations of the plasma (assuming a steady state exists). Thus the coefficients in the expansion (1.3) are time independent, the only time dependence being that imposed by the pump. In a time-dependent inhomogeneous theory, which we will consider in a subsequent paper, the expansion coefficients $A_l(\vec{k}, \omega)$ become slowly varying functions of t, on the time scale $2\pi/\omega_0$, and slowly varying functions of $\vec{r}$, on the scale of $2\pi/k_0$. It should be noted that the issue of accessibility of the final steady-state solutions of this paper is being studied by direct numerical integration of the time-evolution equations in this time-dependent theory and will be reported elsewhere.

II. NONLINEAR PROCESSES

The nonlinear physics is all contained in the generalized susceptibility functions $q^+(1,1')$ and noise source $S(1,1')$. In I we derived expressions for $q^+(1,1')$ and $S(1,1')$ in weak-turbulence theory which are valid for arbitrary macroscopic space and time dependence imposed by coherent fields. The procedure to adapt these expressions to the present problem is straightforward. The propagators $\vec{G}_s(1,2)$ are replaced by the propagator-obeying equation (32) of I for electrons in the oscillating pump field $\vec{E}_0$. The effect of the pump on the heavy ions can be neglected. The space-time integrations in Eqs. (62), (63), (65), and (70) of I can be carried out giving expressions for $q^+(1,1')$ and $S(1,1')$.

The resulting general expressions are very complicated, and we have no need to present them here. We have restricted our considerations to the weak-pump case

$$
\frac{\vec{k}\cdot\vec{u}_0}{\omega_0} \ll 1, \quad \vec{u}_0 = e\vec{E}_0/m\omega_0.
$$

In this case Eq. (32) of I for the propagators can be iterated as follows:

$$
\begin{aligned}
\vec{G}(\bar{1},\bar{2}) &= \vec{G}^0(\bar{1}-\bar{2}) + \frac{e}{m}\int d\bar{3}\, \vec{G}^0(\bar{1}-\bar{3})\cdot \vec{E}_0(3) \vec{G}(\bar{3},\bar{2})\\
&\simeq \vec{G}^0(\bar{1}-\bar{2}) + \frac{e}{m}\int d\bar{3}\, \vec{G}^0(\bar{1}-\bar{3})\cdot \vec{E}_0(3)\vec{G}^0(\bar{3},\bar{2}) + \frac{e^2}{m^2}\int d\bar{3}\int d\bar{4}\, \vec{G}^0(\bar{1}-\bar{3})\cdot \vec{E}_0(3)\vec{G}^0(\bar{3}-\bar{4})\cdot \vec{E}_0(4)\vec{G}^0(\bar{3}-\bar{2}) + \cdots,
\end{aligned} \tag{2.1}
$$

where $\vec{G}^0(\tilde{1}-\tilde{2})$ is the pump-free propagator,

$$\vec{G}^0(\tilde{1}-\tilde{2})=\int\frac{d^3k}{(2\pi)^3}\int\frac{d\omega}{2\pi}\,\frac{\exp[i\vec{k}\cdot(\vec{r}_1-\vec{r}_2)]\exp[-i\omega(t_1-t_2)]}{\omega-\vec{k}\cdot\vec{v}+i0+}\,\delta^3(\vec{v}_1-\vec{v}_2)\vec{\partial}_{v_2}, \tag{2.2}$$

and $\vec{E}_0(1)=\vec{E}_0(\vec{r}_1,t_1)$, the pump field $\vec{E}_0(1)=\vec{E}_0\cos(\omega_0 t_1-\vec{k}_0\cdot\vec{r}_1)$.

This means, for example, that in the matrix elements $v(1,2,3)$ which occur in the "mode coupling" terms $q^+_{\rm mc}$ and $S_{\rm mc}$ [calculated in I, Eqs. (62) and (70)], neglecting terms of order $(\vec{k}\cdot\vec{u}_{0e})/\omega_0$, we can use the matrix elements [I, Eq. (86)] which are appropriate for the system without the pump. Similarly, to this order the propagators $\vec{G}_s$ in the expression for the "nonlinear Landau damping" terms can be replaced by their zero pump values.

The same is obviously *not* true for the field-response functions d^+ and correlation functions I. These functions are modified by the pump in a way which is not expandable in powers of E_0 to produce the parametric instability.

From I, Eq. (62), we have the expression for the mode-coupling nonlinear contribution to q^+

$$q^+(1,1')_{\rm mc}=-\int d2\int d2'\int d3\times\int d3\; v(1,2,3)v(2',1',3')d^+(2,2')I(3,3'). \tag{2.3}$$

We can write (1.3) in the form

$$A(1,2)=\sum_{l=-\infty}^{\infty}\exp\{-i\tfrac{1}{2}l[\omega_0(t_1+t_2)-\vec{k}_0\cdot(\vec{r}_1+\vec{r}_2)]\}A_l(1-2), \tag{2.4}$$

where A_l is a function of space-time difference variables only. Such an expansion can be used for $I(2,2')$ and $d^+(1,1')$ in Eq. (2.3), and we also replace $v(1,2,3)$ by its pump-free value [see Eq. (86) of I]. The result of this substitution is

$$q_l^+(k)_{\rm mc}=-\int\frac{d^4k'}{(2\pi)^4}\int\frac{d^4k''}{(2\pi)^4}v\Big(k+\frac{l}{2}k_0,k'+\frac{l'}{2}k_0,k''+\frac{l''}{2}k_0\Big)\,v^*\Big(k-\frac{l}{2}k_0,k'-\frac{l'}{2}k_0,k''-\frac{l''}{2}k_0\Big)I_{l'}(k')\times d^+_{l''}(k'')(2\pi)^4\delta^4(k-k'-k'')\delta_{l-l'-l''}, \tag{2.5}$$

where the matrix elements v were derived in I:

$$v(k,k',k'')=\frac{4\pi e^3}{m^2}\int d^3v[\vec{G}(k)\cdot\vec{k}'\vec{G}(k'')\cdot\vec{k}''+\vec{G}(k)\cdot\vec{k}''\vec{G}(k')\cdot\vec{k}']F_e(\vec{v}). \tag{2.6}$$

Here we use the redundant notation $v(k,k',k'')=v_{k,k'}$ where $v_{k,k'}$ is given by Eq. (86) of I. In these expressions the 4 notation of Eqs. (85) and (86) of I is used. In the zero-pump case only the $l=0$ components of all functions survive and we recover Eq. (85) of I. Equations (1.6), (2.5), and (2.6) specify the infinitesimal response of the pumped system completely in terms of the components of the wave intensity. Just as $d_{\pm 1}$ is related to d_0 it is possible to relate $I_{\pm 1}$ to I_0. To begin with, we have the definition

$$I(1,1')\equiv\langle\delta U(1)\delta U(1')\rangle=\frac{1}{(2\pi)^8}\int d^3k\int d\omega\int d^3k'\int d\omega'\exp[-i(\omega t_1-\vec{k}\cdot\vec{r}_1)\langle U(\vec{k},\omega)U(\vec{k}',\omega')\rangle]\exp[-i(\omega' t_1'-\vec{k}'\cdot\vec{r}_1')]. \tag{2.7}$$

In the usual random-phase approximation of weak-turbulence theory without the coherent-pump field one takes $\langle U(\vec{k},\omega)U(\vec{k}',\omega')\rangle=(2\pi)^4\delta(\omega+\omega')\delta^3(\vec{k}+\vec{k}')I_{\vec{k}\omega}$, i.e., one assumes that different Fourier components are uncorrelated. An important difference in the present theory is that pump induces strong correlations between Fourier components which differ by multiples of ω_0 and $\vec{k}_0$:

$$\langle U(\vec{k},\omega)U(\vec{k}',\omega')\rangle=\sum_n(2\pi)^4\delta(\omega+\omega'+n\omega_0)\delta^3(\vec{k}+\vec{k}'+n\vec{k}_0)\mathcal{I}_n(\vec{k},\omega). \tag{2.8}$$

When this is substituted into (2.7), we obtain

$$I(1,1')=\frac{1}{(2\pi)^3}\int d^3k\,\frac{1}{2\pi}\int d\omega\sum_n\exp\{\tfrac{1}{2}in[\omega_0(t_1+t_1')-\vec{k}_0\cdot(\vec{r}_1+\vec{r}_1')]\}\times\mathcal{I}_n(\vec{k},\omega)\exp\{-i[(\omega+\tfrac{1}{2}n\omega_0)(t_1-t_1')-(\vec{k}+\tfrac{1}{2}n\vec{k}_0)\cdot(\vec{r}_1-\vec{r}_1')]\}. \tag{2.9}$$

Comparing this with (1.3), we can identify the Fourier coefficients $I_l(\vec{k},\omega)$:

$$I_l(\vec{k},\omega)=\mathcal{I}_l(\vec{k}+\tfrac{1}{2}l\vec{k}_0,\ \omega+\tfrac{1}{2}l\omega_0). \tag{2.10}$$

We need to compute the ratio of various Fourier components. For example,

$$\frac{\langle U(\vec{k},\omega)U^*(\vec{k}-n\vec{k}_0,\ \omega-n\omega_0)\rangle}{\langle U(\vec{k},\omega)U^*(\vec{k},\omega)\rangle}=\frac{\mathcal{I}_n(\vec{k},\omega)}{\mathcal{I}_0(\vec{k},\omega)}\,. \tag{2.11}$$

The ratio

$$U(\vec{k}-l\vec{k}_0,\ \omega-l\omega_0)/U(\vec{k},\omega)\equiv R_l(\vec{k},\omega) \tag{2.12}$$

can be computed from the coupled mode equations $\vec{M}\cdot\vec{U}=0$ if we assume that the nonlinear dispersion relation $\mathrm{Det}\vec{M}=0$ is satisfied. Here $\vec{M}$ is the matrix of Eq. (1.7). For example, from (1.7) we find

$$\begin{aligned}R_1(\vec{k},\omega)&=\frac{U(\vec{k}-\vec{k}_0,\ \omega-\omega_0)}{U(\vec{k},\omega)}\\&=\frac{-q_1^+[\omega-\frac{1}{2}\omega_0]}{(\vec{k}-\vec{k}_0)^2\bar{\epsilon}(\vec{k}-\vec{k}_0,\ \omega-\omega_0)}\,,\end{aligned} \tag{2.13}$$

which is obtained using the bottom row of the matrix. A rigorous proof of this procedure for relating Fourier coefficients is discussed in Ref. 2. The ratio of Fourier coefficients can be found from the homogeneous equations $\vec{M}\cdot\vec{U}=0$ because we are interested in *resonant* excitations for which $D(\vec{k},\omega)=\mathrm{Det}\vec{M}=0$. [Thus the source-driven equations $\vec{M}\cdot\vec{U}=\vec{S}$ (represents the fluctuating-noise-source components) has the solution $U_n=\sum_m C_{nm}S_m/D(\vec{k},\omega)$, where C_{nm} is the cofactor of the element M_{nm}. In the 2×2 approximation for example, we have

$$\frac{U_\omega}{U_{\omega-\omega_0}}=\frac{U_1}{U_2}=\frac{C_{11}S_1+C_{12}S_2}{C_{21}S_1+C_{22}S_2}$$

but since

$$\begin{aligned}\mathrm{Det}\vec{M}&=M_{11}M_{22}-M_{12}M_{21}=M_{11}C_{11}+M_{21}C_{21}\\&=M_{12}C_{12}+M_{22}C_{22}=0,\end{aligned}$$

we find $U_1/U_2=-M_{12}/M_{11}$. Similar results hold in the general case.]

The contributions from $q_{nlld}(1,2)$ [Eq. (65) of I] can be worked out the same way. Again, since only the pump modulation of the factor $I(3,4)$ will be taken into account we find, using (1.3) and Eq. (65) of I:

$$\begin{aligned}q_l^+(k)_{nlld}=-\frac{4\pi e^4(i)^3}{m^2}\int\frac{d^4k'}{(2\pi)^4}\int d^3v\,\vec{G}(k+\tfrac{1}{2}lk_0)\cdot(\vec{k}'+\tfrac{1}{2}l\vec{k}_0)\,[\vec{G}(k'')\cdot(\vec{k}'-\tfrac{1}{2}l\vec{k}_0)\vec{G}(k-\tfrac{1}{2}lk_0)\cdot(\vec{k}-\tfrac{1}{2}l\vec{k}_0)\\+\vec{G}(k'')\cdot(\vec{k}-\tfrac{1}{2}l\vec{k}_0)\mathrm{G}(\tfrac{1}{2}lk_0-k')\cdot(\vec{k}'-\tfrac{1}{2}l\vec{k}_0)]\,F_e(v)I_l(k').\end{aligned} \tag{2.14}$$

III. NONLINEAR DISPERSION RELATION FOR DECAY INSTABILITIES

We will discuss here the application of this formalism to the decay instabilities. In this case, we have only two coupled modes, at ω and $\omega-\omega_0$. That is, in Eq. (1.6) or (1.7), we assume the anti-Stokes response $d_{-1}^+(\omega+\frac{1}{2}\omega_0)$ is small and neglect the first row and column of the matrix. Then,

$$\begin{aligned}d_0^+(\vec{k},\omega)&=(\vec{k}-\vec{k}_0)^2\bar{\epsilon}_{\omega-\omega_0}/D(\vec{k},\omega),\\d_1^+(\vec{k}-\tfrac{1}{2}\vec{k}_0,\ \omega-\tfrac{1}{2}\omega_0)&=\frac{-q_1^+(\omega-\frac{1}{2}\omega_0)}{(\vec{k}-\vec{k}_0)^2\bar{\epsilon}_{\omega-\omega_0}}\,d_0^+(\vec{k},\omega)\\&=R_1(\vec{k},\omega)d_0(\vec{k},\omega),\end{aligned} \tag{3.1}$$

where

$$D(\vec{k},\omega)=k^2(\vec{k}-\vec{k}_0)^2\bar{\epsilon}_\omega\bar{\epsilon}_{\omega-\omega_0}-q_1^+(\omega-\tfrac{1}{2}\omega_0)q_{-1}^+(\omega-\tfrac{1}{2}\omega_0). \tag{3.2}$$

The equation $D(\vec{k},\omega)=0$ has the same general structure as the dispersion relation of a parametric instability in the linear (nonturbulent) theory (see, e.g., Ref. 14).

We write q_0,q_1,q_{-1} as the sum of a linear term plus nonlinear terms, i.e.,

$$q_l^+(\vec{k}_0\omega)=q_l^+(\vec{k},\omega)_0+q_l^+(\vec{k},\omega)_{\mathrm{mc}}+q_l^+(\vec{k},\omega)_{nlld}\,, \tag{3.2a}$$

where the linear contribution $q_l^+(\vec{k},\omega)_0$ can be identified from previous work.[14] If we shift k to $k-\frac{1}{2}lk_0$ in Eq. (2.5) and use the relations (2.12), we can write $q_l^+[k-\frac{1}{2}lk_0]_{\mathrm{mc}}$ entirely in terms of I_0 and d_0^+:

$$\begin{aligned}q_l^+(k-\tfrac{1}{2}lk_0)_{\mathrm{mc}}=-\sum_{l',l''}\int\frac{d^4k'}{(2\pi)^4}\int\frac{d^4k''}{(2\pi)^4}\,v(k,k',k'')v^*(k-lk_0,k'-l'k_0,k''-l''k_0)\\\times R_{l'}^*(k')R_{l''}^*(k'')I_0(k')d_0^+(k'')(2\pi)^4\delta^4(k-k'-k'')\delta_{l-l'-l''}\,.\end{aligned} \tag{3.3}$$

From (3.3) we can obtain an expression for $q_{-l}^+[k-\frac{1}{2}lk_0]$. First take $l\to-l$ in Eq. (3.3). By making the transformations $k\to k-lk_0$, $k'\to k'-l'k_0$, $k''\to k''-lk_0$ in Eq. (3.3) and using the identities

$$R^*_{-l}(k-lk_0)R^*_l(k)=1\,,\quad I_0(k-lk_0)=|R_l(k)|^2 I_0(k),\quad d_0^+(k-lk_0)=|R_l(k)|^2 d_0^+(k),$$

which follow from Eq. (2.12), we find that

$$q^+_{-l}(k-\tfrac{1}{2}lk_0)_{\rm mc}=-\sum_{l',l''}\int\frac{d^4k'}{(2\pi)^4}\int\frac{d^4k''}{(2\pi)^4}\,v^*(k,k',k'')v(k-lk_0,k'-l'k_0,k''-l''k_0)$$
$$\times R_{l'}(k')R_{l''}(k'')I_0^+(k')d_0(k'')(2\pi)^4\delta^4(k-k'-k'')\delta_{l-l'-l''}\,. \tag{3.4}$$

From this we see that $q_l^+[k-\frac{1}{2}lk_0]_{\rm mc}=q^{+*}_{-l}[k-\frac{1}{2}lk_0]_{\rm mc}$ *if* $d_0^+(k'')$ is real, which, as we shall see, is often the case. To obtain explicit expressions for q_0^+ and $q_{\pm1}^+$ we use (3.3) and (3.4) making only the ω dependence explicit for compactness. Then for $l=0$,

$$q_0^+(\omega)_{\rm mc}=-\int\frac{d^4k'}{(2\pi)^4}v(\omega,\omega',\omega'')I_0(\omega')d_0^+(\omega'')$$
$$\times\bigg[v^*(\omega,\omega',\omega'')+v^*(\omega,\omega'-\omega_0,\omega''+\omega_0)\,\frac{q_1^{+*}(\omega'-\frac{1}{2}\omega_0)q^{+*}_{-1}(\omega''+\frac{1}{2}\omega_0)}{(\vec{k}'-\vec{k}_0)^2\bar{\epsilon}^*(\omega'-\omega_0)(\vec{k}''+\vec{k}_0)^2\bar{\epsilon}^*(\omega''+\omega_0)}$$
$$+v^*(\omega,\omega'+\omega_0,\omega''-\omega_0)\,\frac{q_1^{+*}(\omega'+\frac{1}{2}\omega_0)q^{+*}_{-1}(\omega''-\frac{1}{2}\omega_0)}{(\vec{k}'+\vec{k}_0)^2\bar{\epsilon}^*(\omega'+\omega_0)(\vec{k}''-\vec{k}_0)^2\bar{\epsilon}^*(\omega''-\omega_0)}\bigg]. \tag{3.5}$$

Similarly for $l=-1$,

$$q^+_{-1}(\vec{k}-\tfrac{1}{2}\vec{k}_0,\omega-\tfrac{1}{2}\omega_0)_{\rm mc}=\int\frac{d^4k'}{(2\pi)^4}v^*(\omega,\omega',\omega'')I_0(\omega')d_0^+(\omega'')$$
$$\times\bigg[v(\omega-\omega_0,\omega'-\omega_0,\omega'')\,\frac{q^+_{-1}(\omega'-\frac{1}{2}\omega_0)}{(\vec{k}'-\vec{k}_0)^2\bar{\epsilon}(\omega'-\omega_0)}+v(\omega-\omega_0,\omega',\omega''-\omega_0)\,\frac{q^+_{-1}(\omega''-\frac{1}{2}\omega_0)}{(\vec{k}''-\vec{k}_0)^2\bar{\epsilon}(\omega''-\omega_0)}\bigg], \tag{3.6}$$

and, from (3.6), (3.4), and (3.3),

$$q_1^+(\vec{k}-\tfrac{1}{2}\vec{k}_0,\omega-\tfrac{1}{2}\omega_0)_{\rm mc}=\int\frac{d^4k'}{(2\pi)^4}v(\omega,\omega',\omega'')I_0(\omega')d_0^+(\omega'')$$
$$\times\bigg[v^*(\omega-\omega_0,\omega'-\omega_0,\omega'')\,\frac{q_1^+(\omega'-\frac{1}{2}\omega_0)^*}{(\vec{k}'-\vec{k}_0)^2\bar{\epsilon}^*(\omega'-\omega_0)}+v^*(\omega-\omega_0,\omega',\omega_0'')\,\frac{q^+_{-1}(\omega''-\frac{1}{2}\omega_0)^*}{(\vec{k}''-\vec{k}_0)^2\bar{\epsilon}^*(\omega''-\omega_0)}\bigg]. \tag{3.6a}$$

Similarly from (2.14) we can change integration variables from k' to $k'-\frac{1}{2}lk_0$ to obtain

$$q_l^+(k\tfrac{1}{2}lk_0)_{nl\,ld}=(-i)^3\frac{4\pi e^4}{m^2}\int\frac{d^4k'}{(2\pi)^4}\int d^3v\,\vec{G}(k)\cdot\vec{k}'[\vec{G}(k'')\cdot(k'-lk_0)\vec{G}(k-lk_0)\cdot(\vec{k}-l\vec{k}_0)$$
$$+\vec{G}(k'')\cdot(\vec{k}-l\vec{k}_0)\vec{G}(lk_0-k')\cdot(\vec{k}'-l\vec{k}_0)]F_e(v)R_l^*(k')I_0(k'). \tag{3.7a}$$

In particular, for $l=1$,

$$q_1^+(\vec{k}-\tfrac{1}{2}\vec{k}_0,\omega-\tfrac{1}{2}\omega_0)_{nl\,ld}$$
$$=-\frac{4\pi e^4}{m^2}(-i)^3\int\frac{d^4k'}{(2\pi)^4}\int d^3v\,\vec{G}(k)\cdot\vec{k}'[\vec{G}(k'')\cdot(\vec{k}'-\vec{k}_0)\vec{G}(k-k_0)\cdot(\vec{k}-\vec{k}_0)+\vec{G}(k'')\cdot(\vec{k}-\vec{k}_0)\vec{G}(k_0-k')\cdot(\vec{k}'-\vec{k}_0)]$$
$$\times F_e(v)\,\frac{q_1^+(k'-\frac{1}{2}k_0)^*}{(\vec{k}'-\vec{k}_0)^2\epsilon^*_{k-k_0}}I_0(k'). \tag{3.7b}$$

Again it is possible to show that $q_l^+(k-\frac{1}{2}lk_0)_{nl\,ld}=q^+_{-l}(k-\frac{1}{2}lk_0)^*_{nl\,ld}$ if none of the G propagators in (3.7a) or (3.7b) are resonant.

In what follows we will use the notation $q_l^+(k)_{\rm NL}$ to represent the sum $q_l^+(k)_{\rm NL}=q_l^+(k)_{\rm mc}+q_l^+(k)_{nl\,ld}$.

To simplify Eq. (3.2) we first note that if $\omega=\omega_+-i\gamma_{\rm NL}=\omega_{\rm NL}-i\gamma_{\rm NL}$ is a solution of $D(\vec{k},\omega)=0$, from the symmetry of (3.2) there is also a root of $D(\vec{k},\omega)=0$ for $\omega=\omega_- -i\gamma_{\rm NL}=\omega_0-\omega_{\rm NL}-i\gamma_{\rm NL}$. We now assume that $\omega_{\rm NL}-i\gamma_{\rm NL}$ is near one of the roots of $\epsilon(\vec{k},\omega)$ and likewise is near one of the roots of $\epsilon(\vec{k}-\vec{k}_0,\omega-\omega_0)$ so that we can approximate $\epsilon(\vec{k},\omega)$

in $\tilde{\epsilon}(\vec{k},\omega)=\epsilon(\vec{k},\omega)+(1/k^2)q_0(\vec{k},\omega)_{NL}$ by $(\partial\,\mathrm{Re}\,\epsilon/\partial\omega_\nu)\times(\omega-\omega_\nu-i\gamma_\nu)$ [where $\mathrm{Re}\,\epsilon(k,\omega_\nu)=0$]. The equation $D(\vec{k},\omega)=0$ is then just of the form of a linear parametric decay dispersion relation,

$$\left(\frac{\partial\,\mathrm{Re}}{\partial\omega_+}\epsilon_{\vec{k},\omega_+}\cdot\frac{\partial\,\mathrm{Re}}{\partial\omega_-}\epsilon_{\vec{k}-\vec{k}_0\omega_-}\right)(\omega-\tilde{\omega}_1+i\tilde{\gamma}_1)(\omega-\omega_0+\tilde{\omega}_2+i\tilde{\gamma}_2)-\frac{q_1^+(\omega-\frac{1}{2}\omega_0)}{k^2}\frac{q_{-1}^+(\omega-\frac{1}{2}\omega_0)}{(\vec{k}-\vec{k}_0)^2}=0, \quad (3.8)$$

where the renormalized frequency and damping are

$$\tilde{\omega}_1=\omega_1+\left(\frac{\partial\,\mathrm{Re}\,\epsilon_{\vec{k},\omega_+}}{\partial\omega}\right)^{-1}\mathrm{Re}\,q_0(\vec{k},\omega)_{NL}k^{-2},$$
$$\tilde{\gamma}_1=\gamma_1+\left(\frac{\partial\,\mathrm{Re}\,\epsilon_{k,\omega_+}}{\partial\omega}\right)^{-1}\mathrm{Im}\,q_0(\vec{k},\omega)_{NL}k^{-2}, \quad (3.8a)$$

and ω_1 and γ_1 are the linear values. We have similar expressions for $\tilde{\omega}_2,\tilde{\gamma}_2$ where $\vec{k}\to\vec{k}_0-\vec{k}$, $\omega_+\to\omega_0-\omega_+=\omega_-$. Note that the frequency argument of q_0 is ω, the solution frequency. The real and imaginary parts of the solution are

$$\omega_{NL}=\tilde{\omega}_1-\Delta\tilde{\omega}\tilde{\gamma}_1(\tilde{\gamma}_1+\tilde{\gamma}_2)^{-1},$$
$$\gamma_{NL}=\tilde{\gamma}_1\tilde{\gamma}_2(\tilde{\gamma}_1+\tilde{\gamma}_2)^{-1}-\tilde{\Gamma}^2(\tilde{\gamma}_1+\tilde{\gamma}_2)^{-1}[1+(\Delta\tilde{\omega})^2/(\tilde{\gamma}_1+\tilde{\gamma}_2)^2]^{-1}. \quad (3.9)$$

Here

$$\Delta\tilde{\omega}=\tilde{\omega}_1(\vec{k})+\tilde{\omega}_2(\vec{k}_0-\vec{k})-\omega_0, \quad (3.10)$$

$$\tilde{\Gamma}^2=\frac{q_1^+(\omega_{NL}-\frac{1}{2}\omega_0)q_{-1}^+(\omega_{NL}-\frac{1}{2}\omega_0)}{k^2(\partial\,\mathrm{Re}\,\epsilon_\omega/\partial\omega_+)(\vec{k}_0-\vec{k})^2(\partial\,\mathrm{Re}\,\epsilon_{\omega-\omega_0}/\partial\omega_-)}. \quad (3.11)$$

We have assumed $\tilde{\Gamma}^2$ is real, which is a good approximation for the example considered in this paper. In general, an imaginary part of $\tilde{\Gamma}^2$ can occur.

In the absence of the pump, only $q_0^+(\vec{k},\omega)_{mc}$ is nonzero, and in this limit it becomes the familiar nonlinearity related to induced scattering in Kadomstev's theory, as shown in I. The existence of nonzero values of $q_{\pm1}(\vec{k}-\frac{1}{2}\vec{k}_0,\omega-\frac{1}{2}\omega_0)_{mc}$ in the presence of the pump can be seen from the following arguments: The usual form of the induced-scattering nonlinearity arises from a two-step nonlinear process (see, for example, Kadomstev[11] pages 31–33, and 42–43) shown diagrammatically in Fig. 1(a). Here two high-frequency electrostatic fields $E(\vec{k},\omega)$ and $E(\vec{k}',\omega')$ can beat together because of the plasma nonlinearity to produce a low-frequency nonresonant field $\mathcal{E}(\vec{k}-\vec{k}',\omega-\omega')$. This is just the low-frequency ponderomotive force of the two high-frequency waves driving a low-frequency density response. The beat field $\mathcal{E}(\vec{k}-\vec{k}',\omega-\omega')$ interacts again by the inverse process to couple $E(\vec{k}',\omega')$ and $E(\vec{k},\omega)$. Thus Fig.

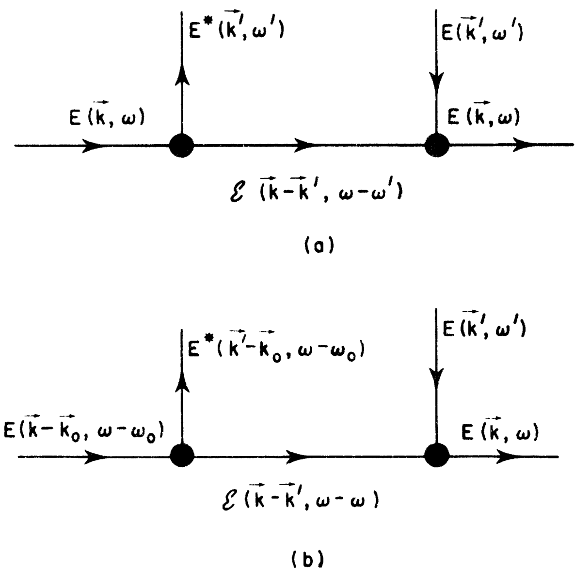

FIG. 1. Mode-coupling diagrams for (a) dielectric function corrections and (b) coupling-saturation corrections. The three-line vertex represents the nonlinear wave interaction with matrix element $v(k,k',k-k')$ for example. Here $\mathcal{E}(\vec{k}-\vec{k}',\omega-\omega)$ represents the low-frequency heat field (ponderomotive force) of the high-frequency fields $E(\vec{k},\omega)$ and $E^*(\vec{k}',\omega')$ in (a) or of the parametrically correlated fields $E(\vec{k}-\vec{k}_0,\omega-\omega_0)$ and $E^*(\vec{k}'-\vec{k}_0,\omega-\omega_0)$ in (b). Diagram (a) provides a diagonal coupling of $E(\vec{k},\omega)$ to itself while diagram (b) provides a cross coupling between $E(\vec{k}-\vec{k}_0,\omega-\omega_0)$ and $E(\vec{k},\omega)$.

1(a) represents a "forward-scattering" amplitude which connects $E(\vec{k},\omega)$ to itself. This then produces a correction to the dielectric function $\epsilon(\vec{k},\omega)$ proportional to $\langle|E(\vec{k}',\omega')|^2\rangle\epsilon(\vec{k}-\vec{k}',\omega-\omega')^{-1}$, where $\epsilon(\vec{k}-\vec{k}',\omega-\omega')$ represent the response of the field $\mathcal{E}(\vec{k}-\vec{k}',\omega-\omega')$. As seen in (3.8), this correction leads to frequency and damping shifts of the linear modes.

In the pump-modulated plasma, however, an electrostatic component $E(\vec{k}'-\vec{k}_0,\omega'-\omega_0)$ is always associated or correlated with a field $E(\vec{k}',\omega')$. The beat interaction of $E(\vec{k}-\vec{k}_0,\omega-\omega_0)$ with $E(\vec{k}'-\vec{k}_0,\omega'-\omega_0)$ produces the difference wave number $\vec{k}-\vec{k}'$ and difference frequency $\omega-\omega'$, precisely the same as the beat interaction between $E(\vec{k},\omega)$ and $E(\vec{k}',\omega')$ considered in the last paragraph. This produces an additional source for the field $\mathcal{E}(\vec{k}-\vec{k}',\omega-\omega')$ and leads to the process in Fig. 1(b), among others. This diagram, in distinction to Fig. 1(a), provides a *coupling* between the Fourier components $E(\vec{k}-\vec{k}_0,\omega-\omega_0)$ and $E(\vec{k},\omega)$ which is proportional to $\langle E(\vec{k}-\vec{k}',\omega'-\omega_0)E(\vec{k}',\omega')\rangle/\epsilon(\vec{k}-\vec{k}',\omega-\omega')$. In the RPA such a cross-correlation coefficient vanishes, but in the presence of the coherent-pump field it does not. The additional

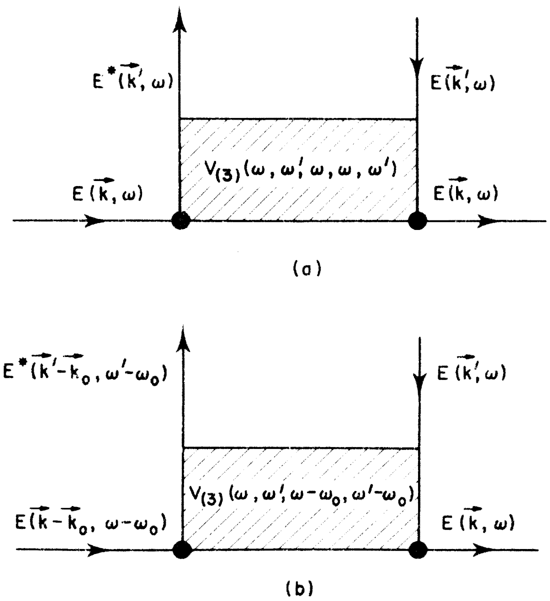

FIG. 2. Direct-coupling diagrams describing the nonlinear contributions arising from q_{nl1d}: (a) diagonal corrects to the dielectric function and (b) coupling saturation corrections. The four-point vertex represents the amplitude $V_{(3)}(\omega, \omega', \omega-\omega_0, \omega'-\omega_0)$ defined in the Appendix.

direct couplings which arise from q_{nl1d} are shown in Fig. 2. These terms do not involve an intermediate beat field.

Thus, as we have stated several times, the coupling-saturation effect arises in precisely the same weak-turbulence approximation as the familiar induced-scattering process, provided pump-induced cross-correlation effects are taken into account.

The marginal stability condition $\gamma_{NL}(\vec{k})=0$ will be used to determine the steady-state enhanced spectrum of the turbulent fields. This condition merits some further comments. In Eqs. (19) and (20) of I we derived a general space-time formula for the electrostatic-potential correlation function. We can apply the same kind of Fourier analysis to this equation and derive an expression for $I_0(\vec{k}, \omega) = \mathcal{G}_0(\vec{k}, \omega)$ [see Eq. (2.10)][2,14,15]

$$I_0(\vec{k}, \omega) = \mathcal{S}_0(\vec{k}, \omega)/|D(\vec{k}, \omega)|^2, \tag{3.12}$$

where $D(\vec{k}, \omega)$ is the determinant defined above and $\mathcal{S}_0(\vec{k}, \omega)$ is an effective noise source term which includes frequency-mixing effects induced by the pump as discussed by DuBois and Goldman[14] for the linear theory. In the present case $\mathcal{S}_0(\vec{k}, \omega)$ is the nonlinear generalization of these effects. The determinant has zeros at $\omega = \omega_{NL} - i\gamma_{NL}$, $\omega = \omega_0 - \omega_{NL} - i\gamma_{NL}$, etc. We can expand $D(\vec{k}, \omega)$ about these zeroes and write, approximately,

$$I_0(\vec{k}, \omega) = \frac{I_+(\vec{k})}{k^2} 2\pi |\omega| \delta(\omega^2 - \omega_{NL}^2) + \frac{I_-(\vec{k})}{k^2} 2\pi |\omega| \delta[\omega^2 - (\omega_0 - \omega_{NL})^2], \tag{3.13}$$

where $\gamma_{NL}/\omega_{NL} \ll 1$, and

$$I_\pm(\vec{k}) = \frac{k^2 \mathcal{S}_0(\vec{k}, \omega_\pm)}{(\partial\, \mathrm{Re} D/\partial \omega_\pm)^2 \gamma_{NL}(\vec{k})}, \tag{3.14}$$

with $\omega_+ = \omega_{NL}$ and $\omega_- = \omega_0 - \omega_{NL}$. Thus the frequency spectrum is dominated by resonances at ω_{NL} and $\omega_0 - \omega_{NL}$. We also see that

$$\gamma_{NL}(\vec{k}) = \frac{k^2 \mathcal{S}_0(\vec{k}, \omega_\pm)}{(\partial\, \mathrm{Re} D/\partial \omega_\pm)^2 I_\pm(\vec{k})}, \tag{3.15}$$

and that when the right-hand side of this equation is much less than unity, for the portion of the spectrum much enhanced over its thermal value, we have $\gamma_{NL}(\vec{k}) \simeq 0$.

Note that the intensities $I_\pm(\vec{k})$ are now *electric-field* fluctuation spectra since we have factored out a factor k^2. It follows from the same type of argument used to derive (2.12) and (2.13) that the ratio of the coefficients in Eq. (3.13) is given by

$$\frac{k^2}{(\vec{k}-\vec{k}_0)^2} \frac{I_-(\vec{k})}{I_+(\vec{k})} = \frac{\langle |U(\vec{k}_0 - \vec{k}, \omega_0 - \omega_{NL})|^2 \rangle}{\langle |U(\vec{k}, \omega_{NL})|^2 \rangle} = |R_1(\vec{k}, \omega_{NL})|^2, \tag{3.16}$$

where R_1 is given by Eq. (2.13). This is formally the same as the ratio of coefficients calculated from the homogeneous set of coupled-mode equations under the condition $D(\vec{k}, \omega_{NL}) = 0$.

IV. SATURATION OF TWO-PLASMON DECAY INSTABILITY

As a first application we consider the nonlinear saturation of the electron-electron decay instability or two-plasmon decay instability involving the decay of an electromagnetic pump wave into two Langmuir waves.[15] The nonlinear theory of this instability has been considered by several authors but without proper account of coupling saturation.[9,10] We will show here that this is in effect the dominant saturation effect.

For this problem the formulas in Sec. III apply where modes 1 and 2 are both Langmuir waves. We use the notation $\omega_l(k) = (\omega_{pe}^2 + 3k^2 v_e^2)^{1/2}$ for the uncorrected Langmuir wave frequency and $\gamma_l(k)$ for the uncorrected damping including both Landau and collisional damping. Thus, in this case, we have

$$\begin{aligned} &\omega_1 = \omega_l(\vec{k}), \quad \gamma_1 = \gamma_l(\vec{k}), \\ &\omega_2 = \omega_l(\vec{k}_0 - \vec{k}), \quad \gamma_2 = \gamma_l(\vec{k}_0 - \vec{k}). \end{aligned} \tag{4.1}$$

The turbulence-corrected ω's and γ's are then calculated using Eq. (3.8a). The coupling parameter $\tilde{\Gamma}^2$ is determined from Eq. (3.11). In the linear (nonturbulent) case for which $q_{\pm 1}(\omega)$ is just $q^*_{\pm 1}(\omega)_0$ we find a familiar result[10,15]:

$$\Gamma_0^2 = \left(\frac{E_0^2}{4\pi n T_e}\right) \frac{\omega_{pe}^4}{\omega_0^4} (\hat{k}_1 \cdot \hat{e}_0)^2 \frac{(\vec{k}_0 \cdot \hat{k}_1)^2}{k_{De}^2} \omega_{pe}^2 . \quad (4.2)$$

The frequency mismatch is

$$\Delta\omega = \tilde{\omega}_l(\vec{k}) + \tilde{\omega}_l(\vec{k}_0 - \vec{k}) - \omega_0. \quad (4.3)$$

For this instability a finite value of $\vec{k}_0$ is necessary. In writing the expression for the coupling parameter Γ_0^2, and in all our considerations here, we will assume that $|\vec{k}_0| \ll |\vec{k}|$ so that $|\vec{k}| \simeq |\vec{k}_0 - \vec{k}|$. In this limit $\tilde{\gamma}_l(\vec{k}) \simeq \tilde{\gamma}_l(\vec{k}_0 - \vec{k})$. Note that if $\gamma_1 \simeq \gamma_2$ we have from Eq. (3.9)

$$\begin{aligned} \omega_{NL}(\vec{k}) &= \tfrac{1}{2}\omega_0 + \tfrac{1}{2}[\tilde{\omega}_l(\vec{k}) - \tilde{\omega}_l(\vec{k}_0 - \vec{k})] \\ &\simeq \tfrac{1}{2}\omega_0 + \tfrac{3}{2}\vec{k}\cdot\vec{k}_0 \omega_{pe}/k_{De}^2, \quad (4.4) \\ \omega_0 - \omega_{NL}(\vec{k}) &\equiv \omega_{NL}(\vec{k}_0 - \vec{k}). \end{aligned}$$

Because of this

$$\begin{aligned} \tilde{\gamma}_l(\vec{k}) &= \gamma_c + \tfrac{1}{2}\omega_{pe}\, \mathrm{Im} q_0[\vec{k}, \omega_{NL}(k)]_{NL} \\ & \quad (4.5) \\ \tilde{\gamma}_l(\vec{k}_0 - \vec{k}) &= \gamma_c + \tfrac{1}{2}\omega_{pe}\, \mathrm{Im} q_0[\vec{k}_0 - \vec{k}, \omega_0 - \omega_{NL}(k)]_{NL} \\ &= \gamma_c + \tfrac{1}{2}\omega_{pe}\, \mathrm{Im} q_0[\vec{k}_0 - \vec{k}, \omega_{NL}(\vec{k}_0 - \vec{k})]_{NL}. \end{aligned}$$

So if $|\vec{k}_0 - \vec{k}| \simeq |\vec{k}|$ it is consistent to assume $\tilde{\gamma}_l(\vec{k}) = \tilde{\gamma}_l(\vec{k}_0 - \vec{k})$. The marginal stability condition in this case (see 3.9) is

$$\gamma_{NL} = \frac{\tilde{\gamma}_l(\vec{k})}{2}\left(1 - \frac{\tilde{\Gamma}^2}{\tilde{\gamma}_l^2(\vec{k}) + \frac{1}{4}(\Delta\tilde{\omega})^2}\right) \simeq 0 \quad (4.6)$$

or

$$\tilde{\gamma}_l = [\tilde{\Gamma}^2 - \tfrac{1}{4}(\Delta\tilde{\omega})^2]^{1/2}. \quad (4.6a)$$

If we write $\frac{1}{2}(\Delta\omega) = \omega_{pe}\bar{\Delta}(1 - k^2/k_m^2)$, where $\bar{\Delta} = (\omega_0 - 2\omega_{pe})/2\omega_{pe}$, we can use (4.6a) to obtain

$$\tilde{\gamma}_l(\vec{k}) = \gamma_l(\vec{k})\left[\frac{\tilde{\Gamma}^2(\vec{k})}{\gamma_l^2(\vec{k})} - \left(\frac{\bar{\Delta}\omega_{pe}}{\gamma_l}\right)^2 \left(1 - \frac{k^2}{k_m^2}\right)^2\right]^{1/2}. \quad (4.7)$$

The growth rate is then strongly dependent on the mismatch $(k^2 - k_m^2)$ but less strongly dependent on the angular behavior determined by the factor $(\hat{k}_1 \cdot \hat{e}_0)^2(\hat{k}_0 \cdot \hat{k}_1)^2$ in Γ^2 [Eq. (4.2)]. Thus it is reasonable to assume, following Pustovalov, Silin, and Tikhonchuk (PST), that the spectrum is localized at $k = k_m$ but spreads in angle because of the nonlinear induced scattering of waves from ions. This requires that the pump strength not be too strong, i.e.,

$$\omega_{pe}^2 \bar{\Delta}^2/\gamma_l^2 \gg \tilde{\Gamma}^2/\gamma_l^2, \quad (4.8)$$

and that $\bar{\Delta}$ is not too small, i.e.,

$$1 \gg \bar{\Delta} \gg \gamma_l/\omega_{pe}. \quad (4.9)$$

PST calculate a steady-state saturated spectrum assuming the only nonlinearity in the contribution to $\tilde{\gamma}_l$ is due to nonlinear induced scattering from ions. The required expression for γ_l is obtained by combining (4.5) and (3.5). They do not consider coupling saturation which we will show is a dominant effect. We can use their results for the total Langmuir wave energy without including coupling saturation to compute the resulting energy with this effect included. Their results are

$$\begin{aligned} W &= \int \frac{d^3k}{(2\pi)^3}[I_+(k) + I_-(k)] \\ &= \frac{243}{\pi\sqrt{2}} \frac{[\tilde{P}^{1/2} - 1]^2}{\tilde{P}^{1/2}} \frac{\gamma_c \omega_{pe}}{\omega_{pi}^2} \frac{v_e^2}{c^2}\left(1 + \frac{T_i}{T_e}\right), \quad (4.10) \end{aligned}$$

provided

$$\tilde{P}^{1/2} - 1 \gg \tfrac{16}{27} c^2 v_i^2/v_e^2. \quad (4.11)$$

The latter condition ensures that the spectrum is broad enough so that the differential (or derivative) approximation [PST's Eq. (2.12)] can be used for the contribution to γ_l arising from induced scattering from ions. In addition to some obvious notational changes, we have replaced their parameter $p = E_0/E_{min}$ by the corresponding *renormalized* pump parameter which in our notation is $\tilde{P}^{1/2}$. Note that PST consider the case of "spontaneously" or unpolarized pump radiation, where

$$E_{0i}E_{0j} = \tfrac{1}{2}E_0^2(\delta_{ij} - k_{0i}k_{0j}/k_0^2). \quad (4.12)$$

In this case $\tilde{\Gamma}^2/\gamma_l^2 = \tilde{P}\sin^2 2\theta$, where $\tilde{P} = \tilde{\Gamma}^2(k_m, \theta = \frac{1}{4}\pi)/\gamma_l^2$ and θ is the angle between $\vec{k}$ and $\vec{E}_0$. We denote by $\vec{k}_m$ the unstable wave vector with $|\vec{k}| = k_m$ and $|\theta| = \frac{1}{4}\pi$. From our point of view these expression are incomplete since $\tilde{P}$ is itself a function of $I(k)$, and as it will turn out, a function of W. Thus Eq. (4.10) is actually a nonlinear equation to be solved for W. To establish these equations, we need to work out the expression for $\tilde{P}(W)$.

To do this return to Eq. (3.13) and (3.16). In the present case

$$\frac{I_+(\vec{k})}{I_-(\vec{k})} = |R_1|^2 = \frac{|q_1(\vec{k} - \frac{1}{2}\vec{k}_0, \omega_{NL}(k) - \frac{1}{2}\omega_0)|^2}{(\vec{k} - \vec{k}_0)^4 |\epsilon(\vec{k} - \vec{k}_0, \omega_{NL}(k) - \omega_0)|^2}. \quad (4.13)$$

Since $\omega_{NL}(k) - \omega_0 \simeq -\omega_{pe}$, we can expand the $\mathrm{Re}\,\epsilon$ about its root near $\omega_{NL} - \omega_0 = \omega_l(k_0 - k)$. Then using expression (4.4) we find

$$\frac{I_+(\vec{k})}{I_-(\vec{k})} = \frac{\tilde{\Gamma}^2}{\frac{1}{4}\Delta\omega^2 + \tilde{\gamma}_l^2}, \quad (4.14)$$

where

$$\tilde{\Gamma}^2 = \frac{\omega_{pe}^2}{4} \frac{|q_1(\vec{k} - \frac{1}{2}\vec{k}_0, \omega_{NL}(k) - \frac{1}{2}\omega_0)|^2}{(\vec{k} - \vec{k}_0)^2 k^2}. \quad (4.15)$$

According to Eq. (4.6) this ratio is unity for marginally stable waves. Thus we can write

$$I_0(\vec{k}, \omega) = 2(2\pi|\omega|/k^2)I(\vec{k})\delta[\omega^2 - \omega_{NL}^2(\vec{k})], \quad (4.16)$$

where $I(\vec{k}) = I_\pm(\vec{k})$.

Next we substitute (4.16) into (3.6) and take $\omega = \omega_{NL}(k)$. Since $\omega' = \omega_{NL}(k')$ we have generally $\omega'' \ll \omega_{pe}$. Thus the arguments of the first two matrix elements in (3.6) have two high-frequency and one low-frequency arguments. If $\omega'' \ll k'' v_e$ we find from Eq. (2.6) (see also Ref. 11)

$$v(\omega, \omega', \omega'') \simeq (e/m)(\vec{k} \cdot \vec{k}') k_{De}^2 / \omega_{pe}^2,$$

provided $|\vec{k}_0| \ll |\vec{k}|, |\vec{k}'|$. The last matrix element in (3.6) has three high-frequency arguments and is smaller than the other two. Furthermore, in the denominator $\epsilon(\vec{k}' - \vec{k}_0, \omega' - \omega_0)$ is near resonance at $\omega_l(\vec{k}_0 - \vec{k}')$. However, the $\epsilon(\vec{k}'' - \vec{k}_0, \omega'' - \omega_0)$ in the second term is far off resonance. Thus, only the first term in large square brackets in (3.6) is important. With the relation

$$\tilde{\Gamma}_-(k) = \tfrac{1}{2}\omega_{pe} q_{-1}(\vec{k} - \tfrac{1}{2}\vec{k}_0, \omega_{NL} - \tfrac{1}{2}\omega_0)(\vec{k} - \vec{k}_0)^{-2}.$$

we then can reduce the contributions of (3.6) and (3.7) to the form

$$\tilde{\Gamma}_-(\vec{k}) = \Gamma_0(\vec{k}) - \frac{e^2 k_{De}^2 (T_e/T_i)}{m^2 \omega_{pe}^4 (1 + T_e/T_i)} \times \int \frac{d^3k'}{(2\pi)^3} \frac{(\hat{k} \cdot \hat{k}')^2 I(\vec{k}') \tilde{\Gamma}_-(\vec{k}')}{\Delta\omega(\vec{k}')/\omega_{pe} - 2i\tilde{\gamma}_l(\vec{k}')/\omega_{pe}} \quad (4.17)$$

where $\Gamma_0(\vec{k})$ is the linear (zero-turbulence) value. Details of this reduction are given in Appendix A. Here we have also taken

$$d_0^+(\vec{k}'', \omega'') \simeq [k''^2 \epsilon(\vec{k}'', \omega'')]^{-1} \simeq (k_{De}^2 + k_{Di}^2)^{-1}$$

since

$$\omega'' = \tfrac{3}{2}\omega_p(\vec{k} - \vec{k}') \cdot \vec{k}_0 k_{De}^{-2} \ll k'' v_i \quad (k'' = |\vec{k} - \vec{k}'|),$$

provided $v_e/c \ll (m_e/m_i)^{1/2}$. Now, in the approximations of PST, $I(\vec{k}')$ is peaked at $\vec{k} = \vec{k}_m$, where $\Delta\omega(\vec{k}')$ is zero, and we can set $\vec{k}' = \vec{k}_m$ in $\Gamma_-(\vec{k}')$. We also have from the marginal stability condition

$$\tilde{\gamma}_l(\vec{k}_m) = |\tilde{\Gamma}^2|^{1/2} = \tilde{P}^{1/2}\gamma_c. \quad (4.18)$$

Since $k_m \ll k_{De}$ we have replaced γ_l by the k-independent collisional damping $\gamma_c = \frac{1}{2}\nu_{ei}$. Finally, we *assume* that $I(k', \theta)$ is strongly peaked about the angle $\theta = \pm\frac{1}{4}\pi$. We can therefore extract from the integral a factor $\tilde{\Gamma}_-(k_m)$, where $\vec{k}_m$ is the most unstable wave vector, and therefore obtain a simple algebraic relation for $\tilde{\Gamma}_-(\vec{k}_m)$:

$$\tilde{\Gamma}_-(\vec{k}_m) \simeq \Gamma_0(\vec{k}_m)\left(1 + \frac{i\omega_{pe} W(T_e/T_i)}{2\gamma_c \tilde{P}^{1/2}(1 + T_e/T_i)}\right)^{-1}, \quad (4.19)$$

where

$$W = \int \frac{d^3k'}{(2\pi)^3} \frac{I(k')}{4\pi n T_e}. \quad (4.20)$$

Since $d+(\vec{k}'', \omega'')$ is real we have $\tilde{\Gamma}_+(\vec{k}) = \tilde{\Gamma}_-(\vec{k})^*$, and

$$\tilde{P} = \frac{\tilde{\Gamma}_+(\vec{k}_m)\tilde{\Gamma}_-(\vec{k}_m)}{\gamma_c^2} = P\left(1 + \frac{\omega_{pe}^2 W^2 (T_e/T_i)^2}{4\gamma_c^2 \tilde{P}(1 + T_e/T_i)^2}\right)^{-1} \quad (4.21)$$

or

$$\tilde{P} = P - \frac{1}{4}\frac{\omega_{pe}^2 W^2 (T_e/T_i)^2}{\gamma_c^2 (1 + T_e/T_i)^2}, \quad (4.22)$$

which relates the renormalized pump parameter $\tilde{P}$ to the unrenormalized $P = \Gamma_0^2(\vec{k}_m)/\gamma_c^2$. We insert this expression into (4.10) to obtain an equation determining W:

$$W = \rho(\tilde{P}^{1/2} - 1)^2/\tilde{P}^{1/2}, \quad (4.23)$$

$$\rho = \frac{243}{\pi\sqrt{2}} \frac{\gamma_c}{\omega_{pe}} \frac{m_i}{m_e} \frac{v_e^2}{c^2}\left(1 + \frac{T_i}{T_e}\right). \quad (4.24)$$

Since $\rho/W \gg 1$ this has a solution

$$W = 2(\gamma_c/\omega_{pe})(P - 1)^{1/2}[(T_e + T_i)/T_e]\ . \quad (4.25)$$

This is generally *much* smaller than the value which we call W_{PST}, obtained by PST

$$\frac{W}{W_{PST}} = \frac{\gamma_c}{\omega_{pe}} \frac{1}{\rho} \frac{(P-1)^{1/2} P^{1/2}}{(P^{1/2} - 1)^2} \sim \frac{\gamma_c}{\omega_{pe}\rho} \ll 1 \quad \text{if } P \gg 1.$$

Note that the weak-turbulence condition $W \ll 1$ now restricts

$$(P - 1)^{1/2} \ll [(T_e + T_i)/T_e]^{-1}\omega_p/\gamma_c$$

so that the theory is applicable for $P \gg 1$, unlike the theory of PST. The condition for the applicability of the derivative approximation must also be tested. The solution (4.25) corresponds to $\tilde{P} = 1$. To find $(\tilde{P}^{1/2} - 1)$ to higher accuracy we iterate Eq. (4.24) to obtain

$$(\tilde{P}^{1/2} - 1)^2 = W/\rho,$$

where W is given by (4.25).

Substituting this into (6.11) we obtain

$$\sqrt{W}/\sqrt{\rho} \gg \tfrac{16}{27} c^2 v_i^2 / v_e^4, \quad (4.26)$$

or, using (4.25) for W this provides a condition on P:

$$(P-1)^{1/4} \gg \sqrt{\rho}\, \frac{16}{27}\left(\frac{\omega_{pe}}{2\gamma_c}\right)^{1/2} \frac{mc^2}{T_e} \frac{m_e}{m_i} \frac{T_i}{T_e}\left(\frac{T_e + T_i}{T_e}\right)^{1/2}, \quad (4.27)$$

which is easily satisfied in practice. We therefore have shown that the coupling saturation *greatly* reduces the energy in the turbulent-wave spectrum and, as a result, greatly extends the range of P values for which the weak-turbulence theory is applicable.

The spectral width is found from

$$\Delta\theta \simeq \left(\frac{\tilde{P}^{1/2}-1}{\tilde{P}^{1/2}}\right)^{1/2} \simeq \left(\frac{W}{\rho}\right)^{1/4}$$

$$\simeq \left(\frac{\gamma_c}{\omega_p \rho}\right)^{1/4} \left(\frac{T_e+T_i}{T_e}\right)^{1/4} (P-1)^{1/8} < 1. \qquad (4.28)$$

The considerations given here demonstrate that coupling saturation plays an important, even dominant, role in the saturation of the $2\omega_{pe}$ decay instability. However, we do not regard the results above as completely representing the nonlinear stage of this instability for several reasons.

(i) The rather narrow spectrum of Langmuir waves peaked near k_m is probably unstable to further decay into Langmuir waves of $k<|k_m|$ and ion acoustic waves. The conditions for this are $\omega_l(\vec{k}_m)=\omega_l(\vec{k})+\omega_a(\vec{k}_m-\vec{k})$, where ω_a is the ion acoustic frequency with the threshold condition $W>16(\gamma_c/\omega_{pe})(\gamma_a/\omega_a)$. If we use (4.25) for W, the condition for further decay instability of the computed spectrum translates into

$$(P-1)^{1/2} > 8\frac{\gamma_a}{\omega_a}\left(\frac{T_e+T_i}{T_e}\right)^{-1}, \qquad (4.29)$$

which is easily satisfied if $T_e \gg T_i$. If $T_e \simeq T_i$, then $\gamma_a/\omega_a \simeq 1$ and we require $P>17$ for further decay instability. We will not attempt here to treat the two- (or three-) dimensional problem which arises in computing the wave spectrum arising from this additional decay (see, however, Ref. 6).

(ii) The steady-state spectrum including the additional decay mentioned above may become modulationally unstable for sufficiently high-turbulence levels. If the resulting Langmuir spectrum is sufficiently broad with a width Δk the condition for modulational instability in the theory of Vedenov *et al.*[16] is

$$W > (\Delta k/k_{\mathrm{De}})^2.$$

In this case energy from the turbulent spectrum is converted into an unstable low-frequency longer-wavelength mode involving ion motion. This unstable mode is conjectured to break up into localized "caviton" excitations.[16] It appears, however, that there may be a range of lower-pump powers where the spectrum is modulationally stable.

(iii) We have assumed that a steady-state wave spectrum is developed. Computer calculations of the time development of the spectrum indicate that a steady state may not be attainable. Instead a time-oscillatory saturated state develops. However, we believe that coupling-saturation nonlinearities must be included to treat the time development of the spectrum properly.

V. SUMMARY AND SUGGESTIONS FOR FURTHER WORK

Let us summarize the main conclusions of this paper:

(a) We have shown that the dispersion relation for small excitations of the turbulent plasma in the presence of the pump modulation can be written in the same form as the linear theory of parametric instabilities. However, in the nonlinear theory the mode frequencies, damping rates *and* mode-coupling coefficients become prescribed functionals of the turbulent-wave intensity.

(b) The coupling-saturation effect arises because of pump-induced cross correlations between modes. This effect arises in the same approximation of weak-turbulence theory which leads to the well-known nonlinear induced-scattering corrections to the mode frequencies and damping rates.

(c) Coupling saturation was shown to greatly reduce the turbulent-wave energy produced by the $2\omega_{pe}$ decay instability when compared to calculations which have not included this effect.

This theory has been applied to the electron-ion decay (EID) instability for $T_e \gg T_i$ and will be reported elsewhere.[17] This application is greatly complicated when one tries to assess the combined effect of coupling saturation and transfer in $\vec{k}$ space due to induced scattering from ions or ion acoustic waves. In this case again coupling saturation appears to play a dominant role in saturating the wave energy.

There undoubtedly are other candidates from the long list of parametric-decay instabilities considered in the literature for which coupling saturation will be important. Hopefully, this paper will stimulate some interest in these applications.

Finally, we want to demonstrate the *inapplicability* of the type of theory presented here to the OTS instability. For $T_e=T_i$ the threshold for OTS is only slightly higher than for EID but for $T_e \gg T_i$ it is higher by a factor of ω_a/γ_a. The OTS instability can be regarded as a electrostatic modulational instability of an electromagnetic pump. It is tempting to try to handle this instability by the same techniques used for the decay instabilities. For this one could go back to the 3×3 system of equations in Eq. (1.7). The linear theory of OTS instability can be obtained from this set. The formulas for q_0, q_{-1}, q still apply. The problem arises with the response functions $d_0^{\pm}(\vec{k}'', \omega'')$ which occur in these expressions. As before $\omega''=\omega_{\mathrm{NL}}(k')-\omega_{\mathrm{NL}}(k)$ but the parametric mode frequencies are now

$$\omega_{\rm NL}(k)=\omega_0+\frac{3}{2}\frac{\vec{k}_0\cdot\vec{k}}{k_{\rm De}^2}\omega_{pe} \tag{5.1}$$

and this means that ω'' will also be a resonant frequency

$$\omega''=\omega_{\rm NL}(k)-\omega_{\rm NL}(k')\equiv\omega_{\rm NL}(\vec{k}-\vec{k}'). \tag{5.2}$$

Thus the response functions $d_0^+(\vec{k}'',\omega'')$ are always on resonance. If, in addition, we assume that there is a finite range of k'' of marginally stable waves, i.e., $\gamma_{\rm NL}(k'')=0$ for $k_m-\frac{1}{2}\Delta k<k''<km+\frac{1}{2}\Delta k$, which correspond to the enhanced portion of the spectrum, we see that the integrals for q_0 and $q_{\pm1}$ are *divergent*. That is, the integrals are of the form

$$\int dk'\frac{F(\vec{k},\vec{k}')}{\gamma_{\rm NL}(\vec{k}-\vec{k}')},$$

where $\gamma_{\rm NL}=0$ over a *finite* range of k'. It would appear that waves with the dispersion relation (5.1) can never be treated by a weak mode-coupling theory. If a large amplitude wave of this type is excited with wave vector $\vec{k}$, it can beat together with itself to form $2\vec{k}$ and $\omega_{\rm NL}(2\vec{k})=2\omega_{\rm NL}(\vec{k})$ will also be a resonant mode. Higher harmonics will grow by a continuation of this process producing a broad spectrum in $\vec{k}$ and permitting the formation of a localized excitation in space. If we start with two waves with $\vec{k}_1$ and $\vec{k}_2$ not colinear a very rich spectrum spreading to higher $\vec{k}$ in two dimensions can be produced.

In Ref. 18, mode-coupling theory was considered and a saturated spectrum was derived from a marginal stability condition. However, instead of the complete response function $d_0^+(\vec{k}'',\omega'')$, which appears in our theory, in Ref. 18 only the unrenormalized $[k''^2\epsilon(\vec{k}''^2,\omega'')]^{-1}$ was used. In this approximation the above divergence was not apparent. In our opinion this divergence, which arises from the *exact* response of the plasma is an intrinsic feature of this type of theory. Note that this divergent behavior will occur for any *modulational* instability as well whose dispersion relation satisfied (5.2).

The question remains as to whether or not this divergence can be removed by further renormalizations in the theory. We believe this divergence is related to the tendency of the plasma to form localized "caviton" excitations[16,19] above the threshold for the OTS instability. If so, the correct treatment of these problems requires a significant change in our assumptions concerning the coherence properties of the system.

ACKNOWLEDGMENTS

The authors wish to acknowledge many conversations and helpful advice and criticism from Dr. M. V. Goldman. This work was performed under the auspices of the U. S. Energy Research and Development Administration and in part by Air Force Contract No. AFWL-TR-72-101 and Air Force Office of Scientific Research Contract No. F44620-73-C-0003.

APPENDIX: INTEGRAL EQUATION FOR SATURATED COUPLING

Here we present some of the details leading to the integral equation (4.17) for the saturated-coupling parameter $\vec{\Gamma}_-(\vec{k})$. As stated just before (4.17), only the first term in (3.6) has a large resonant contribution. If we add this to $q_{-1}(\vec{k}-\frac{1}{2}\vec{k}_0,\omega-\frac{1}{2}\omega_0)_{nl1d}$ from (3.7b) we obtain an integral equation for the total q_{-1}^+ as defined in (3.2a):

$$q_{-1}^+(\omega-\tfrac{1}{2}\omega_0)=q_{-1}^+(\omega-\tfrac{1}{2}\omega_0)_0-\int\frac{d^4k'}{(2\pi)^4}I_0(\omega')\{v^*(\omega,\omega',\omega'')v(\omega-\omega_0,\omega'-\omega_0,\omega'')d_0^+(\omega'')-V_{(3)}(\omega,\omega',\omega-\omega_0,\omega'-\omega_0)\}$$
$$\times\left(\frac{q_{-1}^+(\omega'-\frac{1}{2}\omega_0)}{(\vec{k}'-\vec{k}_0)^2\bar{\epsilon}(\omega'-\omega_0)}\right). \tag{A1}$$

We identify the quantity $V_{(3)}$ from (3.7b) to be

$$V_{(3)}(\omega,\omega',\omega-\omega_0,\omega'-\omega_0)=\frac{4\pi e^4}{m^2}(-i)^3\int d^3v\,\vec{G}(\omega)\cdot\vec{k}'[\vec{G}(\omega'')\cdot(\vec{k}'-\vec{k}_0)\vec{G}(\omega-\omega_0)\cdot(\vec{k}-\vec{k}_0)$$
$$+\vec{G}(\omega'')\cdot(\vec{k}-\vec{k}_0)G(\omega_0-\omega')\cdot(\vec{k}'-\vec{k}_0)]F_e(v). \tag{A2}$$

The matrix elements v are well known,

$$v(\omega,\omega',\omega'')=(e/m)(\vec{k}\cdot\vec{k}')[\chi_e(k'',\omega'')/\omega_{pe}^2](k'')^2,$$
$$v(\omega-\omega_0,\omega'-\omega_0,\omega'')=(e/m)(\vec{k}-\vec{k}_0)\cdot(\vec{k}'-\vec{k}_0)$$
$$\times[\chi_e(k'',\omega'')/\omega_{pe}^2](k'')^2, \tag{A3}$$

where the last equality applies to the case $\omega''\ll k''v_e$. To evaluate $V_{(3)}$, we note $\vec{G}(\omega)=-i(\omega-\vec{k}\cdot\vec{v}+i0+)^{-1}\vec{\partial}_v$ and integrate by parts relative to the $\vec{\partial}_v$ operator in the first factor on the left of the integrand, $\vec{G}(\omega)$. The $\vec{\partial}_v$ operator in the factor $\vec{G}(\omega'')$ operates on the third G in the product as well as on $F_e(v)$. If we assume $\omega_0-\omega'$

$\gg |\vec{k}_0 - \vec{k}'|v_e$ and $\omega_0 - \omega \gg |\vec{k}_0 - \vec{k}|v_e$ which is valid for the EED instability, we can write

$$V_{(3)}(\omega, \omega', \omega - \omega_0, \omega' - \omega_0) = \frac{e^2}{m^2} \frac{(\vec{k}\cdot\vec{k}')(\vec{k} - \vec{k}_0)\cdot(\vec{k}' - \vec{k}_0)}{\omega_p^4} \chi_e(\vec{k}'', \omega'')(k'')^2 . \tag{A4}$$

The form involving the second velocity derivative of $F_e(v)$ vanishes identically in this case. Here

$$\chi_e(\vec{k}, \omega) = \frac{4\pi e^2}{m^2 k^2} \int d^3v\, \vec{G}(\vec{k}, \omega)\cdot \vec{k} F_e(v)$$

is the usual collisionless susceptibility. Next we replace $d_0^+(\vec{k}'', \omega'')$ in (A1) by $(k'')^{-2}\epsilon(\vec{k}'', \omega'')^{-1}$ and combine terms to get

$$q_{-1}^+(\omega - \tfrac{1}{2}\omega_0) = q_{-1}^+(\omega - \tfrac{1}{2}\omega_0)_0 + \frac{e^2}{m^2}\frac{1}{\omega_{pe}^4}\int \frac{d^4k'}{(2\pi)^4} I_0(\omega')(\vec{k}\cdot\vec{k}')(\vec{k} - \vec{k}_0)\cdot(\vec{k}' - \vec{k}_0)\chi_e(\vec{k}'', \omega'') \times \left(\frac{\chi_i(\vec{k}'', \omega'') + 1}{\epsilon(\vec{k}'', \omega'')}\right)(k'')^2 \frac{q_{-1}^+(\omega' - \tfrac{1}{2}\omega_0)}{(\vec{k}' - \vec{k}_0)^2 \tilde{\epsilon}(\omega' - \omega_0)} . \tag{A5}$$

Note the familiar combination $\chi_e^2/\epsilon - \chi_e = \chi_e(\chi_e/\epsilon - 1) = -\chi_e(1 + \chi_i)/\epsilon$ results from the two terms in curly brackets in (A1). The first term we recall arises from q_{mc} nonlinearity whereas the second arises from q_{nlld}.

We use the static approximations $\chi_{e,i}(\vec{k}'', \omega'') = k_{De,i}^2/k''^2$ and assume $k_0 \ll k, k'$ to obtain

$$q_{-1}^+(\omega - \tfrac{1}{2}\omega_0) = q_{-1}^+(\omega - \tfrac{1}{2}\omega_0)_0 + \frac{e^2}{m^2\omega_{pe}^4}\frac{k_{De}^2(T_e/T_i)}{1 + T_e/T_i} \times \int \frac{d^4k'}{(2\pi)^4} \frac{(\vec{k}\cdot\vec{k}')^2 I_0(\omega')}{(\vec{k}' - \vec{k}_0)^2\tilde{\epsilon}(\omega' - \omega_0)} \times q_{-1}^+(\omega' - \tfrac{1}{2}\omega_0) . \tag{A6}$$

Next, we use (4.16) to compute the integral over ω' and use the relation

$$\tilde{\Gamma}_-(\vec{k}) = \tfrac{1}{2}\omega_{pe} q_{-1}(\vec{k} - \tfrac{1}{2}\vec{k}_0, \omega_{NL} - \tfrac{1}{2}\omega_0)(\vec{k} - \vec{k}_0)^{-2} \tag{A7}$$

to change (A6) into the integral equation for $\tilde{\Gamma}_-(\vec{k})$ given in (4.17). To obtain (4.17) we have also used the near-resonance approximation

$$\tilde{\epsilon}(\vec{k}' - \vec{k}_0; \omega'_{NL} - \omega_0) \simeq \frac{\omega'_{NL} - \omega_0 + \tilde{\omega}_l(\vec{k}' - \vec{k}_0) + i\tilde{\gamma}_l(\vec{k}' - \vec{k}_0)}{-\tfrac{1}{2}\omega_{pe}} \simeq \frac{1}{2}\frac{[-\omega_0 + \tilde{\omega}_l(\vec{k}') + \tilde{\omega}_l(\vec{k}' - \vec{k}_0)] + i2\tilde{\gamma}_l(\vec{k}' - \vec{k}_0)}{-\tfrac{1}{2}\omega_{pe}} \simeq \frac{\Delta\omega - 2i\tilde{\gamma}_l(\vec{k}')}{-\omega_{pe}} .$$

[1]D. F. DuBois, Phys. Fluids (to be published).

[2]For an alternate, heuristic treatment of this problem, see B. Bezzerides and D. F. DuBois, Phys. Rev. Lett. 36, 729 (1976); see also, M. V. Goldman and D. F. DuBois, Air Force Technical Report No. AFWL-TR-72-101, 1972 (unpublished).

[3]D. F. DuBois and M. V. Goldman, Phys. Fluids 15, 419 (1972); Phys. Rev. Lett. 28, 218 (1972).

[4]E. Valeo, F. Perkins, and C. Oberman, Phys. Rev. Lett. 28, 340 (1972).

[5]W. L. Kruer and E. Valeo, Phys. Fluids 16, 675 (1973).

[6]Y. Kuo and J. A. Fejer, Phys. Rev. Lett. 29, 1667 (1972).

[7]B. Bezzerides and J. Weinstock, Phys. Rev. Lett. 28, 481 (1972).

[8]V. V. Postovalov and V. P. Silin, Zh. Eksp. Teor. Fiz. 59, 2215 (1970) [Sov. Phys.-JETP 32, 1198 (1971)].

[9]V. V. Postovalov, V. P. Silin, and V. T. Tikhonchuk, Zh. Eksp. Teor. Fiz. 65, 1880 (1973) [Sov. Phys.-JETP 38, 938 (1974)].

[10]D. F. DuBois, M. V. Goldman, and S. Bardwell, Bull. Am. Phys. Soc. 18, 1258 (1973), paper 1D5.

[11]B. B. Kadomstev, *Plasma Turbulence* (Academic, New York, 1965).

[12]R. Z. Sagdeev and A. A. Galeev, *Lectures on Nonlinear Plasma Theory* (Benjamin, New York, 1969).

[13]V. N. Tsytovich, *Nonlinear Effects in Plasmas* (Plenum, New York, 1970).

[14]D. F. DuBois and M. V. Goldman, Phys. Rev. 164, 207 (1967); M. V. Goldman, Ann. Phys. (N.Y.) 38, 95 (1966) (this reference uses techniques for the linear theory similar to those used in this paper); D. F. DuBois, in *Statistical Physics of Charged Particle Systems*, edited by R. Kubo and T. Kihara (Syokabo, Tokyo, 1969).

[15]This instability was studied by M. V. Goldman, thesis (Harvard University, 1965); published in Ann. Phys. (N.Y.) 38, 95 (1966). For further references see Ref. 9.

[16]A. A. Vedenov, A. V. Gordeev, and L. I. Rudakov,

Plasma Phys. 9, 719 (1967); V. E. Zakharov, Zh. Eksp. Teor. Fiz. 62, 1745 (1972) [Soc. Phys.-JETP 35, 908 (1972)].

[17]D. F. DuBois and B. Bezzerides, "Coupling Saturation in the Nonlinear Theory of Parametric Instability", in *Proceedings of the 1976 Nobel Symposium on Nonlinear Effects in Plasmas*, Aspenäsgården, Sweden (Plenum, New York, to be published).

[18]J. Weinstock and B. Bezzerides, Phys. Fluids 18, 251 (1974).

[19]E. J. Valeo and W. L. Kruer, Phys. Rev. Lett. 33, 750 (1974).