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Similarity transformation for explosions in two-component plasmas with thermal energy and heat-flux relaxation*

S. H. Choi and H. E. Wilhelm
Department of Electrical Engineering, Colorado State University, Fort Collins, Colorado 80523
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The nonlinear partial differential equations describing plane, cylindrical, and spherical explosions in a fully ionized electron-ion plasma with heat-flux relaxation and thermal relaxation are reduced to ordinary differential equations by means of novel similarity transformations. The resulting ordinary boundary-value problem for the plasma explosion, with the strong shock conditions as boundary values at the moving shock front, is formulated mathematically. The scaling laws for the plasma fields are presented which show how the plasma properties change with time during the course of the explosion. The importance of electron and ion heat-flux relaxation, which enhances the concentration of thermal energy behind the shock front, is stressed for the understanding of the shock-heating mechanism in fast processes. It is concluded that heat-flux relaxation is an important process for short-time plasma explosions, which determines the discontinuity of the electron and ion temperature fields at the shock front.

I. INTRODUCTION

The theory of blast waves in gas dynamics has been treated by means of similarity theory by a number of investigators.[1-4] Taylor[1] initiated the theoretical research on intense explosions by solving the problem of a spherical blast wave arising from an intense explosion due to instantaneous energy release in air. He showed[5] that the theoretical results are in good agreement with the atomic explosion in New Mexico of 1945. An extension of this work has been given by Jones[6] and Neumann.[7] Courant and Friedrichs[8] and Newton[9] have studied theoretically spherical blast waves by a method known as the progressing-wave approach. Taylor's[1] analysis of a strong spherical explosion has been extended to the plane and cylindrical cases by Sakurai,[10] Lin,[11] and Kompaneets.[12] Rogers[13] and Grag and Siekmann[14] have studied the similarity flows behind strong shock waves when the explosion energy is released from finite spherical charges. In these investigations, gas motions due to the shock-wave phenomenon in strong explosions are described by one-component gas-dynamic equations with all dissipative terms being neglected (including heat conduction).

Korobeinikov[15] has discussed the propagation of a strong spherical blast wave in a heat-conducting gas with a heat conductivity $\lambda \propto T^{1/6}$. He shows that the temperature at the center is finite, in accordance with energy dissipation due to heat conduction. A similar problem was treated by Neuvazhaev,[16] who assumed that the temperature is continuous at the shock front. Plane, cylindrical, and spherical explosions taking temperature-dependent viscosity and heat conductivity into account were analyzed by Sedov[17] for certain temperature dependences of viscosity μ and heat conductivity λ ($\mu, \lambda \propto T^{1/2}$, T^0, and $T^{1/6}$ for plane, cylindrical, and spherical explosions, respectively). These approaches[15-17] assume a relaxation-free heat-flux vector $\vec{q} = -\lambda(T)\nabla T$, which leads to a parabolic diffusion equation for the temperature field.

Extensions of similarity theory to explosions in magnetohydrodynamics and electrogas-dynamics have made by Korobeinikov,[18] Greenspan,[19] Greifinger and Cole,[20] and Oshima.[21] Oshima[21] has treated the propagation of a strong electron blast wave propagating through an electrically neutral plasma. As a model, a collisionless partially ionized plasma composed of electrons, ions, and neutral particles is considered. He gives particular attention to charge separation due to the large mass difference of the electrons and ions. In this case, self-similar solutions of the electrogas-dynamic blast wave no longer exist for constant energy release.

As noted, the explosion theories of Korobeinikov,[15] Sedov,[17] and Neuvazhaev[16] assume a relaxation-free heat-flux vector $\vec{q} = -\lambda\nabla T$, which implies that a temperature gradient ∇T produces quasi-instantaneously a heat flux $-\lambda\nabla T$. As has been shown by the present authors,[22] this assumption leads to physical (false speed of thermal wave front, wrong shape of thermal wave) and mathematical (divergent energy integral) difficulties. In order to avoid the unacceptable consequences of the quasistatic approximation for the heat-flux vector, the relaxation equations for the heat-flux vectors of the electron and ion components of the plasma are applied here to strong explosions in two-component plasmas. The heat-flux relaxation equations, which lead to hyperbolic wave equations for the temperature fields, are based on the Boltzmann equation. The electron and ion temperatures are discontinuous at the shock front because of the hyperbolic character of the partial differential equations describing these fields. In view of the nonisothermal behavior ($T_e \neq T_i$) of the plasma, thermal relaxation between the electron and ion gases through scalar heat flow is considered. A similarity transformation is presented which reduces the nonlinear partial plasma equations to ordinary nonlinear differential equations. The similarity transformation shows how the fields of the electron and ion gases change, at any point in space, with time during strong explosions in two-component plasmas.

II. PHYSICAL PRINCIPLES

Consider an experiment in which a large amount of energy is released in a steady-state fully ionized plasma along a plane, a line, or at a point. A plane, cylindrical, or spherical explosion wave will then be propagated through an infinite region starting from the plane, line, or point where the energy is liberated. The energy release under consideration takes place in a plasma consisting of electrons and ions.

Experiment shows that an abrupt jump in the characteristics of the motion takes place on the boundary of the distrubed region during an explosion, and a shock wave is formed. For strong shock waves which occur for sufficiently large energy releases, it is permitted to neglect the initial gas pressure in comparison with the pressure behind the shock wave.[23] The radius of the shock wave increases from $r=0$ with increasing time, and the wave moves away from the center with time-dependent velocity. The abrupt jump at the shock front makes it possible to assume that the motion can be approximated by a mathematical discontinuity at the wave front. The assumption of discontinuous solutions is justified mathematically by the hyperbolic equations describing the plasma fields.

It is supposed that the plasma behaves as an ideal gas with a constant specific-heat ratio $c_p/c_v \equiv \gamma = \frac{5}{3}$ (c_p and c_v are the specific heats at constant pressure and volume, respectively). The equations of state are, for the electron and ion gases,

$$p_s = n_s k T_s, \quad s = e, i,$$

where p_s, n_s, and T_s are the partial pressure, particle density, and temperature of electrons and ions as indicated by the subscripts, and k is the Boltzmann constant.

Due to the absence of external electric and magnetic fields, the plasma remains quasineutral during the course of the explosion, so that

$$n_e = Z n_i, \quad n_i \equiv n,$$

where Z designates the ionic charge number. Consequently, the electrons and ions move together (collective motion),

$$\vec{v}_e = \vec{v}_i = \vec{v},$$

where $\vec{v}_s$ ($s = e, i$) is the mean mass velocity of the s component (Debye shielding length is assumed small compared to the characteristic dimension of the plasma). Because of the large difference between electron mass and ion mass ($m_e/m_i \ll 1$), one has

$$\rho = n_e m_e + n_i m_i \approx n_i m_i \equiv n m_i .$$

The momentum relaxation times τ_{sr} for the r- and s-particle components ($r, s = e, i$) are, in the absence of electron-ion drift ($\vec{v}_e = \vec{v}_i$), given by[24]

$$\tau_{sr}^{-1} = \tfrac{8}{3}(2kT_{sr}/\pi m_{sr})^{1/2}(m_{sr}/m_s) n_r Q_{sr} . \tag{1}$$

The thermal relaxation times τ_{sr}^T are related to the momentum relaxation times τ_{sr} by[24]

$$\tau_{sr}^T = \tfrac{1}{2}(m_r/m_{sr})\tau_{sr} . \tag{2}$$

The reduced mass m_{sr} and temperature T_{sr} are defined by

$$m_{sr} = \frac{m_s m_r}{m_s + m_r}, \quad T_{sr} = m_{sr}\left(\frac{T_s}{m_s} + \frac{T_r}{m_r}\right). \tag{3}$$

The transport cross sections Q_{sr} are for binary Coulomb interactions in a fully ionized plasma given by

$$Q_{sr} = \tfrac{1}{2}\pi\,(e_s e_r/kT_{sr})^2 \Lambda_{sr}, \tag{4}$$

where $e_{s,r}$ are the electric charges of the particles. Λ_{sr} is the Coulomb logarithm[25]

$$\Lambda_{sr} = \ln(2kT_{sr}D/|e_s e_r|),$$

where

$$D = \left(4\pi \sum_{r=e,i} \frac{n_r e_r^2}{kT_r}\right)^{-1/2}, \tag{5}$$

is the Debye length. The relaxation time τ_s of the heat-flux vector $\vec{q}_s$ is related to the momentum relaxation times by[26]

$$\tau_e^{-1} = \tfrac{4}{5}\tau_{ee}^{-1} + \tfrac{13}{10}\tau_{ei}^{-1} = \tfrac{4}{5}\tau_{ee}^{-1}[1 + \tfrac{13}{8}\sqrt{2}\,Z(\Lambda_{ei}/\Lambda_{ee})], \tag{6}$$

$$\tau_i^{-1} = \tfrac{4}{5}\tau_{ii}^{-1} + 3\tau_{ie}^{-1}$$
$$= \tfrac{4}{5}\tau_{ii}^{-1}\left[1 + \frac{15}{4Z}\left(\frac{2m_e}{m_i}\right)^{1/2}\frac{\Lambda_{ie}}{\Lambda_{ii}}\left(\frac{T_i}{T_e}\right)^{3/2}\right] \cong \tfrac{4}{5}\tau_{ii}^{-1}, \tag{7}$$

since the effect of e-i cross collisions on τ_i is negligible for $T_e \geqslant T_i$.

III. BOUNDARY-VALUE PROBLEM

With $n_i = n_e/Z \equiv n$ and the relaxation times in Eqs. (2), (6), and (7), the coupled nonlinear partial differential equations for plane ($\sigma = 0$), cylindrical ($\sigma = 1$), and spherical ($\sigma = 2$) explosions produced by explosive energy release from a plane ($\sigma = 0$), along a line ($\sigma = 1$), and from a point ($\sigma = 2$) in the plasma are[27] ($\theta_s = kT_s$, $s = e, i$):

$$\frac{\partial n}{\partial t} + v\frac{\partial n}{\partial r} = -n\left(\frac{\partial v}{\partial r} + \sigma\frac{v}{r}\right), \tag{8}$$

$$nm_i\left(\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial r}\right) = -(\theta_i + Z\theta_e)\frac{\partial n}{\partial r} - n\frac{\partial}{\partial r}(\theta_i + Z\theta_e), \tag{9}$$

$$\frac{3Zn}{2}\left(\frac{\partial \theta_e}{\partial t} + v\frac{\partial \theta_e}{\partial r}\right) = -Zn\theta_e\left(\frac{\partial v}{\partial r} + \sigma\frac{v}{r}\right) - \frac{3}{2}\frac{Z}{\tau_0}n^2\frac{\theta_e - \theta_i}{\theta_e^{3/2}} - \left(\frac{\partial}{\partial r}q_e + \sigma\frac{q_e}{r}\right), \tag{10}$$

$$\frac{3n}{2}\left(\frac{\partial \theta_i}{\partial t} + v\frac{\partial \theta_i}{\partial r}\right) = -n\theta_i\left(\frac{\partial v}{\partial r} + \sigma\frac{v}{r}\right) - \frac{3}{2}\frac{Z}{\tau_0}n^2\frac{\theta_i - \theta_e}{\theta_e^{3/2}} - \left(\frac{\partial}{\partial r}q_i + \sigma\frac{q_i}{r}\right), \tag{11}$$

$$\frac{\partial q_e}{\partial t} + \frac{5}{2}\frac{Zn}{m_e}\theta_e\frac{\partial \theta_e}{\partial r} + \frac{7q_e}{5}\left(\frac{\partial v}{\partial r} + \sigma\frac{v}{r}\right) + v\frac{\partial q_e}{\partial r} + \frac{9q_e}{5}\frac{\partial v}{\partial r} = -\frac{Znq_e}{\mu_e\theta_e^{3/2}}, \tag{12}$$

$$\frac{\partial q_i}{\partial t} + \frac{5}{2}\frac{n}{m_i}\theta_i\frac{\partial \theta_i}{\partial r} + \frac{7q_i}{5}\left(\frac{\partial v}{\partial r} + \sigma\frac{v}{r}\right) + v\frac{\partial q_i}{\partial r} + \frac{9q_i}{5}\frac{\partial v}{\partial r} = -\frac{nq_i}{\mu_i\theta_i^{3/2}}, \tag{13}$$

where

$$\begin{aligned}\tau_0^{-1} &= \tfrac{8}{3}(2\pi/m_e)^{1/2}(m_e/m_i)Z^2e^4\Lambda_{ie},\\ \mu_e^{-1} &= \tfrac{16}{15}(\pi/m_e)^{1/2}e^4\Lambda_{ee}[1 + \tfrac{13}{8}\sqrt{2}\,Z(\Lambda_{ei}/\Lambda_{ee})],\\ \mu_i^{-1} &= \tfrac{16}{15}(\pi/m_i)^{1/2}Z^4e^4\Lambda_{ii}\end{aligned} \tag{14}$$

are constants which are related to the relaxation times τ_{ei}^T, τ_{ie}^T, and τ_e, τ_i, respectively. The heat-flux relaxation Eqs. (12) and (13) are derived in the Appendix.

The explosion is initiated by a time-dependent energy release from quasiplane ($\sigma = 0$), quasiline ($\sigma = 1$), and quasipoint ($\sigma = 2$) sources, respectively, of the form (E_0, ν are constant)

$$E = E_0 t^\nu \ \mathrm{erg\,cm}^{\sigma-2}, \quad 0 \leqslant t < \infty. \tag{15}$$

This energy will be, under ideal conditions, conserved in form of kinetic and thermal energies of the electrons and ions. Because of the collective motion ($v_e = v_i$) and the large difference of mass of the electrons and ions ($m_e \ll m_i$), the kinetic energy of the electrons is negligible compared to the kinetic energy of the ions. Accordingly, the solutions of Eqs. (8)–(13) are subject to the integral constraint

$$E = E_0 t^\nu = \zeta_\sigma \int_0^{R(t)} \left(\frac{nm_i v^2}{2} + \frac{3n(Z\theta_e + \theta_i)}{2}\right) r^\sigma dr, \tag{16}$$

$$\zeta_\sigma \equiv \begin{cases} 2, & \sigma = 0,\\ 2\pi, & \sigma = 1,\\ 4\pi, & \sigma = 2,\end{cases}$$

where $R(t)$ is the radius of the explosion front at time $t \geqslant 0$. Since the released energy is concentrated behind the front of the explosion wave, the upper limit of integration in Eq. (16) is given by $R(t)$.

The plasma fields behind (0) and in front of (1) the shock front are related at the shock front by the jump conditions (strong shock assumption, $p_{s1} \gg p_{s0}$, $s = e, i$),

$$n_1 u_1 = n_0 u_0, \tag{17a}$$

$$n_1(Z\theta_e + \theta_i)_1 + n_1 m_i u_1^2 = n_0 m_i u_0^2, \tag{17b}$$

$$[\tfrac{1}{2}n_1 Z m_e u_1^2 + \tfrac{5}{2}n_1 Z\theta_{e1} + (q_{e1}/u_1)]u_1 = \tfrac{1}{2}n_0 Z m_e u_0^2 u_0, \tag{17c}$$

$$[\tfrac{1}{2}n_1\ m_i u_1^2 + \tfrac{5}{2}n_1\theta_{i1} + (q_{i1}/u_1)]u_1 = \tfrac{1}{2}n_0 m_i u_0^2 u_0, \tag{17d}$$

where $q_{s0} = 0$ ahead of the shock front if initially $q_s = 0$ in the unperturbed plasma, since heat propagates with finite speed (hyperbolic theory). The velocities ($u_{1,0}$) in the shock system are related to those ($v_{1,0}$) in the laboratory system by

$$u_1 = v_1 - \dot{R} \quad \text{and}\ u_0 = -\dot{R}, \tag{18}$$

where $v_0 = 0$, since the medium ahead of the shock front is at rest and $\dot{R}$ ($\equiv dR/dt$) designates the velocity of the shock front. Accordingly, Eqs. (17) can be reduced with the help of Eq. (18) and Eq. (17a) to

$$\begin{aligned} &n_1(\dot{R} - v_1) = n_0\dot{R},\\ &n_1(Z\theta_{e1} + \theta_{i1}) + n_0 m_i \dot{R}(\dot{R} - v_1) = n_0 m_i \dot{R}^2,\\ &\tfrac{1}{2}n_0 Z m_e \dot{R}(\dot{R} - v_1)^2 + \tfrac{5}{2}n_0 Z\theta_{e1}\dot{R} - q_{e1} = \tfrac{1}{2}n_0 Z m_e \dot{R}^3,\\ &\tfrac{1}{2}n_0 m_i \dot{R}(\dot{R} - v_1)^2 + \tfrac{5}{2}n_0\theta_{i1}\dot{R} - q_{i1} = \tfrac{1}{2}n_0 m_i \dot{R}^3.\end{aligned} \tag{19}$$

The scalar heat flows $\sim(\theta_e - \theta_i)$ between the electron and ion components do not contribute to the energy-shock conditions in Eqs. (17) and (19) in the limit of vanishing shock thickness, $\Delta \to 0$. In addition to the jump conditions, the plasma fields satisfy for symmetry reasons the boundary conditions

$$v(r=0,t)=0, \quad q_s(r=0,t)=0, \qquad \left.\frac{\partial n(r,t)}{\partial r}\right|_{r=0}=0, \quad \left.\frac{\partial \theta_s(r,t)}{\partial r}\right|_{r=0}=0. \tag{20}$$

The temperature fields θ_e and θ_i are necessarily discontinuous at $R(t)$ as a result of the hyperbolic nature of the equations describing θ_s. In the static or parabolic approximation, however, θ_e and θ_i may be continuous across the shock front.[16,22] Since the partial two-component equations (8)–(13) have similarity solutions with a single shock radius, $R_e(t)=R_i(t)=R(t)$, separate shock fronts do not exist for the electron and ion gases as long as $D \ll R(t)$. Equations (12) and (13) do not lead to shock conditions in the limit $\Delta \to 0$, since these partial differential equations determine only the temporarily and spatially "delayed" heat fluxes $\vec{q}_s$ produced by the temperature gradients $\nabla\theta_s$. Similarly, in the static approximation, the corresponding relaxation-free heat-flux equations, $\vec{q}_s = -\lambda_s \nabla\theta_s/k$, do not yield additional shock conditions.

IV. SIMILARITY TRANSFORMATIONS

A. Reduction to ordinary differential equations

The plasma explosion problem formulated in Eqs. (8)–(13) can be reduced to a boundary-value problem for nonlinear ordinary differential equations by means of a similarity transformation. In this approach, a similarity variable is introduced by

$$\xi = r/R(t), \quad R(t)=At^{\alpha}, \quad [A]=LT^{-\alpha}, \tag{21}$$

where A and α are constants which are determined below. The dimensions of the physical variables are

$$[n]=L^{-3}, \quad [v]=LT^{-1}, \quad [\theta_s]=ML^2T^{-2}, \quad [q_s]=MT^{-3}.$$

Introducing six nondimensional functions $N(\xi),\dots,Q_s(\xi)$ of the similarity variable ξ, a self-similar solution of Eqs. (8)–(13) is sought, for dimensional reasons, in the form (B and ω are constant)

$$\begin{aligned} &n(r,t)=(B/R^{\omega})N(\xi), \quad v(r,t)=\dot{R}V(\xi),\\ &\theta_e(r,t)=m_i\dot{R}^2\Theta_e(\xi), \quad \theta_i(r,t)=m_i\dot{R}^2\Theta_i(\xi),\\ &q_e(r,t)=m_iB(\dot{R}^3/R^{\omega})Q_e(\xi),\\ &q_i(r,t)=m_iB(\dot{R}^3/R^{\omega})Q_i(\xi). \end{aligned} \tag{22}$$

The similarity transformation in Eqs. (8)–(13) exists if the variables t and ξ in Eqs. (8)–(13) can be separated. This can be shown to be the case if

$$\frac{R\ddot{R}}{\dot{R}^2}=\text{const}, \quad \frac{R^{1-\omega}}{\dot{R}^4}=\text{const}, \quad \ddot{R}\equiv\frac{d^2R}{dt^2}. \tag{23}$$

Hence

$$R(t)=At^{4/(\omega+3)}. \tag{24}$$

Recalling $R(t)=At^{\alpha}$, one finds that the similarity exponent α takes the value

$$\alpha = 4/(\omega+3). \tag{25}$$

Equation (23) is therefore reduced to

$$\frac{R\ddot{R}}{\dot{R}^2}=\frac{1-\omega}{4}, \quad \frac{R^{1-\omega}}{\dot{R}^4}=\frac{(\omega+3)^4}{256A^{\omega+3}}. \tag{26}$$

Elimination of $R\ddot{R}/\dot{R}^2$ and $R^{1-\omega}/\dot{R}^4$ in accordance with Eq. (26) reduces the partial differential equations (8)–(13) to the nonlinear ordinary coupled differential equations:

$$(V-\xi)\frac{dN}{d\xi}-\omega N=-N\left(\frac{dV}{d\xi}+\sigma\frac{V}{\xi}\right), \tag{27}$$

$$N\left(\frac{1-\omega}{4}V+(V-\xi)\frac{dV}{d\xi}\right)=-(\Theta_i+Z\Theta_e)\frac{dN}{d\xi}-N\frac{d}{d\xi}(\Theta_i+Z\Theta_e), \tag{28}$$

$$\frac{3ZN}{2}\left(\frac{1-\omega}{2}\Theta_e+(V-\xi)\frac{d\Theta_e}{d\xi}\right)=-ZN\Theta_e\left(\frac{dV}{d\xi}+\sigma\frac{V}{\xi}\right)-\frac{3\kappa_1}{2}\frac{\Theta_e-\Theta_i}{\Theta_e^{3/2}}N^2-\left(\frac{dQ_e}{d\xi}+\sigma\frac{Q_e}{\xi}\right), \tag{29}$$

$$\frac{3N}{2}\left(\frac{1-\omega}{2}\Theta_i+(V-\xi)\frac{d\Theta_i}{d\xi}\right)=-N\Theta_i\left(\frac{dV}{d\xi}+\sigma\frac{V}{\xi}\right)-\frac{3\kappa_1}{2}\frac{\Theta_i-\Theta_e}{\Theta_e^{3/2}}N^2-\left(\frac{dQ_i}{d\xi}+\sigma\frac{Q_i}{\xi}\right), \tag{30}$$

$$\frac{3-7\omega}{4}Q_e-\xi\frac{dQ_e}{d\xi}+\frac{5Z}{2}\frac{m_i}{m_e}N\Theta_e\frac{d\Theta_e}{d\xi}+\frac{7Q_e}{5}\left(\frac{dV}{d\xi}+\sigma\frac{V}{\xi}\right)+V\frac{dQ_e}{d\xi}+\frac{9Q_e}{5}\frac{dV}{d\xi}=-\kappa_2\frac{NQ_e}{\Theta_e^{3/2}}, \tag{31}$$

$$\frac{3-7\omega}{4}Q_i-\xi\frac{dQ_i}{d\xi}+\frac{5N\Theta_i}{2}\frac{d\Theta_i}{d\xi}+\frac{7Q_i}{5}\left(\frac{dV}{d\xi}+\sigma\frac{V}{\xi}\right)+V\frac{dQ_i}{d\xi}+\frac{9Q_i}{5}\frac{dV}{d\xi}=-\kappa_3\frac{NQ_i}{\Theta_i^{3/2}}, \tag{32}$$

where

$$\kappa_1\equiv\frac{Z(\omega+3)^4}{256\tau_0 m_i^{3/2}}\frac{B}{A^{\omega+3}},$$

$$\kappa_2 \equiv \frac{Z(\omega+3)^4}{256\mu_e m_i^{3/2}} \frac{B}{A^{\omega+3}}, \tag{33}$$

$$\kappa_3 \equiv \frac{(\omega+3)^4}{256\mu_i m_i^{3/2}} \frac{B}{A^{\omega+3}}$$

are nondimensional constant coefficients which are related to the known constants τ_0, μ_e, and μ_i and the parameters A, B, and ω.

B. Determination of the parameters A, B, and ω

Consideration of the dimensions of the fundamental constants m_i, E_0 and τ_0 of the system $\{[\mu_s] = [\tau_0]$ in Eq. (14)$\}$,

$$[E_0] = ML^{\sigma}T^{-(\nu+2)}, \quad [\tau_0] = M^{-3/2}L^{-6}T^4,$$

$$[A] = LT^{-4/(\omega+3)}, \quad [B] = L^{\omega-3},$$

permits formation of two quantities which may serve as the characteristic length and time scale, i.e.,

$B^{1/(\omega-3)}$, with dimension L,

$E_0^{3/(2-3\nu)}\tau_0^{2/(2-3\nu)}B^{3(\sigma-4)/(\omega-3)(3\nu-2)}$,

with dimension T.

Furthermore, these two scales make it possible to construct the parameter A, by the method of combination of characteristic scales, which does not contain the unit of mass,[17]

$$A = E_0^{12/(3\nu-2)(\omega+3)} \tau_0^{8/(3\nu-2)(\omega+3)} \times B^{[(\omega+3)(3\nu-2)+12(4-\sigma)]/(\omega^2-9)(3\nu-2)}. \tag{34}$$

Hence Eq. (34) gives for the radius of the shock front $R(t)$ in Eq. (24)

$$R(t) = E_0^{12/(3\nu-2)(\omega+3)} \tau_0^{8/(3\nu-2)(\omega+3)} \times B^{[(\omega+3)(3\nu-2)+12(4-\sigma)]/(\omega^2-9)(3\nu-2)} t^{4/(\omega+3)}. \tag{35}$$

In order to determine the parameters B and ω, it is necessary to use the equation of energy conservation: Substitution of Eq. (22) into Eq. (16) gives

$$E_0 t^{\nu} = \zeta_{\sigma} m_i B \dot{R}^2 R^{\sigma-\omega+1} J, \tag{36}$$

where

$$J = \int_0^1 \left(\frac{NV^2}{2} + \frac{3N(Z\Theta_e + \Theta_i)}{2}\right) \xi^{\sigma} d\xi \tag{37}$$

is a nondimensional numerical constant. Substitution for $R(t)$ in accordance with Eq. (24) gives for Eq. (36)

$$E_0 t^{\nu} = \zeta_{\sigma} J[4/(\omega+3)]^2 m_i B A^{\sigma-\omega+3} t^{2(2\sigma-3\omega+3)/(\omega+3)}. \tag{38}$$

This equation can be satisfied for all times $0 \leq t < \infty$ only if

$$\nu = 2(2\sigma - 3\omega + 3)/(\omega+3), \tag{39}$$

or

$$\omega = (4\sigma - 3\nu + 6)/(\nu+6). \tag{40}$$

Equations (34) and (38) represent, under consideration of Eq. (39), two independent relations for the constants A and B from which one obtains by elimination

$$A = \left(\frac{\sigma+6}{\nu+6}\right)^{5/(\sigma+6)} \left(\frac{1}{\zeta_{\sigma} m_i J}\right)^{5/2(\sigma+6)} \left(\frac{E_0}{\tau_0}\right)^{1/(\sigma+6)}, \tag{41}$$

$$B = \{\zeta_{\sigma}[(\nu+6)/(\sigma+6)]^2 m_i J\}^{(3\nu-2)/2(\nu+6)} \times E_0^{4/(\nu+6)} \tau_0^{(\nu+2)/(\nu+6)}. \tag{42}$$

Consequently, the radius of the shock front in Eq. (35) becomes, by consideration of Eq. (42),

$$R(t) = \left(\frac{\sigma+6}{\nu+6}\right)^{5/(\sigma+6)} \left(\frac{1}{\zeta_{\sigma} m_i J}\right)^{5/2(\sigma+6)} \left(\frac{E_0}{\tau_0}\right)^{1/(\sigma+6)} t^{4/(\omega+3)}. \tag{43}$$

Thus all parameters which appear in the system of differential equations have been obtained by dimensional considerations and from the energy constraint (16).

The power ω in the similarity statement of Eq. (22) is restricted by the shock conditions in Eq. (19). Accordingly, ω takes the value zero when energy is liberated in a homogeneous plasma with constant initial ion density n_0, while ω is arbitrary for energy release arising in an inhomogeneous plasma with an initial ion density proportional to $r^{-\omega}$.

For explosions in a plasma model with homogeneous initial ion density n_0, one has by Eq. (39)

$$\omega = 0, \quad \nu = \tfrac{2}{3}(2\sigma+3). \tag{44}$$

In this case, the parameters A and B can be determined in a simpler way. The parameter B has the dimension of L^{-3}, so that the initial ion density n_0 can take the place of B for dimensional reasons. Thus the parameter B in Eq. (22) is replaced by n_0, and the dimension of A is

$$[A] = LT^{-4/3}, \quad \text{for } \omega = 0.$$

This parameter A, for $\omega = 0$, is determined by Eq. (38),

$$E_0 t^{\nu} = \zeta_{\sigma} J(\tfrac{4}{3})^2 m_i n_0 A^{\sigma+3} t^{2(2\sigma+3)/3}.$$

Hence

$$A = (\tfrac{9}{16} E_0/\zeta_{\sigma} J m_i n_0)^{1/(\sigma+3)}, \tag{45}$$

by Eq. (44), where J is given in Eq. (37), i.e., J changes numerically with A and B [Eqs. (27)–(33)]. Thus the radius of the shock front becomes

$$R(t) = (\tfrac{9}{16} E_0/\zeta_{\sigma} J m_i n_0)^{1/(\sigma+3)} t^{4/3}, \tag{46}$$

which implies that the radius is proportional to $t^{4/3}$ for plane, cylindrical, and spherical explosions. Thus all parameters have been defined for the particular case $\omega = 0$.

Equation (39) ($\nu \neq 0$) implies that for the special case of constant energy release in strong plasma explosions, taking heat conduction into account, the self-similar solutions no longer exist in an initially homogeneous plasma. The existence for self-similar solutions for constant energy release ($\nu = 0$) is, in accordance with Eq. (40), possible only for $\omega = \frac{1}{3}(2\sigma + 3)$, i.e., for explosions arising in originally inhomogeneous plasmas with varying initial ion density $n_0 \propto r^{-(2\sigma+3)/3}$.

V. APPLICATION TO HOMOGENEOUS PLASMAS ($\omega = 0$)

For $\omega = 0$, the boundary conditions in Eqs. (19) and (20) are readily reduced with the help of the self-similar transformation in Eq. (22) to nondimensional form:

$$\begin{aligned} &N_1(1 - V_1) = n_0/B, \\ &(1 - V_1) + (B/n_0)N_1(Z\Theta_{e1} + \Theta_{i1}) = 1, \\ &\tfrac{1}{2}m_e Z(1 - V_1)^2 + \tfrac{5}{2} m_i Z\Theta_{e1} - m_i (B/n_0)Q_{e1} = \tfrac{1}{2} Z m_e, \\ &\tfrac{1}{2}(1 - V_1)^2 + \tfrac{5}{2}\Theta_{i1} - (B/n_0)Q_{i1} = \tfrac{1}{2}, \quad \text{at } \xi = 1, \end{aligned} \tag{47}$$

and

$$\begin{aligned} &V_{\xi=0} = 0, \quad [Q_s]_{\xi=0} = 0, \\ &\left[\frac{dN}{d\xi}\right]_{\xi=0} = 0, \quad \left[\frac{d\Theta_s}{d\xi}\right]_{\xi=0} = 0. \end{aligned} \tag{48}$$

For the case $\omega = 0$, in which n_0 is used as a characteristic parameter, B in Eq. (33) is replaced by n_0 and A is defined by Eq. (45), and $n_0/B \equiv 1$ in Eq. (47). In accordance with Eqs. (45) and (46), the self-similar solution in Eq. (22) is written in the form

$$\begin{aligned} &n(r,t) = n_0 N(\xi), \\ &v(r,t) = \tfrac{4}{3}(\tfrac{9}{16}E_0/\zeta_\sigma J m_i n_0)^{1/(\sigma+3)} t^{1/3} V(\xi), \\ &\theta_e(r,t) = \tfrac{16}{9} m_i (\tfrac{9}{16}E_0/\zeta_\sigma J m_i n_0)^{2/(\sigma+3)} t^{2/3} \Theta_e(\xi), \\ &\theta_i(r,t) = \tfrac{16}{9} m_i (\tfrac{9}{16}E_0/\zeta_\sigma J m_i n_0)^{2/(\sigma+3)} t^{2/3} \Theta_i(\xi), \\ &q_e(r,t) = \tfrac{64}{27} m_i n_0 (\tfrac{9}{16}E_0/\zeta_\sigma J m_i n_0)^{3/(\sigma+3)} t Q_e(\xi), \\ &q_i(r,t) = \tfrac{64}{27} m_i n_0 (\tfrac{9}{16}E_0/\zeta_\sigma J m_i n_0)^{3/(\sigma+3)} t Q_i(\xi), \end{aligned} \tag{49}$$

where

$$\xi = (\tfrac{9}{16}E_0/\zeta_\sigma J m_i n_0)^{-1/(\sigma+3)} r t^{-4/3} \tag{50}$$

for the plane ($\sigma = 0$), cylindrical ($\sigma = 1$), and spherical ($\sigma = 2$) explosions.

In order to find the plasma fields at points $0 \leq r \leq R(t)$ behind the shock front, the distributions $N(\xi)$, $V(\xi)$, $\Theta_e(\xi)$, $\Theta_i(\xi)$, $Q_e(\xi)$, and $Q_i(\xi)$ have to be determined for $0 \leq \xi \leq 1$. This requires numerical integration of Eqs. (27)–(32) with the boundary conditions at $\xi = 1$ given by the shock conditions in Eq. (47) and at $\xi = 0$ by the symmetry conditions in Eq. (48).

As an illustration, an electron-deuteron plasma ($Z = 1$) is considered with the average Coulomb logarithms: $\Lambda_{ie} = \Lambda_{ei} = \Lambda_{ee} = \Lambda_{ii} = 12.12429$ for $\overline{T} = 10^6$ °K, $\overline{n}_i = 10^{15}$ cm^{-3}. The constants related to the thermal (τ_0) and heat-flux (μ_e, μ_i) relaxation times are given in Table I.

In accordance with Eq. (46), the radius of the shock front $R(t)$ at time t is proportional to

$$R(t) \propto (E_0/m_i n_0)^{1/(\sigma+3)} t^{4/3}. \tag{51}$$

This means that the radius of the shock front increases with time more rapidly than in ordinary gas dynamics, in which[1] $R(t) \propto t^{2/5}$ for spherical explosions. Figure 1 shows the time dependence of the radius of the shock front. Furthermore, the propagation velocity of the shock wave increases slowly with time, i.e.,

$$\dot{R}(t) \propto (E_0/m_i n_0)^{1/(\sigma+3)} t^{1/3}. \tag{52}$$

This is in contrast to the velocity of the shock front in an ordinary blast wave, where $\dot{R}(t) \propto t^{-3/5}$ for spherical geometry.[1] This is due to Eqs. (15) and (44), i.e., the total energy increase with time. In accordance with Eqs. (51) and (52),

$$\dot{R} = \frac{dR}{dt} \propto \frac{R}{t} \propto \left(\frac{E_0}{n_0 m_i}\right)^{3/4(\sigma+1)} R^{1/4}. \tag{53}$$

It is seen that the velocity of the shock wave increases slowly as the shock wave propagates.

The similarity transformations show how the fields of the electrons and ions change with time. Since the motion is self-similar, the shapes of the distributions $N(\xi)$, $V(\xi)$, $\Theta_e(\xi)$, $\Theta_i(\xi)$, $Q_e(\xi)$, and $Q_i(\xi)$ do not change with time. Accordingly, the density, velocity, electron temperature, ion temperature, electron heat flux, and ion heat flux at the shock front, for plane, cylindrical, and spherical explosions, depend on time t as

$$\begin{aligned} &n \propto n_0 t^0, \quad v \propto (E_0/m_i n_0)^{1/(\sigma+3)} t^{1/3}, \\ &\theta_e \propto \left(\frac{E_0}{m_i n_0}\right)^{2/(\sigma+3)} t^{2/3}, \quad \theta_i \propto \left(\frac{E_0}{m_i n_0}\right)^{2/(\sigma+3)} t^{2/3}, \\ &q_e \propto m_i n_0 \left(\frac{E_0}{m_i n_0}\right)^{3/(\sigma+3)} t^1, \quad q_i \propto m_i n_0 \left(\frac{E_0}{m_i n_0}\right)^{3/(\sigma+3)} t^1. \end{aligned} \tag{54}$$

TABLE I. Constants related to thermal and heat-flux relaxation times.

τ_0 (erg$^{-3/2}$cm^{-3} sec)	2.561×10^{25}
μ_e (erg$^{-3/2}$cm^{-3} sec)	7.480×10^{21}
μ_i (erg$^{-3/2}$cm^{-3} sec)	1.495×10^{24}

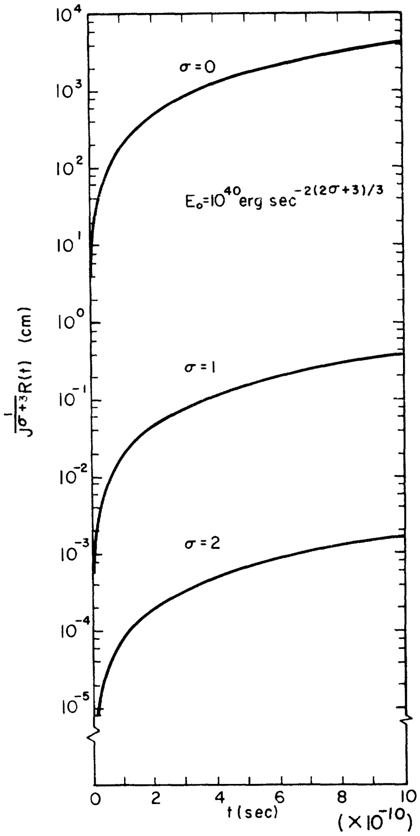

FIG. 1. Shock-wave radius for plane ($\sigma = 0$), cylindrical ($\sigma = 1$), and spherical ($\sigma = 2$) explosions for $E_0 = 10^{40}$ erg sec$^{-2(2\sigma+3)/3}$.

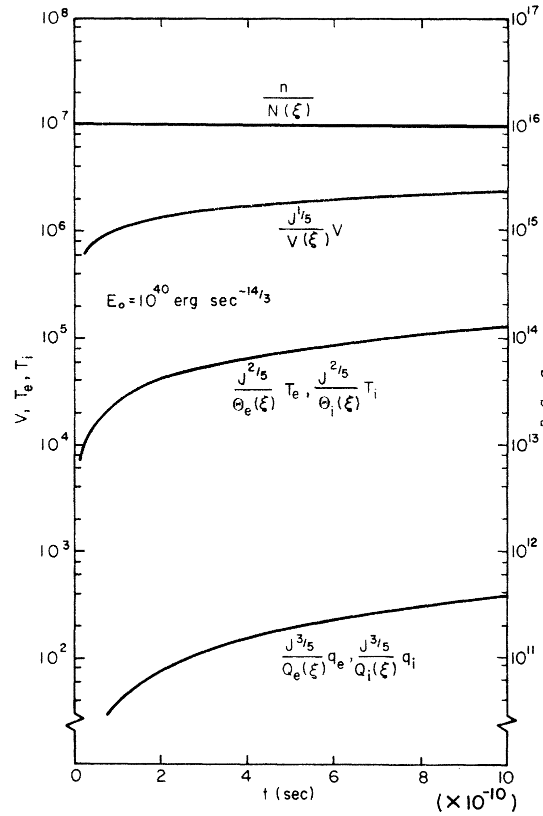

FIG. 2. Fields of the electron and ion components in a strong spherical explosion ($\sigma = 2$, deuterium plasma) in dependence of time.

It is seen, e.g., that the ion density at the shock front ($\xi = 1$) does not change with time and has the value $4n_0$, while the velocity, temperatures, and heat fluxes at the shock front increase with t, as shown in Fig. 2. The physical meaning of the time dependence of the electron and ion temperature fields in Eq. (54) is readily understood. At a time t the shock wave reaches a radius $R(t)$, and encompasses a volume $\frac{4}{3}\pi R^3$ for spherical explosions. The thermal energy density is proportional to the average energy per unit volume, i.e.,

$$(\rho c T)_{e,i} \propto E/R^3 \propto E_0 t^{\nu} R^{-3} \propto t^{14/3} t^{-4} \propto t^{2/3}, \quad \sigma = 2,$$

i.e., $T_{e,i} \propto t^{2/3}$ where $\nu = \frac{14}{3}$ for spherical explosions. Analogous consideration holds for plane and cylindrical explosions.

Equations (54) give

$$n \propto t^0 \propto R^0, \quad v \propto t^{1/3} \propto R^{1/4}, \qquad \theta_{e,i} \propto t^{2/3} \propto R^{1/2}, \quad q_{e,i} \propto t \propto R^{3/4}. \tag{55}$$

The heat-flux field of the electron and ion components increases more rapidly than the velocity or temperature fields as the shock wave propagates. Furthermore, the velocity, temperatures, and heat fluxes would go to infinity as $R \to \infty$. Since the total explosion energy is limited, $E(t) < \infty$, one has $R(t) < \infty$ and $t < \infty$.

In Eq. (46), the radius of the shock front R can be considered as a characteristic distance of the plasma. Then, for this self-similar solution, the Debye shielding distance D in Eq. (5) is proportional to $T_s^{1/2}$ and by Eq. (54)

$$D \propto T_s^{1/2} \propto \theta_s^{1/2} \propto t^{1/3}, \tag{56}$$

whereas the characteristic distance R is, by Eq. (51), $R \propto t^{4/3}$. This implies that $R \gg D$ as $t \to \infty$. Therefore the assumed quasineutrality of the plasma in the absence of external magnetic and electric fields is justified for $t > 0$, if the plasma was quasineutral at $t = 0$.

The effect of heat-flux relaxation is qualitatively important at all times of the explosion, since it affects considerably the electron and ion temperature distributions; in particular, it determines their discontinuous behavior at the shock front. According to Eq. (12), heat-flux relaxation is negligible only for extremely large times

$$t \gg \tau_e, \quad \tau_e = \mu_e (kT_e)^{3/2}/Zn. \tag{57}$$

VI. CONCLUSION

New similarity transformations for the two-component plasma-fluid equations with thermal and heat-flux relaxations governing strong plasma explosions have been derived. This explosion theory is important for the understanding of the dynamics of plasma heating by strong blast waves generated in plasmas by means of laser beams[28,29] and electron[30,31] and ion[30,31] beams for the purpose of producing thermonuclear fusion.[32] Plasma heating by means of these mechanisms requires consideration of the energy exchange between the electron and ion components and heat conduction in the plasma components with heat-flux relaxation. When ordinary hydrodynamical effects are dominant, the well-known self-similar solution[1,17,33-35] for a strong explosion in one-component gases can be used for the explanation of plasma heating.

The replacement of the static approximation, $\vec{q}_s = -\lambda_s \nabla T_s$, by the heat-flux relaxation equations (12) and (13) avoids the difficulties of the phenomenological parabolic heat-conduction theory[22] and permits introduction of unified shock conditions for all plasma fields, i.e., including the electron and ion temperatures. The usual assumption of a continuous temperature distribution[16] at the shock front is due to the (inconsistent) static heat-flux equation.

As a result of heat-flux relaxation, the thermal energy in the explosion remains concentrated behind the shock front, i.e., it is not dissipated partly ahead of the shock front, as follows from the parabolic heat-conduction equation.[16] An illustration of the effect of heat-flux relaxation on the transient energy distributions due to an instantaneous energy release has been given elsewhere.[22]

In the present plasma explosion theory, significant approximations made in the previous approaches to the problem have been removed. The transport coefficients are based on the Boltzmann equation for binary Coulomb interactions.[24-27] The Lenard-Balescu equation, which considers also particle-wave interactions, gives essentially the same transport coefficients for momentum[36] and energy[37] exchange as the Boltzmann equation for fast processes, which are still slow compared to high-frequency wave phenomena. In this connection, it is assumed that the plasma explosion is laminar, since for turbulent plasma motions particle-wave interactions at distances larger than the Debye radius become important.[38] It should be noted, however, that energy transport by (nonequilibrium) radiation has been disregarded for mathematical reasons to make a similarity solution feasible.

APPENDIX

The basic equations (8)–(11) are the (one-dimensional) conservation equations for mass, momentum, and energy which are derived as moments of the Boltzmann equation.[27] Since the heat flux $\vec{q}_s$ is defined as the moment,

$$\vec{q}_s = \int \int \int_{-\infty}^{+\infty} \frac{m_s c_s^2 \vec{c}_s f_s}{2} \, d\vec{v}_s, \tag{A1}$$

the conservation equation for the heat flux $\vec{q}_s$ is obtained as the corresponding moment of the Boltzmann equation ($\vec{E}, \vec{H}$ is the electromagnetic field):

$$\int \int \int_{-\infty}^{+\infty} \frac{\partial f_s}{\partial t} \frac{m_s c_s^2 \vec{c}_s}{2} \, d\vec{v}_s + \int \int \int_{-\infty}^{+\infty} \frac{\partial}{\partial \vec{r}} \cdot (\vec{v}_s f_s) \frac{m_s c_s^2 \vec{c}_s}{2} \, d\vec{v}_s + \int \int \int_{-\infty}^{+\infty} \frac{\partial}{\partial \vec{v}_s} \cdot \left[\frac{e_s}{m_s} \left(\vec{E} + \frac{\vec{v}_s}{c} \times \vec{H} \right) f_s \right] \frac{m_s c_s^2 \vec{c}_s}{2} \, d\vec{v}_s = \sum_r \int \int \int_{-\infty}^{+\infty} \left[\frac{\delta f_s}{\delta t} \right]_r \frac{m_s c_s^2 \vec{c}_s}{2} \, d\vec{v}_s, \tag{A2}$$

where

$$\left[\frac{\delta f_s}{\delta t} \right]_r \equiv \int \cdots \int [f_r^* f_s^* - f_r f_s] |\vec{v}_r - \vec{v}_s| \sigma_{rs}(\vec{v}_r - \vec{v}_s, \theta) \, d\Omega \, d\vec{v}_r \tag{A3}$$

is the collision integral for binary (r-s) Coulomb interactions[24] and

$$\vec{c}_s = \vec{v}_s - \langle \vec{v}_s \rangle \tag{A4}$$

is the thermal velocity ($\vec{v}_s$ is the actual velocity and $\langle \vec{v}_s \rangle$ the mean mass velocity) of the s particles. Since $\vec{r}$, $\vec{v}_s$, and t are independent variables and $\vec{c}_s = \vec{c}_s(\vec{r}, t)$ [since $\langle \vec{v}_s \rangle = \langle \vec{v}_s(\vec{r}, t) \rangle$], (A2) yields ($\vec{\delta}$ is the unit tensor: $\delta_{ii} = 1$, $\delta_{i \neq j} = 0$)

$$\frac{\partial \vec{q}_s}{\partial t}+\frac{e_s}{m_s c}\vec{H}\times\vec{q}_s+\frac{\frac{5}{2}p_s\vec{\delta}+\vec{p}_s}{n_s m_s}\cdot\sum_r\int\int\int_{-\infty}^{+\infty}\left[\frac{\delta f_s}{\delta t}\right]_r m_s\vec{v}_s\,d\vec{v}_s+\vec{q}_s\nabla\cdot\langle\vec{v}_s\rangle$$
$$+\langle\vec{v}_s\rangle\cdot\nabla\vec{q}_s+\vec{q}_s\cdot\nabla\langle\vec{v}_s\rangle+\tfrac{2}{5}(\vec{q}_s\cdot\nabla\langle\vec{v}_s\rangle+\nabla\langle\vec{v}_s\rangle\cdot\vec{q}_s+\vec{q}_s\nabla\cdot\langle\vec{v}_s\rangle)+\frac{p_s\vec{\delta}-\vec{p}_s}{n_s m_s}\cdot\nabla\cdot\vec{p}_s$$
$$-\frac{kT_s}{m_s}\vec{p}_s\cdot\nabla\ln n_s+\frac{5k}{2m_s}(p_s\vec{\delta}+\vec{p}_s)\cdot\nabla T_s=\sum_r\int\int\int_{-\infty}^{+\infty}\left[\frac{\delta f_s}{\delta t}\right]_r\frac{m_s c_s^2\vec{c}_s}{2}\,d\vec{v}_s, \quad \text{(A5)}$$

where

$$n_s=\int\int\int_{-\infty}^{+\infty}f_s\,d\vec{v}_s,\quad \langle\vec{v}_s\rangle=\int\int\int_{-\infty}^{+\infty}\vec{v}_s f_s\frac{d\vec{v}_s}{n_s},$$
$$p_s=\frac{2}{3}\int\int\int_{-\infty}^{+\infty}\frac{m_s c_s^2 f_s}{2}\,d\vec{v}_s,$$
$$\vec{p}_s=\int\int\int_{-\infty}^{+\infty}m_s\left(\vec{c}_s\vec{c}_s-\frac{c_s^2\vec{\delta}}{3}\right)f_s\,d\vec{v}_s,$$

are the density, mean mass velocity, static pressure, and viscous stress tensor ($p_{s,ii}=0$) of the s particles. The third term in (A5) is derived by means of the moment equation for the variable $m_s\vec{v}_s$ (heat flux due to barodiffusion). The third- and fourth-order moments in (A2) have been evaluated in accordance with the 13-moment approximation,[39]

$$\int\int\int_{-\infty}^{+\infty}m_s c_{s,i}c_{s,j}c_{s,k}f_s\,d\vec{v}_s=\tfrac{2}{5}(q_{s,i}\delta_{jk}+q_{s,j}\delta_{ki}+q_{s,k}\delta_{ij}), \quad \text{(A6)}$$

$$\int\int\int_{-\infty}^{+\infty}m_s c_{s,i}c_{s,i}c_{s,j}c_{s,k}f_s\,d\vec{v}_s=(kT_s/m_s)(7p_{s,jk}+5p_s\delta_{jk}). \quad \text{(A7)}$$

For the explosion in the nonviscous electron-ion plasma under consideration one has $\vec{p}_{e,i}=\vec{0}$, $\vec{E}=\vec{0}$, $\vec{H}=\vec{0}$, and $\langle\vec{v}_e\rangle=\langle\vec{v}_i\rangle$. For this case, (A5) reduces, if terms of third order in the inhomogeneities are neglected, to

$$\frac{\partial\vec{q}_s}{\partial t}+\vec{q}_s\nabla\cdot\langle\vec{v}_s\rangle+\langle\vec{v}_s\rangle\cdot\nabla\vec{q}_s+\vec{q}_s\cdot\nabla\langle\vec{v}_s\rangle$$
$$+\tfrac{2}{5}(\vec{q}_s\cdot\nabla\langle\vec{v}_s\rangle+\nabla\langle\vec{v}_s\rangle\cdot\vec{q}_s+\vec{q}_s\nabla\cdot\langle\vec{v}_s\rangle)$$
$$+(\tfrac{5}{2}k/m_s)p_s\nabla T_s\cong-\vec{q}_s/\tau_s. \quad \text{(A8)}$$

In the 13-moment approximation,[24] the collision integrals give $-\vec{q}_s/\tau_s$ if terms of order m_e/m_i are disregarded. The cross-collision integrals ($rs=ei, ie$) contribute to the relaxation times τ_s for the heat fluxes $\vec{q}_s$ as indicated in Eqs. (6) and (7). From (A8) follow the heat-flux relaxation equations (12) and (13) for radial flow of heat. The last two terms in (A8) together form the static heat-flux–force relation $\vec{q}_s=-\lambda_s\nabla T_s$, since the heat conductivity is given by $\lambda_s=(\frac{5}{2}k/m_s)p_s\tau_s$.

Equation (A8) indicates that the phenomenological heat-flux relation $\vec{q}_s=-\lambda_s\nabla T_s$ is to be replaced by a nonlinear partial differential equation for the heat-flux field $\vec{q}_s(\vec{r},t)$. On principle, $\vec{q}_s(\vec{r},t)$ can be eliminated from Eq. (A8) as an integral functional of the driving forces producing the heat flow. If a similarity transformation exists, the problem reduces to eliminating $Q_s(\xi)$ from the corresponding ordinary nonlinear differential equation [Eqs. (31) and (32)] as an integral functional. Let this be demonstrated as follows:

(i) In the lowest approximation, it is sufficient to consider only temporal relaxation. Equation (A8) then reduces to

$$\frac{\partial\vec{q}_s}{\partial t}+\frac{\vec{q}_s}{\tau_s}\cong-\frac{\lambda_s}{\tau_s}\nabla T_s\ . \quad \text{(A9)}$$

Accordingly,

$$\vec{q}_s(\vec{r},t)=\left[\vec{q}_s(\vec{r},t=0)-\int_0^t\lambda_s\tau_s^{-1}\nabla T_s\exp\left(+\int_0^t\tau_s^{-1}dt\right)dt\right]\exp\left(-\int_0^t\tau_s^{-1}dt\right), \quad \text{(A10)}$$

where λ_s and τ_s are functions of t through their dependence on n_s and T_s.

(ii) In general, the velocity divergence terms in Eqs. (31) and (32) are quantitatively of subordinate importance. Hence an approximate form of Eqs. (31) and (32) is

$$(V-\xi)\frac{dQ_s}{d\xi}+\left(\frac{3-7\omega}{4}+\kappa_s N\Theta_s^{-3/2}\right)Q_s\cong-\frac{5Z_s}{2}\frac{m_i}{m_s}N\Theta_s\frac{d\Theta_s}{d\xi}. \quad \text{(A11)}$$

Accordingly,

$$Q_s(\xi) = \left[Q_s(\xi=0) - \frac{5Z_s}{2}\frac{m_i}{m_s}\int_0^\xi \frac{N\Theta_s}{V-\xi}\frac{d\Theta_s}{d\xi}\exp\left(+\int_0^\xi w_s(\xi)\,d\xi\right)d\xi\right]\exp\left(-\int_0^\xi w_s(\xi)\,d\xi\right),$$
$$w_s(\xi) \equiv [\tfrac{1}{4}(3-7\omega) + \kappa_s N\Theta_s^{-3/2}]/(V-\xi), \qquad \text{(A12)}$$

where $0 \leq \xi \leq 1$ and $\xi = 1$ at the shock front, and $\kappa_e = \kappa_2$ and $\kappa_i = \kappa_3$ [Eq. (33)].

Equations (A10) and (A12) show that the solutions for $\vec{q}_s(\vec{r}, t)$ and $Q_s(\xi)$ are complicated integral functionals of the driving forces $\sim\nabla T_s$ and $\sim d\Theta_s/d\xi$, respectively. The simple proportionality between flux and driving force exists only in the phenomenological theory as a rough approximation for slow processes.

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