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Approximation schemes for strongly coupled two-component plasmas*

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A formalism appropriate for obtaining a self-consistent handling of the strongly coupled two-component plasma problem is discussed. The divergence of the classical electron-ion pair correlation function is removed with the aid of a phenomenological "soft-core" potential. The important formal development is the introduction of partial linear and nonlinear polarizabilities, their interrelations, and the linear and nonlinear fluctuation-dissipation theorems satisfied by them. The formalism is used to generalize existing strongly coupled one-component plasma theories for the two-component situations. Both the schemes based on the first Bogoliubov-Born-Green-Kirkwood-Yvon (BBGKY) equation and the fluctuation-dissipation theorems, and the scheme based on the second BBGKY equation and the decomposition of the triplet correlation function are developed into self-consistent two-component equations. Algebraically, the equations appear as a set of three, coupled, nonlinear integral equations for pair correlation functions or polarizabilities.

I. INTRODUCTION

The problem of strongly coupled electron plasmas has been attacked by various groups[1-6] and through different approaches with a considerable success. The methods can be classified according to whether they rely on the first Bogoliubov-Born-Green-Kirkwood-Yvon (BBGKY) or on the second BBGKY equation. The former can be characterized as considering the dielectric function as the central object, deriving an expression for it from the first BBGKY equation, and then guaranteeing self-consistency through the use of fluctuation-dissipation-theorem (FDT) -type relations. Hubbard,[1] Singwi, Tosi, Sjolander, and Land[2] (STSL), and the present authors and Silevitch[5] (GKS) followed this approach in different ways. In the second BBGKY equation approach the central object is the equilibrium pair correlation function for which the equation is made self-consistent by the introduction of a decomposition of the triplet correlation function into clusters of the pair correlation functions. Ichimaru[3] and Totjusi and Ichimaru[4] (TI) have pursued this method; it has, however, been shown by the present authors and Datta[6] that the fundamental equation of the GKS approach can also be derived through the second BBGKY equation without any further approximation.

All of the different approaches listed above have achieved important results. The equilibrium pair correlation function has been calculated by numerically solving integral equations which result from the theory[2,4,7,8]; equations of state have been computed[4,8]; conditions for phase transition have been estimated[4]; and the theories have been refined to the point that original inconsistencies concerning the satisfaction of sum-rule requirements can be reduced or removed.[4,5,8]

In the light of the important progress made in the field of one-component strongly coupled plasmas, the investigation of the equivalent two-component plasma systems is much less developed and our knowledge about their physical properties is considerably more modest. Nevertheless, many physical systems, in particular some which have acquired importance recently, cannot be approximated as a one-component system. Laser-compressed plasmas,[9] electron-hole liquids,[10] and high-Z stellar interiors[11] are in this category. Since for an ion gas the coupling parameter γ (inverse number of particles in the Debye sphere) is proportional to $Z^{5/2}$, in certain cases it can be argued that the approximation which treats the ion gas as strongly coupled and the electron gas as a smeared-out background is reasonable. However, the electrons are certainly not dynamically inert, and even if they were, it is not *a priori* evident how a strongly coupled ion gas in a weakly coupled electron background should be treated.

The purpose of the present paper is to develop the formal theory of strongly coupled two-component plasmas. We will adopt the ideas already existing for electron plasmas and adapt them to the two-component situations. No attempt will be made here to introduce new approximation schemes or physical simplifications. The thrust of the paper is to rewrite the three main electron plasma approximation schemes, i.e., the Singwi, GKS, and Ichimaru schemes, into their two-component equivalents. Attempts in this direction have already been made by Vashishta, Bhattacaryya, and Singwi[12] with regard to the Singwi scheme; they have not, however, derived full self-consistent relations. Rather, they used an *ad hoc* assumption for the correlation functions.

The two-component system is obviously more complex than the one-component one, indeed to a substantial degree so, which already makes the task of this paper more than trivial. In addition, the two-component system is fraught with some special difficulties. The foremost among them is the divergence of the classical electron-ion pair correlation functions. This is, of course, a defect of the classical theory, and properly taking into account the atomic bound states would remove it. However, in the framework of the classical description a modification of the interaction is required. Such a new phenomenological interaction will be introduced and employed in this work. A problem on a different level arises from the fact that when the number of plasma species is greater than one (say, K), the number of physical polarizabilities is K, but the number of pair correlation functions in the system is $\frac{1}{2}K(K+1)$. Thus even for $K=2$ the straightforward application of linear FDT's becomes impossible. Instead of polarizabilities, however, one can work in terms of density-density response functions, or alternatively, follow Vashishta *et al.*,[12] who suggested the use of "partial" polarizabilities, describing the response of the system to fictitious fields which act on the electrons or on the ions only. This latter method will be adopted in this paper and will be fully developed and generalized.

The results of any of the three schemes appear in the form of a *set* of nonlinear integral equations which can be formulated either in terms of polarizabilities or pair correlation functions. The STLS and TI schemes are fully self-consistent, but the GKS theory, as presented here, is not, since it requires a further specified approximation for the quadratic polarizabilities. Although such an approximation scheme is fairly straightforward, its discussion will be relegated to a subsequent publication, together with the similar problem for the one-component system.

The plan of the paper now can be sketched as follows: In Sec. II we introduce the partial linear and nonlinear polarizabilities, their relations to the physical full polarizabilities, and connections between external (responding to an external field) and total (responding to a total field) polarizabilities. Linear and nonlinear FDT's connecting pair and triplet correlation functions with the partial polarizabilities are established. At the same time the phenomenological interactions accounting for quantum effects and eliminating the electron-ion divergence are introduced. In Sec. III the derivations of the STLS and GKS schemes from the first BBGKY equations are presented. The GKS expressions that can also be derived from the second BBGKY equation are exact, and exhibit a structure for the partial linear polarizabilities in terms of the full physical linear quadratic polarizabilities which represent the first establishment of such exact relations. In Sec. III the TI scheme is developed from the second BBGKY equation.

As we have already emphasized, the results of this work are formal. Solution or approximations of the equations derived will be the objective of later works.

II. ELECTRODYNAMIC AND FLUCTUATION-DISSIPATION RELATIONS

In this section we summarize and display relations which play a key role in the development of the formalism for a two-component system. These relations are partly connections between "partial" and "full," or "external" and "total," response functions (both linear and quadratic), and partly FDT-type relations.

First, as was noted in the Introduction, a physical problem which does not arise in the one-component theory relates to the negatively divergent interaction energy of an electron-ion pair at a small separation distance. As a consequence, the pair and higher-order correlation functions diverge as the separation distance $r\to 0$. Such a divergence is not only unphysical, but, in an approximation scheme such as the present one, where the pair correlation function is ultimately determined through an integral equation, it also renders the consistent solution of the integral equation impossible. On the other hand, it is well known that this divergence difficulty, which also manifests itself in the divergence of the classical plasma partition function, is the defect of the classical theory, and properly taking into account the atomic bound states removes it. A simple phenomenological treatment of the problem has been suggested by Dunn and Broyles.[13] It consists of "softening" the interaction potential by multiplying it by factor $1-e^{-\mu r}$, where μ^{-1} is of the order of the Bohr radius a. As pointed out by Gombert and Deutsch[14] a similar modification of the interaction potential between *like* particles, where μ^{-1} is of the order of the de Broglie wavelength λ, fairly well describes quantum-diffraction effects. Thus in this vein we introduce the effective interaction potentials $\psi_{AB}(k)$ between particles of species A and B as

$$\psi_{AB}(k)=B_{AB}(k)\phi_{AB}(k),$$
$$\phi_{AB}(k)=Z_A Z_B\, 4\pi e^2/k^2,$$
$$B_{AB}(k)=\mu^2_{AB}/(\mu^2_{AB}+k^2),\tag{1}$$
$$Z_e=-1,\quad Z_i=Z.$$

Although the three characteristic distances μ_{ee}^{-1}, μ_{ii}^{-1}, and μ_{ei}^{-1} are physically certainly different (order of λ_e, λ_i and a, respectively), not much harm is done by setting $\mu_{ee}^{-1}=\mu_{ii}^{-1}=a$, since under normal conditions the effect of quantum corrections on the particle interaction is small. Such a simplification leads to a very substantial enhancement of the transparency of the otherwise fairly cumbersome formalism, and it will be followed throughout our derivations, although final results will be displayed for the general $B_{ee}(k)\neq B_{ii}(k)\neq B_{ei}(k)$ case.

Linear and quadratic (longitudinal) external polarizabilities ${}_1\hat{\alpha}_A\equiv\hat{\alpha}$, ${}_2\hat{\alpha}_A$ are defined, as usual, as the coefficients of the expansion of the plasma field $\mathcal{E}_A$ due to species A $(=e, i)$ in terms of an applied external field $\hat{E}$.

In a further publication we shall further employ the so-called partial external polarizabilities $\hat{\alpha}_A^B$ defined by contemplating a perturbation caused by a partial field $\hat{E}^B$ which acts on species B only.[12] They are useful, however, in the equal-B case only, when[15]

$$\mathcal{E}_A^{(1)}(\vec{k}\omega)=-\sum_B \hat{\alpha}_A^B(\vec{k}\omega)\hat{E}^B(\vec{k}\omega),\tag{2a}$$

$$\mathcal{E}_A^{(2)}(\vec{k}\omega)=-2\pi V\sum_{BC}\sum_{\vec{q}\mu}\sum_{\vec{p}\nu}{}_2\hat{\alpha}_A^{BC}(\vec{q}\mu,\vec{p}\nu)\hat{E}^B(\vec{q}\mu)\times\hat{E}^C(\vec{p}\nu)\delta_{\vec{k}-\vec{p}-\vec{q}}\delta(\omega-\mu-\nu).\tag{2b}$$

To deal with the case where the $B(k)$ functions are unequal it is useful to introduce partial external charge-density response functions $\hat{\chi}_A^B$,

$$\rho_A^{(1)}(\vec{k}\omega)=-\sum_B\hat{\chi}_A^B(\vec{k}\omega)\hat{E}^B(\vec{k}\omega),\tag{3a}$$

$$\rho_A^{(2)}(\vec{k}\omega)=-2\pi V\sum_{BC}\sum_{\vec{q}\mu}\sum_{\vec{p}\nu}{}_2\hat{\chi}_A^{BC}(\vec{q}\mu,\vec{p}\nu)\,\hat{E}^B(\vec{q}\mu)\times\hat{E}^C(\vec{p}\nu)\delta_{\vec{k}-\vec{q}-\vec{p}}\delta(\omega-\mu-\nu).\tag{3b}$$

We note that only the projections symmetric under the interchange of the superscripts and of the arguments of ${}_2\hat{\alpha}_A^{BC}$, ${}_2\hat{\chi}_A^{BC}$ are physically meaningful.

We note the simple relation between partial and full (physical) polarizabilities,

$$\hat{\alpha}_A=\sum_B\hat{\alpha}_A^B,\quad \hat{\alpha}=\sum_A\hat{\alpha}_A,\tag{4a}$$

$${}_2\hat{\alpha}_A=\sum_{BC}{}_2\hat{\alpha}_A^{BC},\quad {}_2\hat{\alpha}=\sum_A{}_2\hat{\alpha}_A.\tag{4b}$$

Total (linear and quadratic) polarizabilities ${}_1\alpha_A\equiv\alpha_A$, ${}_2\alpha_A$ are the expansion coefficients of the plasma field in terms of the total (plasma plus external) field, and are related to the external polarizabilities by

$$\hat{\alpha}_A(\vec{k}\omega)=\alpha_A(\vec{k}\omega)/\epsilon(\vec{k}\omega),\tag{5a}$$

$${}_2\hat{\alpha}_A(\vec{q}\mu,\vec{p}\nu)=\frac{[1+\alpha_B(\vec{q}+\vec{p},\mu+\nu)]{}_2\alpha_A(\vec{q}\mu,\vec{p}\nu)-\alpha_A(\vec{q}+\vec{p},\mu+\nu){}_2\alpha_B(\vec{q}\mu,\vec{p}\nu)}{\epsilon(\vec{q}\mu)\epsilon(\vec{p}\nu)\epsilon(\vec{q}+\vec{p},\mu+\nu)},\quad B\neq A,\tag{5b}$$

$$\epsilon(\vec{k}\omega)=1+\alpha(\vec{k}\omega).$$

Relations similar to (4a) and (4b) exist between the partial and full total polarizabilities, and thus in view of (5b) the well-known relation

$${}_2\hat{\alpha}(\vec{q}\mu,\vec{p}\nu)=\frac{{}_2\hat{\alpha}(\vec{q}\mu,\vec{p}\nu)}{\epsilon(\vec{q}\mu)\epsilon(\vec{p}\nu)\epsilon(\vec{q}+\vec{p},\mu+\nu)}\tag{6}$$

is satisfied.

Next we list static ($\omega=0$) fluctuation-dissipation theorems which will be used in Sec. III. These are relations between pair and triplet correlation functions [$g_{AB}(\vec{k})$ and $h_{ABC}(\vec{q}\vec{p})$] on the one hand and the previously defined external polarizabilities on the other hand.

The equilibrium pair and triplet correlation functions g_{AB}, h_{ABC} will be used in the usual way. With F_A, G_{AB}, and H_{ABC} being the full one-, two-, and three-particle distribution functions, they are introduced through the relations

$$G_{AB}^{(0)}(\vec{x}_1\vec{v}_1,\vec{x}_2\vec{v}_2)=F_A^{(0)}(\vec{v}_1)F_B^{(0)}(\vec{v}_2)[1+g_{AB}(12)],$$
$$H_{ABC}^{(0)}(\vec{x}_1\vec{v}_1,\vec{x}_2\vec{v}_2,\vec{x}_3\vec{v}_3)=F_A^{(0)}(\vec{v}_1)F_B^{(0)}(\vec{v}_2)F_C^{(0)}(\vec{v}_3)\,[1+g_{AB}(12)+g_{AC}(13)+g_{BC}(23)+h_{ABC}(123)],$$
$$g_{AB}(12)\equiv g_{AB}(|\vec{x}_1-\vec{x}_2|),\qquad (7)$$
$$h_{ABC}(123)\equiv h_{ABC}(|\vec{x}_1-\vec{x}_2|,|\vec{x}_1-\vec{x}_3|),$$

and $g_{AB}(\vec{k})$ and $h_{ABC}(\vec{q}\vec{p})$ are the Fourier transforms[15] of $g_{AB}(12)$ and $h_{ABC}(123)$, respectively.

For the case where $B_{ee}(k)=B_{ei}(k)=B_{ii}(k)=B(k)$, the linear[16] and nonlinear[16,17] static FDT's are

$$\hat{\alpha}_A^B(\vec{k})=\alpha_{A0}(\vec{k})[\delta_{AB}+(Z_AZ_B/|Z_AZ_B|)n_Ag_{AB}(\vec{k})],\qquad (8a)$$
$${}_2\hat{\alpha}_C^{AB}(\vec{q}\vec{p})={}_2\alpha_{C0}(\vec{q}\vec{p})(Z_AZ_B/Z_C^2)[\delta_{AC}\delta_{BC}+n_A\delta_{BC}g_{AC}(\vec{q})+n_B\delta_{AC}g_{BC}(\vec{p})+n_A\delta_{AB}g_{AC}(\vec{p}+\vec{q})+n_An_Bh_{CAB}(\vec{q}\vec{p})],\qquad (8b)$$

where

$$\alpha_{A0}(\vec{k})=Z_A^2 4\pi\beta n_Ae^2B(k)/k^2,\quad {}_2\alpha_{C0}(\vec{q}\vec{p})=-iZ_C^3 2\pi\beta^2n_Ce^2B(k)/qp|\vec{q}+\vec{p}|,$$
$$Z_e=-1,\quad Z_i=Z,\quad \alpha(\vec{k})\equiv\alpha(\vec{k},\omega=0),\quad \text{etc.},\qquad (9)$$

are the linear and quadratic Vlasov polarizabilities of the plasma fluid with the effects of the "soft core" included. We note that evidently $\hat{\alpha}_A^B=\hat{\alpha}_B^A$, and that in (8b) ${}_2\hat{\alpha}_C^{AB}$ is the *unsymmetrized* external polarizability. We further observe that in view of the relations (4a) and (4b) the following *total* FDT's are formed from (8a) and (8b):

$$\hat{\alpha}_A(\vec{k})=\alpha_{A0}(\vec{k})[1+n_Ag_{AA}(\vec{k})-n_Ag_{AB}(\vec{k})],\quad B\neq A,\qquad (10a)$$
$${}_2\hat{\alpha}_e(\vec{q}\vec{p})={}_2\alpha_{e0}(\vec{q}\vec{p})\Big(1+n_e\sum_{\vec{l}=\vec{q},\vec{p},\vec{q}+\vec{p}}g_{ee}(\vec{l})-n_eg_{ei}(\vec{q})-n_eg_{ei}(\vec{p})+Zn_eg_{ei}(\vec{q}+\vec{p})$$
$$+n_e^2[h_{eee}(\vec{q}\vec{p})-h_{eei}(\vec{q}\vec{p})-h_{eie}(\vec{q}\vec{p})+h_{eii}(\vec{q}\vec{p})]\Big),\qquad (10b)$$
$${}_2\hat{\alpha}_i(\vec{q}\vec{p})={}_2\alpha_{i0}(\vec{q}\vec{p})\Big(1+n_i\sum_{\vec{l}=\vec{q},\vec{p},\vec{p}+\vec{q}}g_{ii}(\vec{l})-n_ig_{ie}(\vec{q})-n_ig_{ie}(\vec{p})+\frac{n_i}{Z}g_{ei}(\vec{q}+\vec{p})$$
$$+n_i^2[h_{iii}(\vec{q}\vec{p})-h_{iie}(\vec{q}\vec{p})-h_{iei}(\vec{q}\vec{p})+h_{iee}(\vec{q}\vec{p})]\Big).$$

For the case of the three $B(k)$ functions being unequal, the $\hat{\alpha}$'s are supplanted by the $\hat{\chi}$'s, and the more generalized FDT relations now read

$$\hat{\chi}_A^B(\vec{k})=(\beta n_AZ_A{}^2e^2/k)[\delta_{AB}+(Z_AZ_B/|Z_AZ_B|)n_Ag_{AB}(k)],\qquad (11a)$$
$${}_2\hat{\chi}_C^{AB}(\vec{q}\vec{p})=Z_AZ_BZ_C(\beta^2n_Ce^3/2qp)[\delta_{AC}\delta_{BC}+n_A\delta_{BC}g_{AC}(\vec{q})+n_B\delta_{AC}g_{BC}(\vec{p})+n_A\delta_{AB}g_{AC}(\vec{p}+\vec{q})+n_An_Bh_{CAB}(\vec{q}\vec{p})].\qquad (11b)$$

The expressions for the partial polarizabilities in terms of the full polarizabilities are not obvious and are model dependent. A set of such relations has been derived by Vashishta *et al.*[12] In spite of its appealing simplicity and transparent structure, it suffers from the general shortcomings of the Singwi approximation scheme. In contrast, in Sec. III we derive a general relation from the GKS approximation scheme which is *exact*.

III. FIRST BBGKY APPROACH: STSL AND GKS SCHEMES

In this section we present the derivations of the relations for the partial polarizabilities in both the STSL and GKS schemes. In conjunction with the linear FDT relations, the (approximate) STSL relations provide a full self-consistent set of equations for the equilibrium pair correlation functions. In contrast, the GKS relations, which are exact, still contain the triplet correlation functions which can be traded, with the aid of the nonlinear FDT relations, for nonlinear polarizabilities. In order to achieve self-consistency, however, further hypotheses eliminating the latter would be needed—quite analogously to the situation in the one-component system.

The derivation is based on the introduction of a small perturbation in the first BBGKY equation. The two approaches differ from each other in the handling of the perturbed part of the correlation function; however, no explicit use of the second BBGKY equation is made—hence the distinction indicated by the titles of this and the next sections.

The first BBGKY equation for type-A plasma particles is

$$-i(\omega-\vec{k}\cdot\vec{v})F_A(\vec{k}\omega,\vec{v})+\frac{Z_Ae}{m_A}\sum_{\vec{q}\mu}\hat{E}(\vec{q}\mu)\cdot\frac{\partial F_A(\vec{k}-\vec{q},\omega-\mu,\vec{v})}{\partial\vec{v}}$$
$$=\frac{i}{Vm_A}\frac{\partial}{\partial\vec{v}}\cdot\sum_B\sum_{\vec{q}}\vec{q}\psi_{AB}(q)\int d^3v'G_{AB}(\vec{k}-\vec{q},\vec{q};\vec{v},\vec{v}';\omega)$$
$$=\frac{i}{m_A}\frac{1}{V}\sum_B\sum_{\vec{q}\mu}\sum_{\vec{p}}\vec{q}\psi_{AB}(q)\cdot\vec{\Gamma}_{AB}(\vec{q}\mu;\vec{k}\omega;\vec{p})\,. \quad (12)$$

It is the structures of the respective Γ_{AB}'s which distinguish the two approximation schemes from each other.

A. STSL scheme

The main feature of the approach of STSL consists of approximating the nonequilibrium equivalent of the pair correlation function, $g_{AB}(\vec{x}_1\vec{v}_1,\vec{x}_2\vec{v}_2,t)$, by the equilibrium pair correlation $g_{AB}(12)$, so that to all orders in $\hat{E}$

$$G_{AB}(\vec{x}_1\vec{v}_1,\vec{x}_2\vec{v}_2,t)=F_A(\vec{x}_1\vec{v}_1,t)F_B(\vec{x}_2\vec{v}_2,t)[1+g_{AB}(12)]\,. \quad (13)$$

Γ_{AB} in Eq. (12) then becomes

$$\Gamma_{AB}^{\mathrm{STSL}}(\vec{q}\mu,\vec{k}\omega;\vec{p})=[V\delta_{\vec{p}}+g_{AB}(\vec{p})]n_B(\vec{q}+\vec{p},\mu)\times\frac{\partial F_A(\vec{k}-\vec{q}-\vec{p},\omega-\mu,\vec{v})}{\partial\vec{v}}\,. \quad (13a)$$

In this paper only static perturbations will be contemplated; thus frequency arguments are set equal to zero and subsequently dropped.

The perturbing electric field can now be specialized to drive type-C plasma particles only, and the responding distribution function $F_A^{C(1)}(\vec{k},\vec{v})$ and density $n_A^{C(1)}$ can be calculated to first order in the perturbing $\hat{E}^C$.

For the case where the three $B(k)$'s are assumed to be equal, this yields, with the application of the linear electrodynamic and FDT relationships (5a) and (8a), the following external polarizability expressions:

$$\hat{\alpha}_e^e(\vec{k})=\Delta^{-1}\alpha_{e0}[1+\alpha_{i0}+u_i^i(\vec{k})]\,, \quad (14a)$$

$$\hat{\alpha}_e^i(\vec{k})=-\Delta^{-1}\alpha_{e0}[\alpha_{i0}+u_e^i(\vec{k})]=-\Delta^{-1}\alpha_{i0}[\alpha_{e0}+u_i^e(\vec{k})]=\hat{\alpha}_i^e(\vec{k})\,, \quad (14b)$$

$$\hat{\alpha}_i^i(\vec{k})=\Delta^{-1}\alpha_{i0}[1+\alpha_{e0}+u_e^e(\vec{k})]\,. \quad (14c)$$

Also, the full electron and ion polarizabilities appear as

$$\hat{\alpha}_e(\vec{k})=\hat{\alpha}_e^e(\vec{k})+\hat{\alpha}_e^i(\vec{k})=\Delta^{-1}\alpha_{e0}[1+u_i^i(\vec{k})-u_e^i(\vec{k})]\,, \quad (15a)$$

$$\hat{\alpha}_i(\vec{k})=\hat{\alpha}_i^e(\vec{k})+\hat{\alpha}_i^i(\vec{k})=\Delta^{-1}\alpha_{i0}[1+u_e^e(\vec{k})-u_i^e(\vec{k})]\,, \quad (15b)$$

where

$$\Delta=[1+\alpha_{e0}+u_e^e(\vec{k})][1+\alpha_{i0}+u_i^i(\vec{k})]-\alpha_{e0}\alpha_{i0}-u_e^i(\vec{k})u_i^e(\vec{k})-[\alpha_{e0}u_i^e(\vec{k})+\alpha_{i0}u_e^i(\vec{k})]\,, \quad (16)$$

$$u_A^B(\vec{k})=\frac{Z_AZ_B}{|Z_AZ_B|}\frac{\kappa_A^2}{k^2}\frac{1}{n_BV}\sum_{\vec{q}}\frac{\vec{k}\cdot\vec{q}}{q^2}B(q)\frac{\alpha_A^B(\vec{k}-\vec{q})}{\alpha_{A0}(\vec{k}-\vec{q})}\,, \quad (17)$$

$$\kappa_A^2=4\pi\beta n_AZ_A^2e^2\,,$$

and for brevity of notation we have dropped the $\vec{k}$ argument in the Vlasov polarizabilities, $\alpha_{A0}\equiv\alpha_{A0}(\vec{k})$.

Alternately, the results can be cast into the form of a set of coupled integral equations for the unknown pair correlation functions:

$$n_e g_{ee}(\vec{k}) = \Delta^{-1}[1+\alpha_{i0}+u_i^i(\vec{k})] - 1\,, \qquad (18a)$$

$$(n_e+n_i)g_{ei}(\vec{k}) = \Delta^{-1}[\alpha_0 + u_e^i(\vec{k}) + u_i^e(\vec{k})]\,, \qquad (18b)$$

$$n_i g_{ii}(\vec{k}) = \Delta^{-1}[1+\alpha_{e0}+u_e^e(\vec{k})] - 1\,, \qquad (18c)$$

with $\alpha_0 = \alpha_{e0} + \alpha_{i0}$ and

$$u_A^B(\vec{k}) = \frac{\kappa_B^2}{k^2}\frac{1}{V}\sum_{\vec{q}} \frac{\vec{k}\cdot\vec{q}}{q^2} B(q) g_{AB}(\vec{k}-\vec{q})\,. \qquad (19)$$

For $Z=0$ ($n_i \to \infty$, electron plasma in smeared out infinitely heavy ion background), one readily recovers the one-component results of STSL,[2] viz.,

$$g_{ee}(\vec{k}) = -\frac{1}{n_e}\frac{\kappa_e^2[1+u_{\rm STSL}(\vec{k})]}{k^2+\kappa_e^2[1+u_{\rm STSL}(\vec{k})]}\,,$$
$$g_{ei}(\vec{k}) = g_{ii}(\vec{k}) = 0\,, \qquad (20)$$
$$u_{\rm STSL}(\vec{k}) = \frac{1}{V}\sum_{\vec{q}}\frac{\vec{k}\cdot\vec{q}}{q^2} B(q) g_{ee}(\vec{k}-\vec{q})\,.$$

For the more general case when the three $B(k)$'s are unequal, application of the linear electrodynamic and FDT relations (3a) and (11a) leads to

$$\hat{\chi}_e^e(\vec{k}) = (\kappa_e^2/4\pi k\Delta)[1+\alpha_{i0}^{(i)} + u_i^i(\vec{k})]\,, \qquad (21a)$$

$$\hat{\chi}_i^e(\vec{k}) = \hat{\chi}_e^i(\vec{k}) = -(\kappa_e^2/4\pi k\Delta)[\alpha_{i0}^{(e)} + u_e^i(\vec{k})]\,, \qquad (21b)$$

$$\hat{\chi}_i^i(\vec{k}) = (\kappa_i^2/4\pi k\Delta)[1+\alpha_{e0}^{(e)} + u_e^e(\vec{k})]\,, \qquad (21c)$$

where now

$$u_A^B(\vec{k}) = \frac{Z_A Z_B}{|Z_A Z_B|}\frac{1}{V n_B}\sum_{\vec{q}}\frac{\vec{k}\cdot\vec{q}}{q^2}\frac{\hat{\chi}_A^B(\vec{k}-\vec{q})}{\chi_{A0}(\vec{k}-\vec{q})}\,,$$
$$\alpha_{A0}^{(B)} = \alpha_{A0} B_{AB}(k)\,, \qquad (22)$$
$$\chi_{A0} = (i/4\pi)\kappa_A^2/k\,.$$

Equation (19), however, remains unchanged.

B. GKS scheme

The GKS approach[5] to strongly coupled plasmas relates the linear and quadratic polarizabilities through self-consistency requirements. The ingredients of the scheme are (i) replacement of the G_{AB}'s on the right-hand side of (12) by their velocity averages, (ii) establishment of the relationships between the nonequilibrium two-point functions and equilibrium three-point functions by use of statistical mechanical perturbation theory and (iii) linking the three-point functions through the nonlinear FDT's with the quadratic polarizabilities.

In order to be able to express the right-hand side of (12) in terms of nonequilibrium two-point functions—binary correlations of microscopic charge densities, $\langle \rho^A_{\vec{k}} \rho^B_{\vec{p}}\rangle(\omega)$—it is necessary to invoke the so-called velocity average approximation (VAA),

$$G_{AB}(\vec{x}_1\vec{v}_1, \vec{x}_2\vec{v}_2; t)$$
$$= \frac{1}{2}\left(f_A(\vec{x}_1\vec{v}_1; t)\int d^3v'\, G_{AB}(\vec{x}_1\vec{v}_1', \vec{x}_2\vec{v}_2; t) + f_B(\vec{x}_2\vec{v}_2; t)\int d^3v_2'\, G_{AB}(\vec{x}_1\vec{v}_1, \vec{x}_2\vec{v}_2', t)\right)\,,$$
$$f_A(\vec{x}\vec{v}; t) \equiv F_A(\vec{x}\vec{v}; t)/n_A(\vec{x}; t)\,. \qquad (23)$$

This is the main assumption of the GKS scheme. Upon applying (23) to (12) and taking note of the fact that in terms of Fourier transforms

$$\int d^3v_1 \int d^3v_2\, G_{AB}(\vec{k}-\vec{q}-\vec{p}, \vec{q}, \vec{v}_1, \vec{v}_2, \omega-\nu)$$
$$= (Z_A Z_B e^2)^{-1}\langle \rho^A_{\vec{k}-\vec{q}-\vec{p}}\rho^B_{\vec{q}}\rangle(\omega-\nu) - n_A(\vec{k}-\vec{p}, \omega-\nu)\delta_{AB}\,, \qquad (24)$$

one obtains Eq. (12) with

$$\vec{\Gamma}_{AB}^{\rm GKS} = (Z_A Z_B e^2)^{-1}\langle \rho^A_{\vec{k}-\vec{q}-\vec{p}}\rho^B_{\vec{q}}\rangle(\omega-\nu)\frac{\partial f_A(\vec{p}\nu, \vec{v})}{\partial \vec{v}}\,. \qquad (25)$$

This result is valid to all orders in $\hat{E}$.

Contemplating now a static perturbation induced by a weak external field $\hat{E}^C$ acting only on C-type plasma particles, one finds

$$F_A^{C(1)}(\vec{k}, \vec{v}) = \frac{\partial F_A^{(0)}(\vec{v})/\partial\vec{v}}{\vec{k}\cdot\vec{v}\, m_A}\cdot\left(\frac{i\vec{k}Z_A e}{k}\hat{E}^C(\vec{k})\delta_{AC} + \frac{1}{e^2 n_A V}\sum_B\sum_{\vec{q}}\vec{q}\phi(q)B_{AB}(q)\langle\rho^A_{\vec{k}-\vec{q}}\rho^B_{\vec{q}}\rangle^{C(1)}\right)\,. \qquad (26)$$

Statistical mechanical perturbation calculations provide the link between the nonequilibrium two-point function and the equilibrium three-point function:

$$\langle\rho^A_{\vec{k}-\vec{q}}\rho^B_{\vec{q}}\rangle^{C(1)} = (-i\beta/Vk)\hat{E}^C(\vec{k})\langle\rho^A_{\vec{k}-\vec{q}}\rho^B_{\vec{q}}\rho^C_{-\vec{k}}\rangle^{(0)}. \qquad (27)$$

Upon combining (26) and (27) and integrating the result over velocity space, one readily obtains the partial density response

$$n_A^{C(1)}(\vec{k}) = \frac{-i\beta}{k}\hat{E}^C(\vec{k})\left(Z_A e n_A \delta_{AC} - \frac{\beta\phi(k)}{e^2V^2}\sum_B\sum_{\vec{q}}\frac{\vec{k}\cdot\vec{q}}{q^2}B_{AB}(\vec{q})\langle\rho^C_{-\vec{k}}\rho^B_{\vec{q}}\rho^A_{\vec{k}-\vec{q}}\rangle^{(0)}\right)\,. \qquad (28)$$

We next expand the ternary correlation function as follows:

$$\langle \rho^C_{-\vec{k}} \rho^B_{\vec{q}} \rho^A_{\vec{k}-\vec{q}} \rangle^{(0)} = \langle \rho^C_{-\vec{k}} \rho^B_{\vec{q}} \rho^A_{\vec{k}-\vec{q}} \rangle^{(0)} \Big|_{\vec{k},\vec{q},\vec{k}-\vec{q}\neq 0} + V(n_C Z_C e \langle \rho^B_{\vec{q}} \rho^A_{-\vec{q}} \rangle^{(0)}|_{\vec{q}\neq 0}\, \delta_{\vec{k}} + n_B Z_B e \langle \rho^C_{-\vec{k}} \rho^A_{\vec{k}} \rangle^{(0)}|_{\vec{k}\neq 0}\, \delta_{\vec{q}}$$
$$+ n_A Z_A e \langle \rho^C_{-\vec{k}} \rho^B_{\vec{k}} \rangle^{(0)}|_{\vec{k},\vec{q}\neq 0}\, \delta_{\vec{k}-\vec{q}} + V^2 n_A n_B n_C Z_A Z_B Z_C e^3 \delta_{\vec{q}} \delta_{\vec{k}}) . \quad (29)$$

The right-hand-side correlations in (29) are, in turn, related to the equilibrium pair and triplet correlation functions as follows:

$$\langle \rho^C_{-\vec{k}} \rho^B_{\vec{q}} \rho^A_{\vec{k}-\vec{q}} \rangle^{(0)}|_{\vec{k},\vec{q},\vec{k}-\vec{q}\neq 0} = Z_A Z_B Z_C e^3 n_C V[\delta_{AB}\delta_{BC} + n_B \delta_{AB} g_{BC}(\vec{k}) + n_B \delta_{AC} g_{BC}(\vec{q})$$
$$+ n_A \delta_{BC} g_{AC}(\vec{k}-\vec{q}) + n_B n_A h_{CBA}(\vec{q}, \vec{k}-\vec{q})] , \quad (30)$$

$$\langle \rho^A_{\vec{k}} \rho^B_{-\vec{k}} \rangle|_{\vec{k}\neq 0} = Z_A Z_B e^2 n_A V[\delta_{AB} + n_B g_{AB}(\vec{k})] . \quad (31)$$

From (28)–(31) we then have for the partial density responses

$$n_e^{e(1)}(\vec{k}) = (i\beta e n_e/k)\hat{E}^e(\vec{k})\{1 - \alpha_{e0}^{(e)}[1 + n_e g_{ee}(\vec{k})] + \alpha_{e0}^{(i)} n_e g_{ei}(\vec{k}) + v_e^e(\vec{k})\} , \quad (32a)$$

$$n_e^{i(1)}(\vec{k}) = (i\beta e n_e/k)\hat{E}^i(\vec{k})\{\alpha_{e0}^{(e)} n_e g_{ei}(\vec{k}) - \alpha_{i0}^{(e)}[1 + n_i g_{ii}(\vec{k})] + v_e^i(\vec{k})\} , \quad (32b)$$

$$n_i^{e(1)}(\vec{k}) = -(i\beta Z e n_i/k)\hat{E}^e(\vec{k})\{\alpha_{i0}^{(i)} n_i g_{ei}(\vec{k}) - \alpha_{e0}^{(i)}[1 + n_e g_{ee}(\vec{k})] + v_i^e(\vec{k})\} , \quad (32c)$$

$$n_i^{i(1)}(\vec{k}) = -(i\beta Z e n_i/k)\hat{E}^i(\vec{k})\{1 - \alpha_{i0}^{(i)}[1 + n_i g_{ii}(\vec{k})] + \alpha_{i0}^{(e)} n_i g_{ei}(\vec{k}) + v_i^i(\vec{k})\} , \quad (32d)$$

where the strong-coupling effects are now introduced through the $v_A^B(\vec{k})$ functions, which, in terms of the pair and triplet correlation functions, are

$$v_e^e(\vec{k}) = -\frac{\kappa_e^2}{k^2}\frac{1}{V}\sum_{\vec{q}} \frac{\vec{k}\cdot\vec{q}}{q^2}[B_{ee}(q) g_{ee}(\vec{k}-\vec{q}) + B_{ee}(q) n_e h_{eee}(\vec{q}, \vec{k}-\vec{q}) - B_{ei}(q) n_e h_{eie}(\vec{q}, \vec{k}-\vec{q})] , \quad (33a)$$

$$v_e^i(\vec{k}) = -\frac{\kappa_i^2}{k^2}\frac{1}{V}\sum_{\vec{q}} \frac{\vec{k}\cdot\vec{q}}{q^2}[B_{ei}(q) g_{ei}(\vec{k}-\vec{q}) + B_{ei}(q) n_i h_{iie}(\vec{q}, \vec{k}-\vec{q}) - B_{ee}(q) n_i h_{iee}(\vec{q}, \vec{k}-\vec{q})] , \quad (33b)$$

$$v_i^e(\vec{k}) = -\frac{\kappa_e^2}{k^2}\frac{1}{V}\sum_{\vec{q}} \frac{\vec{k}\cdot\vec{q}}{q^2}[B_{ei}(q) g_{ei}(\vec{k}-\vec{q}) + B_{ei}(q) n_e h_{eei}(\vec{q}, \vec{k}-\vec{q}) - B_{ii}(q) n_e h_{eii}(\vec{q}, \vec{k}-\vec{q})] , \quad (33c)$$

$$v_i^i(\vec{k}) = -\frac{\kappa_i^2}{k^2}\frac{1}{V}\sum_{\vec{q}} \frac{\vec{k}\cdot\vec{q}}{q^2}[B_{ii}(q) g_{ii}(\vec{k}-\vec{q}) + B_{ii}(q) n_i h_{iii}(\vec{q}, \vec{k}-\vec{q}) - B_{ei}(q) n_i h_{iei}(\vec{q}, \vec{k}-\vec{q})] . \quad (33d)$$

When the $B(k)$'s are equal, application of the linear electrodynamic and FDT relations (5a) and (8a) to Eq. (33) readily gives

$$\hat{\alpha}_e^e(\vec{k}) = \alpha_{e0}[1 - \hat{\alpha}_e(\vec{k}) + v_e^e(\vec{k})]$$
$$= [\alpha_{e0}/\epsilon(\vec{k})][1 + \alpha_i(\vec{k}) + r_e^e(\vec{k})] , \quad (34a)$$

$$[\hat{\alpha}_e^i(\vec{k})] = -\alpha_{e0}[\hat{\alpha}_i(\vec{k}) - v_e^i(\vec{k})]$$
$$= -[\alpha_{e0}/\epsilon(\vec{k})][\alpha_i(\vec{k}) - r_e^i(\vec{k})] , \quad (34b)$$

$$[\hat{\alpha}_i^e(\vec{k})] = -\alpha_{i0}[\hat{\alpha}_e(\vec{k}) - v_i^e(\vec{k})]$$
$$= -[\alpha_{i0}/\epsilon(\vec{k})][\alpha_e(\vec{k}) - r_i^e(\vec{k})] , \quad (34c)$$

$$\hat{\alpha}_i^i(\vec{k}) = \alpha_{i0}[1 - \hat{\alpha}_i(\vec{k}) + v_i^i(\vec{k})]$$
$$= [\alpha_{i0}/\epsilon(\vec{k})][1 + \alpha_e(\vec{k}) + r_i^i(\vec{k})] . \quad (34d)$$

where the $r_A^B(\vec{k})$ functions are related to the $v_A^B(\vec{k})$'s by

$$v_A^B(\vec{k}) = r_A^B(\vec{k})/\epsilon(\vec{k}) ; \quad (35)$$

they are the generalizations of the function $r(\vec{k})$ used in Ref. 5. Note that if one eliminates the $\hat{\alpha}(\vec{k})$'s in (34a) and (34d) in favor of the correlation functions by use of the linear FDT's (8a) and takes account of the symmetry conditions,

$$h_{CBA}(\vec{q}, \vec{k}-\vec{q}) = h_{ACB}(\vec{k}, -\vec{q}) , \quad (36)$$

one recovers precisely the equilibrium second BBGKY equations (52a) and (52c) for $g_{ee}(\vec{k})$ and $g_{ii}(\vec{k})$ (to be considered in Sec. IV). However, as to (34b) and (34c), it is clear that these relations do not satisfy the linear FDT interchange symmetry requirement that $\hat{\alpha}_e^i(\vec{k}) = \hat{\alpha}_i^e(\vec{k})$. {Hence the notation $[\hat{\alpha}_e^i(\vec{k})]$, $[\hat{\alpha}_i^e(\vec{k})]$.} This defect is apparently due to the VAA, although it does not arise in the case of a one-component plasma. A clue to its rectification is provided by the second BBGKY equation (52b) for g_{ie} in Sec. IV. If we symmetrize $\hat{\alpha}_e^i$ (and $\hat{\alpha}_i^e$),

$$\hat{\alpha}_e^i(\vec{k}) = -\tfrac{1}{2}\{\alpha_{i0}\hat{\alpha}_e(\vec{k}) + \alpha_{e0}\hat{\alpha}_i(\vec{k})$$
$$- [\alpha_{e0} v_i^e(\vec{k}) + \alpha_{i0} v_i^e(\vec{k})]\}$$
$$= -\tfrac{1}{2}[1/\epsilon(\vec{k})]\{\alpha_{e0}\alpha_i(\vec{k}) + \alpha_{i0}\alpha_e(\vec{k})$$
$$- [\alpha_{e0} r_e^i(\vec{k}) + \alpha_{i0} r_e^i(\vec{k})]\} , \quad (37)$$

agreement with the second BBGKY equation result is achieved. The full electron and ion polarizabilities can then also be obtained as

$$\hat{\alpha}_e(\vec{k}) = [\alpha_{e0}/\epsilon(\vec{k})][1 + r_e(\vec{k})], \tag{38a}$$

$$\hat{\alpha}_i(\vec{k}) = [\alpha_{i0}/\epsilon(\vec{k})][1 + r_i(\vec{k})], \tag{38b}$$

where the evaluation of $r_A(\vec{k})$ yields the expression

$$r_A(\vec{k}) = [1/(2+\alpha_0)]\{(2+\alpha_{A0}) r_A^A(\vec{k}) + (1+\alpha_{A0})[r_A^B(\vec{k}) + (\alpha_{B0}/\alpha_{A0}) r_B^A(\vec{k})] + \alpha_{B0} r_B^B(\vec{k})\}, \tag{39a}$$

for $A \neq B$.

Equations (34a), (34d), (37), (38a), and (38b) should be compared with the corresponding (approximate) Singwi results, Eqs. (14) and (15). The difference in the structure, especially $r_A(\vec{k}) \neq \sum_B r_A^B(\vec{k})$, should be noted. Nevertheless, it is easily demonstrated that the relationship

$$\hat{\alpha}(\vec{k}) = [1/\epsilon(\vec{k})]\{\alpha_{e0} + [1 + r_e^e(\vec{k}) + r_e^i(\vec{k})] + \alpha_{i0}[1 + r_i^i(\vec{k}) + r_i^e(\vec{k})]\} \tag{39b}$$

still holds.

Equation (34) now can be rewritten in terms of linear and nonlinear polarizabilities. By first explicitly solving Eqs. (34a)–(34d), one finds

$$\hat{\alpha}_e^e(\vec{k}) = \frac{\alpha_{e0}}{\epsilon_0}\left(1 + \alpha_{i0} + \frac{\epsilon_0}{1+\alpha_{e0}} v_e^e(\vec{k}) + \frac{\alpha_{i0}\alpha_{e0}}{(1+\alpha_{e0})(2+\alpha_0)}[(1+\alpha_{i0}) v_e^e(\vec{k}) + (1+\alpha_{e0}) v_i^i(\vec{k})] - \frac{1+\alpha_{i0}}{2+\alpha_0}[\alpha_{i0} v_i^e(\vec{k}) + \alpha_{e0} v_e^i(\vec{k})]\right), \tag{40a}$$

$$\hat{\alpha}_e^i(\vec{k}) = -\frac{\alpha_{e0}\alpha_{i0}}{\epsilon_0}\left[1 + \frac{(1+\alpha_{i0}) v_e^e(\vec{k}) + (1+\alpha_{e0}) v_i^i(\vec{k})}{2+\alpha_0} - \frac{(1+\alpha_{e0})(1+\alpha_{i0})}{2+\alpha_0}\left(\frac{v_i^e(\vec{k})}{\alpha_{e0}} + \frac{v_e^i(\vec{k})}{\alpha_{i0}}\right)\right] = \hat{\alpha}_i^e(\vec{k}), \tag{40b}$$

$$\hat{\alpha}_i^i(\vec{k}) = \frac{\alpha_{i0}}{\epsilon_0}\left(1 + \alpha_{e0} + \frac{\epsilon_0}{1+\alpha_{i0}} v_i^i(\vec{k}) + \frac{\alpha_{i0}\alpha_{e0}}{(1+\alpha_{i0})(2+\alpha_0)}[(1+\alpha_{i0}) v_e^e(\vec{k}) + (1+\alpha_{e0}) v_i^i(\vec{k})] - \frac{1+\alpha_{e0}}{2+\alpha_0}[\alpha_{i0} v_i^e(\vec{k}) + \alpha_{e0} v_e^i(\vec{k})]\right). \tag{40c}$$

The full electron and ion polarizabilities are now

$$\hat{\alpha}_e(\vec{k}) = \frac{\alpha_{e0}}{\epsilon_0}\left\{1 + v_e(\vec{k}) + \frac{\alpha_{i0}}{2+\alpha_0}\left[v_e(\vec{k}) - v_i(\vec{k}) + \epsilon_0\left(\frac{v_i^e(\vec{k})}{\alpha_{e0}} - \frac{v_e^i(\vec{k})}{\alpha_{i0}}\right)\right]\right\}, \tag{41a}$$

$$\hat{\alpha}_i(\vec{k}) = \frac{\alpha_{i0}}{\epsilon_0}\left\{1 + v_i(\vec{k}) + \frac{\alpha_{e0}}{2+\alpha_0}\left[v_i(\vec{k}) - v_e(\vec{k}) + \epsilon_0\left(\frac{v_e^i(\vec{k})}{\alpha_{i0}} - \frac{v_i^e(\vec{k})}{\alpha_{e0}}\right)\right]\right\}, \tag{41b}$$

where the $v_A^B(\vec{k})$ functions are then expressed with the aid of the nonlinear FDT relation (8b):

$$v_A^B(\vec{k}) = -\frac{\kappa_A^2}{k^2}\frac{1}{n_A}\frac{1}{V}\sum_C\sum_{\vec{q}}\frac{\vec{k}\cdot\vec{q}}{q^2}B(q)\frac{{}_2\hat{\alpha}_A^{BC}(\vec{k},-\vec{q})}{{}_2\alpha_{A0}(\vec{k},-\vec{q})}, \tag{42}$$

$$v_A(\vec{k}) = \sum_B v_A^B(\vec{k}) = -\frac{\kappa_A^2}{k^2}\frac{1}{n_A}\frac{1}{V}\sum_{\vec{q}}\frac{\vec{k}\cdot\vec{q}}{q^2}B(q)\frac{{}_2\hat{\alpha}_A(\vec{k},-\vec{q})}{{}_2\alpha_{e0}(\vec{k},-\vec{q})}.$$

The formulations of the linear external polarizabilities in Eqs. (41a) and (41b) in terms of the quadratic polarizabilities become all the more concise by noting that

$$v_e(\vec{k}) - v_i(\vec{k}) = -\frac{\kappa_e^2}{k^2}\frac{1}{n_e}\sum_{\vec{q}}\frac{\vec{k}\cdot\vec{q}}{q^2}B(q)\frac{{}_2\hat{\alpha}(\vec{k},-\vec{q})}{{}_2\alpha_{e0}(\vec{k},-\vec{q})}. \tag{43}$$

From (41) and (42) we further note that, similar to (39b),

$$\hat{\alpha}(\vec{k}) = (1/\epsilon_0)\{\alpha_{e0}[1 + v_e(\vec{k})] + \alpha_{i0}[1 + v_i(\vec{k})]\}. \tag{44}$$

As we have already emphasized, the GKS results are exact, but require further specifications of the nonlinear polarizabilities. No such statement can be made for the results of the STSL scheme which, however, is self-contained. We adhere to the philosophy that the GKS method proceeds not from the second BBGKY equation, but rather from the first.[6] This is important inasmuch as a self-consistent set of relations among the linear and quadratic polarizabilities can, in principle, be guaranteed by calculating quadratic polarizabilities from appropriately symmetrized second-order perturbation expansions of Eqs. (12) and (25).

It is immediately obvious that in the weak-coupling limit ($|v_A^B| \ll 1$) both STSL and GKS theories go over into the Vlasov theory, i.e.,

$$\begin{aligned}
&\hat{\alpha}_e^e(\vec{k}) = (\alpha_{e0}/\epsilon_0)(1+\alpha_{i0}),\\
&\hat{\alpha}_i^i(\vec{k}) = (\alpha_{i0}/\epsilon_0)(1+\alpha_{e0}),\\
&\hat{\alpha}_e^i(\vec{k}) = -\alpha_{e0}\alpha_{i0}/\epsilon_0 = \hat{\alpha}_i^e(\vec{k}),\\
&\hat{\alpha}_e(\vec{k}) = \alpha_{e0}/\epsilon_0, \quad \hat{\alpha}_i(\vec{k}) = \alpha_{i0}/\epsilon_0 .
\end{aligned} \tag{45}$$

It is seen that in this limit the pair correlation functions reduce to the well-known Debye-Hückel results

$$\begin{aligned}
&g_{ee}(\vec{k}) = \frac{-\kappa_e^2/n_e}{k^2+\kappa^2},\\
&g_{ei}(\vec{k}) = \frac{\kappa^2/n}{k^2+\kappa^2},\\
&g_{ii}(\vec{k}) = \frac{-\kappa_i^2/k^2}{k^2+\kappa^2}.
\end{aligned} \tag{46}$$

In the $Z=0$ limit, the GKS strongly coupled electron plasma results[5] are recovered, i.e.,

$$\hat{\alpha}_e^e(\vec{k}) = (\alpha_{e0}/\epsilon_0)[1+v_e^e(\vec{k})] = (\alpha_{e0}/\epsilon(\vec{k}))[1+r_e^e(\vec{k})] = \hat{\alpha}_e(\vec{k}) = \hat{\alpha}(\vec{k}) \quad \hat{\alpha}_e^i(\vec{k}) = 0 = \hat{\alpha}_i^i(\vec{k}), \tag{47}$$

We now display the general case where the three B's are different. It can be shown from Equations (3), (11), (32), and (33) that

$$\frac{\hat{\chi}_e^e(\vec{k})}{\chi_{e0}} = \frac{1+v_e^e(\vec{k})}{1+\alpha_{e0}^{(e)}} + \frac{\alpha_{e0}^{(i)}\alpha_{i0}^{(e)}}{(1+\alpha_{e0}^{(e)})[(1+\alpha_{e0}^{(e)})(1+\alpha_{i0}^{(i)}) - \alpha_{i0}^{(e)}\alpha_{e0}^{(i)}]} \times\left(1+\frac{(1+\alpha_{i0}^{(i)})v_e^e(\vec{k}) + (1+\alpha_{e0}^{(e)})v_i^i(\vec{k}) - (1+\alpha_{e0}^{(e)})(1+\alpha_{i0}^{(i)})[\chi_{e0}v_e^i(\vec{k}) + \chi_{i0}v_i^e(\vec{k})]/\chi_{e0}\alpha_{i0}^{(e)}}{2+\alpha_{e0}^{(e)}+\alpha_{i0}^{(i)}}\right), \tag{48a}$$

$$\frac{1}{2}\,\frac{\chi_e^i(\vec{k})}{\chi_{e0}\alpha_{i0}^{(e)} + \chi_{i0}\alpha_{e0}^{(i)}} = -\frac{1}{(1+\alpha_{e0}^{(e)})(1+\alpha_{i0}^{(i)}) - \alpha_{i0}^{(e)}\alpha_{e0}^{(i)}} \times\left(1+\frac{(1+\alpha_{i0}^{(i)})v_e^e(\vec{k}) + (1+\alpha_{e0}^{(e)})v_i^i(\vec{k}) - 2(1+\alpha_{e0}^{(e)})(1+\alpha_{i0}^{(i)})[\chi_{e0}v_e^i(\vec{k}) + \chi_{i0}v_i^e(\vec{k})]/(\chi_{e0}\alpha_{i0}^{(e)} + \chi_{i0}\alpha_{e0}^{(i)})}{2+\alpha_{e0}^{(e)}+\alpha_{i0}^{(i)}}\right), \tag{48b}$$

$$\frac{\chi_i^i(\vec{k})}{\chi_{i0}} = \frac{1+v_i^i(\vec{k})}{1+\alpha_{i0}^{(i)}} + \frac{\alpha_{i0}^{(e)}\alpha_{e0}^{(i)}}{(1+\alpha_{i0}^{(i)})[(1+\alpha_{e0}^{(e)})(1+\alpha_{i0}^{(i)}) - \alpha_{i0}^{(e)}\alpha_{e0}^{(i)}]} \times\left(1+\frac{(1+\alpha_{i0}^{(i)})v_e^e(\vec{k}) + (1+\alpha_{e0}^{(e)})v_i^i(\vec{k}) - (1+\alpha_{e0}^{(e)})(1+\alpha_{i0}^{(i)})[\chi_{e0}v_e^i(\vec{k}) + \chi_{i0}v_i^e(\vec{k})]/\chi_{i0}\alpha_{e0}^{(i)}}{2+\alpha_{e0}^{(e)}+\alpha_{i0}^{(i)}}\right). \tag{48c}$$

The expressions for the full response functions are

$$\frac{\hat{\chi}_e(\vec{k})}{\chi_{e0}} = \frac{1+\alpha_{i0}^{(i)} - \alpha_{i0}^{(e)}}{(1+\alpha_{e0}^{(e)})(1+\alpha_{i0}^{(i)}) - \alpha_{i0}^{(e)}\alpha_{e0}^{(i)}} + \frac{v_e^e(\vec{k})}{1+\alpha_{e0}^{(e)}} - \frac{\alpha_{i0}^{(e)}(1+\alpha_{e0}^{(e)} - \alpha_{e0}^{(i)})[(1+\alpha_{i0}^{(i)})v_e^e(\vec{k}) + (1+\alpha_{e0}^{(e)})v_i^i(\vec{k}) - (1+\alpha_{e0}^{(e)})(1+\alpha_{i0}^{(i)})(\chi_{e0}v_e^i + \chi_{i0}v_i^e)/\chi_{e0}\alpha_{i0}^{(e)}]}{(1+\alpha_{e0}^{(e)})(2+\alpha_{e0}^{(e)}+\alpha_{i0}^{(i)})[(1+\alpha_{e0}^{(e)})(1+\alpha_{i0}^{(i)}) - \alpha_{i0}^{(e)}\alpha_{e}^{(i)}]}, \tag{49a}$$

$$\frac{\hat{\chi}_i(\vec{k})}{\chi_{i0}} = \frac{1+\alpha_{e0}^{(e)} - \alpha_{e0}^{(i)}}{(1+\alpha_{e0}^{(e)})(1+\alpha_{i0}^{(i)}) - \alpha_{i0}^{(e)}\alpha_{e0}^{(i)}} + \frac{v_i^i(\vec{k})}{1+\alpha_{i0}^{(i)}} - \frac{\alpha_{e0}^{(i)}(1+\alpha_{i0}^{(i)} - \alpha_{i0}^{(e)})[(1+\alpha_{i0}^{(i)})v_e^e(\vec{k}) + (1+\alpha_{e0}^{(e)})v_i^i(\vec{k}) - (1+\alpha_{e0}^{(e)})(1+\alpha_{i0}^{(i)})[\chi_{e0}v_e^i(\vec{k}) + \chi_{i0}v_i^e(\vec{k})]/\chi_{i0}\alpha_{e0}^{(i)}]}{(1+\alpha_{i0}^{(i)})(2+\alpha_{i0}^{(i)}+\alpha_{e0}^{(e)})[(1+\alpha_{e0}^{(e)})(1+\alpha_{i0}^{(i)}) - \alpha_{i0}^{(e)}\alpha_{e0}^{(i)}]}, \tag{49b}$$

and finally

$$\hat{\chi}(k)=\hat{\chi}_e(\vec{k})+\hat{\chi}_i(\vec{k})$$

$$=\frac{\chi_0+\chi_{e0}(\alpha_{i0}^{(i)}-\alpha_{i0}^{(e)})+\chi_{i0}(\alpha_{e0}^{(e)}-\alpha_{e0}^{(i)})}{(1+\alpha_{e0}^{(e)})(1+\alpha_{i0}^{(i)})-\alpha_{i0}^{(e)}\alpha_{e0}^{(i)}}+\frac{\chi_{e0}v_e(\vec{k})}{1+\alpha_{e0}^{(e)}}+\frac{\chi_{i0}v_i(\vec{k})}{1+\alpha_{i0}^{(i)}}$$

$$-\frac{2-\alpha_{e0}^{(i)}/(1+\alpha_{e0}^{(e)})-\alpha_{i0}^{(e)}/(1+\alpha_{i0}^{(i)})}{(2+\alpha_{e0}^{(e)}+\alpha_{i0}^{(i)})[(1+\alpha_{e0}^{(e)})(1+\alpha_{i0}^{(i)})-\alpha_{i0}^{(e)}\alpha_{e0}^{(i)}]}$$

$$\times\{\tfrac{1}{2}(\chi_{e0}\alpha_{i0}^{(e)}+\chi_{i0}\alpha_{e0}^{(i)})[(1+\alpha_{i0}^{(i)})v_e^e(\vec{k})+(1+\alpha_{e0}^{(e)})v_i^i(\vec{k})]-(1+\alpha_{e0}^{(e)})(1+\alpha_{i0}^{(i)})[\chi_{e0}v_e^i(\vec{k})+\chi_{i0}v_i^e(\vec{k})]\}. \tag{50}$$

The $v_A^B(\vec{k})$ functions now are given as

$$v_e^e(\vec{k})=\frac{8\pi}{\beta n_e eV}\sum_{\vec{q}}\frac{\vec{k}\cdot\vec{q}}{kq}[B_{ee}(q)_2\hat{\chi}_e^{ee}(\vec{k},-\vec{q})+B_{ei}(q)_2\hat{\chi}_e^{ei}(\vec{k},-\vec{q})], \tag{51a}$$

$$v_e^i(\vec{k})=\frac{8\pi}{\beta n_e eV}\sum_{\vec{q}}\frac{\vec{k}\cdot\vec{q}}{kq}[B_{ei}(q)_2\hat{\chi}_e^{ii}(\vec{k},-\vec{q})+B_{ee}(q)_2\hat{\chi}_e^{ie}(\vec{k},-\vec{q})], \tag{51b}$$

$$v_i^e(\vec{k})=\frac{-8\pi}{\beta n_e eV}\sum_{\vec{q}}\frac{\vec{k}\cdot\vec{q}}{kq}[B_{ei}(q)_2\hat{\chi}_i^{ee}(\vec{k},-\vec{q})+B_{ii}(q)_2\hat{\chi}_i^{ei}(\vec{k},-\vec{q})], \tag{51c}$$

$$v_i^i(\vec{k})=\frac{-8\pi}{\beta n_e eV}\sum_{\vec{q}}\frac{\vec{k}\cdot\vec{q}}{kq}[B_{ii}(q)_2\hat{\chi}_i^{ii}(\vec{k},-\vec{q})+B_{ei}(q)_2\hat{\chi}_i^{ie}(\vec{k},-\vec{q})]. \tag{51d}$$

IV. SECOND BBGKY APPROACH: TI SCHEME

The STSL[1,2] and GKS[5] approaches to strongly coupled plasmas involve perturbation (in E) expansions of the first BBGKY kinetic equations coupled with fluctuation-dissipation theorems to guarantee self-consistency. While only static ($\omega=0$) results were presented in Sec. III, it is clear that these schemes are amenable to perturbation expansions of the dynamical first BBGKY equations coupled to dynamical fluctuation-dissipation theorems. Such is not the case for the approach of TI,[3,4] which proceeds from the equilibrium second BBGKY equation linking the static pair and triplet correlation functions. The principal approximation in the scheme is that the triplet correlation has a Debye-like structure and can be approximated by an expansion of Mayer pair-correlation clusters. Fluctuation-dissipation theorems have no place in the TI scheme, except if one wishes to change from pair correlation language to linear partial polarizability language.

In this section, we extend the electron liquid scheme of TI[3,4] to the case of ion-electron plasmas. We start with the following equilibrium second BBGKY equations for g_{ee}, g_{ei}, and g_{ii}:

$$(1+\alpha_{e0}^{(e)})n_e g_{ee}(\vec{k})-\alpha_{e0}^{(i)}n_e g_{ei}(\vec{k})=-\alpha_{e0}^{(e)}+v_e^e(\vec{k}), \tag{52a}$$

$$\left(1+\frac{\alpha_{e0}^{(e)}+\alpha_{i0}^{(i)}}{2}\right)g_{ei}(\vec{k})-\tfrac{1}{2}[\alpha_{i0}^{(e)}g_{ee}(\vec{k})+\alpha_{e0}^{(i)}g_{ii}(\vec{k})]$$

$$=\frac{1}{2}\left(\frac{\alpha_{e0}^{(i)}}{n_i}+\frac{\alpha_{i0}^{(e)}}{n_e}\right)-\frac{1}{2}\left(\frac{v_e^i(\vec{k})}{n_e}+\frac{v_i^e(\vec{k})}{n_i}\right), \tag{52b}$$

$$(1+\alpha_{i0}^{(i)})n_i g_{ii}(\vec{k})-\alpha_{i0}^{(e)}n_i g_{ei}(\vec{k})=-\alpha_{i0}^{(i)}+v_i^i(\vec{k}), \tag{52c}$$

where the strong-coupling $v_A^B(\vec{k})$ functions are defined in terms of the pair and triplet correlation functions $g(\vec{k})$ and $h(\vec{q}\,\vec{p})$ by Eq. (33) subject to the symmetry condition (36).

Solving these equations without any further approximations one obtains the symmetrized GKS expressions, relations (34a), (34d), and (37), derived from the first BBKGY equation.

In order to obtain the Ichimaru approximate solution we next assume that the h's can be expressed in terms of g clusters. It is helpful to refer to a graphical representation as given in Fig. 1 corresponding to the algebraic relation

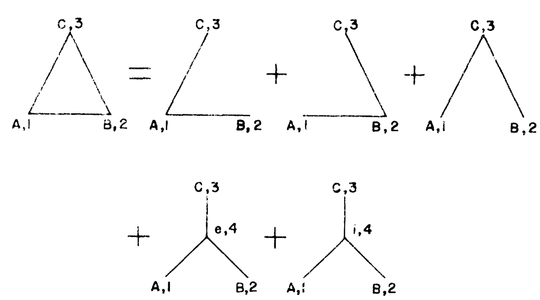

FIG. 1. Diagrammatic representation of the cluster decomposition of the three-particle correlation function $h_{ABC}(123)$ [cf. Eq. (53)].

$$h_{ABC}(123)=g_{AB}(12)g_{AC}(13)+g_{AB}(12)g_{BC}(23)+g_{AC}(13)g_{BC}(23)+\sum_D\int d^3x_4 g_{AD}(14)g_{BD}(24)g_{CD}(34)\,, \tag{53}$$

with Fourier transform

$$h_{ABC}(\vec{q}\vec{p})=g_{AB}(\vec{q})g_{BC}(\vec{p})+g_{AB}(\vec{q}+\vec{p})g_{AC}(\vec{p})+q_{AC}(\vec{q}+\vec{p})g_{BC}(\vec{q})+\sum_D n_D g_{AD}(\vec{q}+\vec{p})g_{BD}(\vec{q})g_{CD}(\vec{p})\,. \tag{54}$$

Bonds between vertices, say $(A,1)$ and $(B,2)$ are associated with $g_{AB}(12)\equiv g_{AB}(|\vec{x}_1-\vec{x}_2|)$.

Upon eliminating the h's in (52) in favor of the g clusters by use of (54), we ultimately obtain

$$g_{ee}(\vec{k})=-\Lambda^{-1}nn_i\left\{\frac{\kappa_e^2}{k^2}w_{ee}\left(1+\frac{\kappa_i^2}{k^2}w_{ii}\right)\left(1+\frac{\kappa_e^2}{2k^2}w_{ee}+\frac{\kappa_i^2}{2k^2}w_{ii}\right)-\frac{\kappa_i^2\kappa_e^2}{2k^4}w_{ie}\left[w_{ei}+w_{ie}+w_{ei}\left(\frac{\kappa_e^2}{k^2}w_{ee}+\frac{\kappa_i^2}{k^2}w_{ii}\right)\right]\right\}\,, \tag{55a}$$

$$g_{ei}(\vec{k})=\Lambda^{-1}n_e n_i\frac{\kappa^2}{2k^2}\left(w_{ei}+w_{ie}+\frac{\kappa_e^2}{k^2}w_{ie}+\frac{\kappa_i^2}{k^2}w_{ei}\right)\,, \tag{55b}$$

$$g_{ii}(\vec{k})=-\Lambda^{-1}nn_e\left\{\frac{\kappa_i^2}{k^2}w_{ii}\left(1+\frac{\kappa_e^2}{k^2}w_{ee}\right)\left(1+\frac{\kappa_e^2}{2k^2}w_{ee}+\frac{\kappa_i^2}{2k^2}w_{ii}\right)-\frac{\kappa_i^2\kappa_e^2}{2k^4}w_{ei}\left[w_{ei}+w_{ie}+w_{ie}\left(\frac{\kappa_e^2}{k^2}w_{ee}+\frac{\kappa_i^2}{k^2}w_{ii}\right)\right]\right\}\,. \tag{55c}$$

The newly introduced $w_A^B(\vec{k})$ screening functions are

$$w_{AB}\equiv w_{AB}(\vec{k})=B_{AB}(\vec{k})+\frac{1}{V}\sum_{\vec{q}}\frac{\vec{k}\cdot\vec{q}}{q^2}\{B_{AB}(\vec{q})[1+n_Ag_{AA}(\vec{q})]-B_{BC}(\vec{q})n_Ag_{AC}(\vec{q})\}\,g_{AB}(\vec{k}-\vec{q}),\quad A\neq C\,, \tag{56}$$

$$\Lambda=n_en_in\left(1+\frac{\kappa_e^2}{2k^2}w_{ee}+\frac{\kappa_i^2}{2k^2}w_{ii}\right)\left[\left(1+\frac{\kappa_e^2}{k^2}w_{ee}\right)\left(1+\frac{\kappa_i^2}{k^2}w_{ii}\right)-\frac{\kappa_e^2\kappa_i^2}{k^4}w_{ei}w_{ie}\right]\,. \tag{57}$$

In general (i.e., for $Z\neq 1$), $w_{ei}\neq w_{ie}$.

One can easily verify that when $Z=0$ (electron plasma), $B=1$, Eq. (55) and (56) become

$$w_{ee}=1+\frac{1}{V}\sum_{\vec{q}}\frac{\vec{k}\cdot\vec{q}}{q^2}[1+n_eg_{ee}(\vec{q})]g_{ee}(\vec{k}-\vec{q})$$
$$\equiv 1+u_{\rm TI}\,, \tag{58}$$

$$g_{ee}(\vec{k})=-\frac{1}{n_e}\frac{(1+u_{\rm TI})\kappa_e^2}{k^2+(1+u_{\rm TI})\kappa_e^2}\,,$$

i.e., the one-component Ichimaru[3] results are recovered. The correct Debye pair correlations (46) are also recovered for the two-component plasma in the weak-coupling limit.

V. CONCLUSIONS

The purpose of the present paper has been twofold: first, to present a formalism appropriate for the handling of the two-component strongly-coupled plasma problem; second, to generalize the existing formal one-component results for the case of a two-component system. The first objective has been achieved by introducing partial linear and nonlinear polarizabilities and the fluctuation-dissipation theorems (again, linear and nonlinear) they satisfy, and finally by connecting the partial polarizabilities with their full physical counterparts.

The second objective has been accomplished by the formulations of Eqs. (14), (16), (17) and (21), (22), which represent the two-component STSL theory, of Eqs. (55)–(57), representing the two-component TI theory, and by the formulation of Eqs. (40), (42) and (48), (51), which yield the two-component version of the GKS scheme originally derived for a one-component plasma. All of these relations appear as nonlinear integral equations for either the three pair correlations functions or partial linear polarizabilities. In addition, the GKS scheme still contains the unknown quadratic partial polarizabilities; how they are to be expressed in terms of the linear polarizabilities hasn't been specified in this paper.

The full physical polarizabilities are sums of the partial polarizabilities, as shown in Eq. (4). Conversely, however, in general there exists no simple relation expressing the partial polarizabilities in terms of the full physical polarizabilities. It is one of the results of the present paper that an exact relationship, but one which is explicitly dependent on the pair and triplet correlations functions, does exist and has here been derived.

Algebraically, each of the three theories leads to a set of three nonlinear coupled integral equations which have to be solved by computer. Discussion of such calculations have not been given in this paper and will be presented in a later work.

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[15]We use the following, not quite self-explanatory, notational conventions: $\not{\sum}_{\vec{k}\omega} \equiv (1/2\pi V)\int d\omega \sum_{\vec{k}}$, V is the volume of the system, $\delta_{\vec{k}} \equiv \delta_{\vec{k},0}$, $\mathcal{E}$ stands for the plasma field, $\hat{E}$ for the external field, and E for the total field. All fields are longitudinal (with respect to their wave-vector argument), and with this understanding their vector character is ignored. Superscripts indicate the order on the $\hat{E}$ expansion. A 0 subscript designates Vlasov values. Capital subscripts or superscripts refer to the plasma species (e or i). The Fourier transform of $h(123)$ is defined as

$$\begin{aligned} h(123) &= \frac{1}{V^2}\sum_{\vec{p}\vec{q}} h(\vec{p}\vec{q}) \exp\{i[\vec{q}\cdot(\vec{x}_3-\vec{x}_1)+\vec{p}\cdot(\vec{x}_1-\vec{x}_2)]\} \\ &= \frac{1}{V^2}\sum_{\vec{p}\vec{q}} h(\vec{p}\vec{q}) \exp\{i[(\vec{p}-\vec{q})\cdot(\vec{x}_1-\vec{x}_2) \\ &\qquad -\vec{q}\cdot(\vec{x}_2-\vec{x}_3)]\} \\ &= \frac{1}{V^2}\sum_{\vec{p}\vec{q}} h(\vec{p}\vec{q}) \exp\{i[-p\cdot(\vec{x}_2-\vec{x}_3) \\ &\qquad +(\vec{p}-\vec{q})\cdot(\vec{x}_3-\vec{x}_1)]\}. \end{aligned}$$

[16]See, e.g., G. Kalman, in *Plasma Physics–Les Houches Lectures, 1972* (Gordon and Breach, New York, 1975).

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