Nuclear polarization in muonic helium*

G. A. Rinker†
Los Alamos Scientific Laboratory, University of California, Los Alamos, New Mexico 87545
(Received 9 February 1976)

Nuclear-polarization energy shifts are calculated for the low-lying levels of muonic ^{3}He and ^{4}He. In ^{4}He, the shift of the $2s_{1/2}$ level is found to be -3.1 meV, in agreement with the result of Bernabeau and Jarlskog. The corresponding shift in ^{3}He is -4.9 meV. Other corrections to the energy levels are calculated, with the improved result that the total theoretical $2s_{1/2}$-$2p_{3/2}$ energy difference in ^{4}He is $1.8131 - 0.102r^2 \pm 0.001$ eV, where r is the nuclear rms charge radius (in femtometers). Together with the measured value 1.5274 ± 0.0009 eV, this implies $r = 1.674 \pm 0.004$ fm, which is consistent with available electron scattering data.

There has been considerable recent interest in the energy levels of very light muonic atoms. This interest has arisen principally from the promise that modern muon beams and tunable lasers hold for precisely measuring differences between some of these energy levels. These experiments are quite similar in character to the classic atomic tests of quantum electrodynamics (QED); however, the region of space explored is quite different. The Lamb shift ($2s$-$2p$ energy difference) is here dominated by vacuum polarization rather than vertex corrections, and the level ordering is reversed. In addition, electron-muon universality is tested, as well as the nuclear polarizability.[1]

At the present time, the $2p_{3/2}$-$2s_{1/2}$ energy difference in ^{4}He has been measured[2] and compared to theoretical predictions. The major uncertainty in these predictions arises from the nuclear charge radius, which must be obtained independently (e.g., by electron scattering) if a test of the other components of the theory is to be made. Another relatively uncertain contribution is the nuclear polarization,[3-5] which is the primary subject of the present work. This correction is the subject of some dispute in the literature. Two detailed calculations have so far been made, by Bernabeu and Jarlskog[4] (hereinafter referred to as BJ), and by Henley, Krejs, and Wilets[5] (hereinafter referred to as HKW). The results of these two calculations are in substantial disagreement. In an effort to illuminate the sources of this disagreement, the present work provides an independent calculation of this effect. In addition, it considers both the computational methods employed and the nuclear properties which are input into the previous calculations, so far as practical. The computational method of HKW is investigated by repeating that calculation with a somewhat simplified version of their nuclear model. The assumed nuclear properties are addressed through comparison with the He photoabsorption cross sections, which provide the most directly pertinent experimental information available from other sources.

I. THEORY

In order to compute muonic-atom energy levels, it is useful to first approximate the true muon-nuclear electromagnetic interaction by a suitable electrostatic central potential, in which the muon radial Dirac equations can be solved to provide the zeroth-order eigenvalues and eigenfunctions. These solutions may then be used to account for various neglected effects in perturbation theory. Many of these effects are quantum electrodynamical in nature, for which unambiguous calculational prescriptions have been given in the literature. Others arise from the zero-order static, central-field approximation to the true electromagnetic interaction. The most important of this second class of corrections is the nuclear polarization, which is here taken to be the two-photon exchange diagram in Fig. 1, in the Coulomb gauge with transverse interactions neglected. Implications of this approximation are discussed later, and similar higher-order processes are assumed to be negligible. The intermediate states in Fig. 1 include all internal nuclear excitations.

In the present work, the nuclear polarization is computed in noncovariant second-order perturbation theory. The sum over intermediate nuclear states is carried out explicitly. The sum over intermediate muon states is carried out implicitly

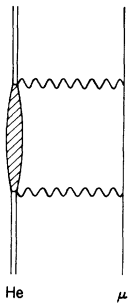

FIG. 1. Nuclear-polarization correction.

using knowledge of the exact muon Hamiltonian, which is equivalent to knowledge of the complete muon spectrum. The method used has been attributed to various people, ranging from Green to Dalgarno and Lewis.[6] It appears to have been used first in the present context by Chen.[7] It is prescribed by writing down the ordinary perturbation equations for a perturbation H_1 on H_0:

$$(H_0 - E_0)\Psi_0 = 0\,, \tag{1}$$

$$(H_0 - E_0)\Psi_1 + (H_1 - E_1)\Psi_0 = 0\,, \tag{2}$$

$$E_1 = (\Psi_0, H_1\Psi_0)\,,$$
$$E_2 = (\Psi_0, (H_1 - E_1)\Psi_1)\,, \tag{3}$$

where Ψ_i and E_i are the ith-order wave function and energy shift. The method consists of solving the (differential) equation (2) directly for Ψ_1 rather than expanding Ψ_1 in terms of unperturbed wave functions. The second-order energy E_2 is obtained from Eq. (3). Since the muon is assumed to interact only electromagnetically, it is always possible to formulate a muon Hamiltonian, given a nuclear model which specifies form factors, transition strengths, and excited-state energies and quantum numbers. Since the nuclear Hamiltonian is not as well known as its spectrum, the nuclear perturbation is constructed in the more conventional way as a sum (or integral) over unperturbed excited states. Thus Eq. (2) is regarded as a differential equation in the muon coordinates only, with the interactions in H_0 and H_1 depending upon the nuclear states under consideration.

The unperturbed muon wave function is constructed from two-component spinors $|\kappa m\rangle$, with (following McKinley's[8] notation)

$$\vec{l}\cdot\vec{l}\,|\kappa m\rangle = l(l+1)\,|\kappa m\rangle\,, \tag{4}$$

$$\vec{\sigma}\cdot\vec{\sigma}\,|\kappa m\rangle = 3\,|\kappa m\rangle\,, \tag{5}$$

$$\vec{j}\cdot\vec{j}\,|\kappa m\rangle = (\vec{l} + \tfrac{1}{2}\vec{\sigma})\cdot(\vec{l} + \tfrac{1}{2}\vec{\sigma})\,|\kappa m\rangle$$
$$= j(j+1)\,|\kappa m\rangle\,, \tag{6}$$

$$j_z\,|\kappa m\rangle = m\,|\kappa m\rangle\,, \tag{7}$$

$$-(1 + \vec{\sigma}\cdot\vec{l})\,|\kappa m\rangle = \kappa\,|\kappa m\rangle\,, \tag{8}$$

$$\vec{\sigma}\cdot\hat{r}\,|\kappa m\rangle = -\,|-\kappa m\rangle\,. \tag{9}$$

The four-component muon spinor is

$$\psi^{(n\kappa)} = \begin{pmatrix} (1/r)G^{(n\kappa)}(r)\,|\kappa m\rangle \\ (i/r)F^{(n\kappa)}(r)\,|-\kappa m\rangle \end{pmatrix} \tag{10}$$

and the unperturbed muon Hamiltonian is ($\hbar = c = 1$)

$$H_\mu = \begin{pmatrix} V^{(0)}(r) + M_\mu & \vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p} & V^{(0)}(r) - M_\mu \end{pmatrix}, \tag{11}$$

where

$$\vec{\sigma}\cdot\vec{p} = -i\frac{\vec{\sigma}\cdot\vec{r}}{r^2}\left(\frac{\partial}{\partial r}r - (1 + \vec{\sigma}\cdot\vec{l})\right), \tag{12}$$

and where $V^{(0)}(r)$ is the (central) potential which provides the zero-order muon-nucleus interaction. We require $(H_\mu - E_\mu^{(n\kappa)})\psi^{(n\kappa)} = 0$, which yields the usual radial Dirac equations

$$\frac{d}{dr}F^{(n\kappa)}(r) = \frac{\kappa}{r}F^{(n\kappa)}(r) + [M_\mu - E_\mu^{(n\kappa)} + V^{(0)}(r)]G^{(n\kappa)}(r)\,,$$
$$\frac{d}{dr}G^{(n\kappa)}(r) \tag{13}$$
$$= -\frac{\kappa}{r}G^{(n\kappa)}(r) + [M_\mu + E_\mu^{(n\kappa)} - V^{(0)}(r)]F^{(n\kappa)}(r)\,.$$

Solution of these equations provides the spectrum of muon eigenvalues $E_\mu^{(n\kappa)}$.

The nuclear states are denoted by $|\gamma IM\rangle$, with

$$H_N\,|\gamma IM\rangle = E_N^{(\gamma I)}\,|\gamma IM\rangle\,, \tag{14}$$

$$\vec{I}\cdot\vec{I}\,|\gamma IM\rangle = I(I+1)\,|\gamma IM\rangle\,, \tag{15}$$

$$I_z\,|\gamma IM\rangle = M\,|\gamma IM\rangle\,. \tag{16}$$

Coupled spinor states are

$$|\kappa\gamma I; \mathcal{F}\mathfrak{M}\rangle = \langle \kappa m I M\,|\,\kappa I; \mathcal{F}\mathfrak{M}\rangle\,|\kappa m\rangle\,|\gamma IM\rangle\,, \tag{17}$$

with

$$\vec{\mathcal{F}}\cdot\vec{\mathcal{F}}\,|\kappa\gamma I; \mathcal{F}\mathfrak{M}\rangle = (\vec{j} + \vec{I})\cdot(\vec{j} + \vec{I})\,|\kappa\gamma I; \mathcal{F}\mathfrak{M}\rangle$$
$$= \mathcal{F}(\mathcal{F}+1)\,|\kappa\gamma I; \mathcal{F}\mathfrak{M}\rangle\,, \tag{18}$$

$$\mathcal{F}_z\,|\kappa\gamma I; \mathcal{F}\mathfrak{M}\rangle = \mathfrak{M}\,|\kappa\gamma I; \mathcal{F}\mathfrak{M}\rangle\,. \tag{19}$$

Since it is the total muon angular momentum $j = |\kappa| - \frac{1}{2}$ which couples to I, the coupling coefficients are the same for $|\pm\kappa m\rangle$. We use the following shorthand:

$$|\pm k\rangle = |\pm\kappa_k\gamma_k I_k; \mathcal{F}_k\mathfrak{M}_k\rangle\,. \tag{20}$$

The initial zero-order coupled state is constructed from $|\pm 0\rangle \equiv |\pm\kappa_0\gamma_0 I_0; \mathcal{F}_0\mathfrak{M}_0\rangle$:

$$\Psi_0^{(n0)} = \begin{pmatrix} (1/r)G^{(n0)}(r)\,|0\rangle \\ (i/r)F^{(n0)}(r)\,|-0\rangle \end{pmatrix}. \tag{21}$$

The general first-order correction, which depends upon n and $|0\rangle$, is

$$\Psi_1^{(n0)} = \sum_k \begin{pmatrix} (1/r)g^{(nk)}(r)\,|k\rangle \\ (i/r)f^{(nk)}(r)\,|-k\rangle \end{pmatrix}, \tag{22}$$

with $g^{(nk)}(r)$ and $f^{(nk)}(r)$ to be determined. We now need to solve Eq. (2), with

$$H_0\Psi_0^{(n0)} = (H_\mu + H_N)\Psi_0^{(n0)} = (E_\mu^{(n0)} + E_N^{(0)})\Psi_0^{(n0)} \tag{23}$$

and $H_1 = V - V^{(0)}(r)$, where V is the true muon-nucleus interaction. One straightforwardly obtains the radial equations for the first-order corrections

$$\frac{d}{dr}f^{(nk)}(r)=\frac{K_k}{r}f^{(nk)}(r)+(M_\mu+V^{(0)}(r)-E_\mu^{(n0)}+E_N^{(k)}-E_N^{(0)})g^{(nk)}(r)+\langle\kappa\,|\,V-V^{(0)}(r)-E_1^{(n0)}\,|\,0\rangle G^{(n0)}(r)\,,$$
$$\frac{d}{dr}g^{(nk)}(r)=-\frac{K_k}{r}g^{(nk)}(r)+(M_\mu-V^{(0)}(r)+E_\mu^{(n0)}-E_N^{(k)}+E_N^{(0)})f^{(nk)}(r)-\langle\kappa\,|\,V-V^{(0)}(r)-E_1^{(n0)}\,|\,0\rangle F^{(n0)}(r)\,, \tag{24}$$

where we have used $\langle -k\,|\,V-V^{(0)}(r)-E_1^{(n0)}\,|\,-0\rangle = \langle k\,|\,V-V^{(0)}(r)-E_1^{(n0)}\,|\,0\rangle$. In terms of the functions $F^{(n0)}(r)$ and $G^{(n0)}(r)$, which are appropriate solutions of Eqs. (13), and the solutions $f^{(nk)}(r)$ and $g^{(nk)}(r)$ of Eqs. (24), the second-order energy shift due to the intermediate state $|k\rangle$ is

$$E_2^{(nk)}=\int_0^\infty dr\,\langle k\,|\,V-V^{(0)}(r)-E_1^{(n0)}\,|\,0\rangle \times[F^{(n0)}(r)f^{(nk)}(r)+G^{(n0)}(r)g^{(nk)}(r)]\,, \tag{25}$$

where the normalization is $\int_0^\infty dr\,[F^{(n0)}(r)^2+G^{(n0)}(r)^2]=1$. As usual, any multiple of $F^{(n0)}$ and $G^{(n0)}$ may be added to $f^{(nk)}$ and $g^{(nk)}$ without affecting this second-order energy shift, since

$$E_1^{(n0)}=\int_0^\infty dr\,\langle k\,|\,V-V^{(0)}(r)\,|\,0\rangle[F^{(n0)}(r)^2+G^{(n0)}(r)^2]\,. \tag{26}$$

So far the interaction V is unspecified. We now take it to be the static Coulomb interaction

$$V(\vec{r})=-\alpha\int d^3r'\,\frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|}\,, \tag{27}$$

where $\rho(\vec{r}')$ is the one-body nuclear charge-density operator and $\alpha\approx\frac{1}{137}$ is the fine-structure constant. This neglect of the transverse interaction is generally a good approximation for muonic atoms, which are inherently low-momentum systems. Implications of this approximation are discussed later. The needed matrix elements are obtained by expanding $|\vec{r}-\vec{r}'|^{-1}$ in spherical harmonics, yielding

$$\begin{aligned}\langle k\,|\,V\,|\,0\rangle&=\sum_{l=0}^\infty\langle k\,|\,V_l\,|\,0\rangle\\&=-\alpha\sum_{l=0}^\infty(-1)^{I_0+\mathfrak{F}_0+j_k+j_0-1/2}\tfrac{1}{2}[1+(-1)^{l_0+l+l_k}]\left(\frac{4\pi(2j_0+1)(2j_k+1)}{2l+1}\right)^{1/2}\begin{Bmatrix}j_0 & I_0 & \mathfrak{F}_0\\ I_k & j_k & l\end{Bmatrix}\begin{pmatrix}j_0 & l & j_k\\ -\frac{1}{2} & 0 & \frac{1}{2}\end{pmatrix}\delta_{\mathfrak{F}_0\mathfrak{F}_k}\delta_{\mathfrak{M}_0\mathfrak{M}_k}\\&\quad\times\langle\gamma_0 I_0\,||\int d^3r'\,\rho(\vec{r}')Y_l(\hat{r}')\frac{r_<^l}{r_>^{l+1}}\,||\,\gamma_k I_k\rangle\,.\end{aligned} \tag{28}$$

We now define the potential $V^{(0)}(r)$ to be

$$V^{(0)}(r)\equiv\langle 0\,|\,V_{l=0}\,|\,0\rangle=-\alpha\int d^3r'\,\rho^{(0)}(\vec{r}')r_<^{-1}\,, \tag{29}$$

where $\rho^{(0)}(\vec{r}')=\langle I_0M_0\,|\rho(\vec{r}')\,|\,I_0M_0\rangle$ is the initial-state nuclear charge distribution. The coupling matrix elements $\langle k\,|\,V-V^{(0)}(r)-E_1^{(n0)}\,|\,0\rangle$ distinguish themselves by whether they are diagonal in the nuclear states. If not, then $V^{(0)}(r)$ and $E_1^{(n0)}$ do not contribute and may be discarded. If $|k\rangle$ and $|0\rangle$ do include the same nuclear state, then

$$\langle k\,|\,V-V^{(0)}(r)-E_1^{(n0)}\,|\,0\rangle=\sum_{l=1}^\infty\langle k\,|\,V_l\,|\,0\rangle-E_1^{(n0)}\delta_{k0}\,. \tag{30}$$

For nuclei with $I_k=I_0\geq 1$, these matrix elements may be nonzero and thus contribute to the second-order energy shift. For $I_k=I_0<1$, $E_1^{(n0)}$ is identically zero, as are all $\langle k\,|\,V_l\,|\,0\rangle$ with $l\neq 0$. Thus in the helium isotopes ($I_0=0,\frac{1}{2}$) we ignore perturbations diagonal in the nuclear states and discard the coupling terms $\langle k\,|\,V^{(0)}(r)+E_1^{(n0)}\,|\,0\rangle$.

For muon coordinates outside the nucleus, the reduced nuclear matrix element appearing in Eq. (28) is the same as that which specifies the electric transition strength

$$B(El;0\to k)=(2I_0+1)^{-1}\times\left|\langle\gamma_k I_k\,||\int d^3r'\,\rho(\vec{r}')r'^l Y_l(\hat{r}')\,||\,\gamma_0 I_0\rangle\right|^2. \tag{31}$$

If this quantity is known experimentally, the reduced matrix element in Eq. (28) resulting from any model calculation can be renormalized to produce the appropriate $B(El)$. In this way the asymptotic form of the transition potential can be specified uniquely by experiment. A more uncertain part of the transition potential concerns the way in which it is cut off inside the nucleus. This de-

pends upon the shape of the transition charge density, which may be specified rather incompletely by inelastic electron scattering, or by a nuclear model. For excitations with $l \neq 0$, this uncertainty is relatively unimportant to the extent that the muon is excluded from the nucleus. For $l = 0$, however, the entire contribution comes from the uncertain interior part, since the exterior potential is identically zero. The most reliable experimental information about monopole excitations comes from the electron scattering matrix element $\langle \gamma_k I_k || \int d^3r' \rho(\vec{r}') r'^2 || \gamma_0 I_0 \rangle$. This is not very similar to that appearing in Eq. (28), so that any relationship between the two must be rather model dependent.

II. CALCULATIONS

The computational exercise is to solve Eqs. (13) and (24) for each muon state and nuclear excitation of interest, using transition potentials given by Eq. (28). The energy shifts [Eq. (25)] for each excitation are then summed to give the total nuclear polarization correction for a given initial muon state. In the present work, Eqs. (13) and (24) are solved numerically using techniques outlined in Appendixes A and B. We take the initial nuclear state to be the ground state with $E_N^{(0)} = 0$, although the calculation may be carried out in the same way for excited initial nuclear states, if these are of interest.

It is useful to express the energy shift for a given l as

$$E_2^{(n,l)} = \sum_k W_l(E_N^{(k)}) B(El; 0 \to k) , \tag{32}$$

where $W_l(E)$ is the energy shift due to a nuclear excitation of unit $B(El)$ at energy E. The excitations in the helium isotopes are all in the continuum. Their transition strengths are known in the long-wavelength limit through the photoabsorption cross sections[9]

$$\sigma(l, E) = (2\pi)^3 \alpha \frac{(l+1)E^{2l-1}}{l[(2l+1)!!]^2} B(El, E) \frac{df}{dE} , \tag{33}$$

where df/dE is the density of final states. Thus we may write

$$E_2^{(n,l)} = \frac{l}{l+1} \frac{[(2l+1)!!]^2}{(2\pi)^3 \alpha} \int_0^\infty \frac{dE}{E} \sigma(l, E) W_l(E) E^{2-2l} . \tag{34}$$

A. Estimates

An estimate of the perturbation may be obtained by making an extreme nonrelativistic reduction of Eqs. (24). If we neglect the small component f with respect to g and G, the first of these equations gives an explicit formula for the perturbation:

$$g^{(nk)}(r) \approx \frac{-V_l(r) G^{(n0)}(r)}{M_\mu + V^{(0)}(r) - E_\mu^{(n0)} + E_N^{(k)}} , \tag{35}$$

where $V_l(r) = \langle k | V_l | 0 \rangle$. The validity of such an approximation is demonstrated in Fig. 2 for a typical dipole excitation, the $p_{1/2}$ perturbation on the $2s_{1/2}$ state in muonic ^{4}He. The true perturbation is the curve labeled g, while the approximation Eq. (35) is the curve labeled g_a. This approximation is clearly poor inside several muon Compton wavelengths. If we further neglect all except the nuclear excitation energy in the denominator of Eq. (35), we get the closure approximation, which is the curve labeled A in Fig. 2. This is the same as term A in HKW, and its large size relative to the true solution demonstrates the considerable degree of cancellation which must be accounted for properly in their calculation. For comparison, the unperturbed muon wave function $G(r)$ is also shown in Fig. 2, scaled smaller by a factor of 500.

It is instructive to consider the closure approximation in more detail. The energy shift in closure is

$$E_2^{(nk)} \approx -\frac{1}{E_N(k)} \int_0^\infty dr\, V_l^2(r) [G^{(n0)}(r)]^2 . \tag{36}$$

In order to avoid the region in which this approximation is bad, we cut off the lower limit of the above integral at some radius R approximately equal to the nuclear radius and set $V_l(r) = q_l / r^{l+1}$, where q_l is determined by the matrix element Eq. (28). This procedure is unfortunately quite sensi-

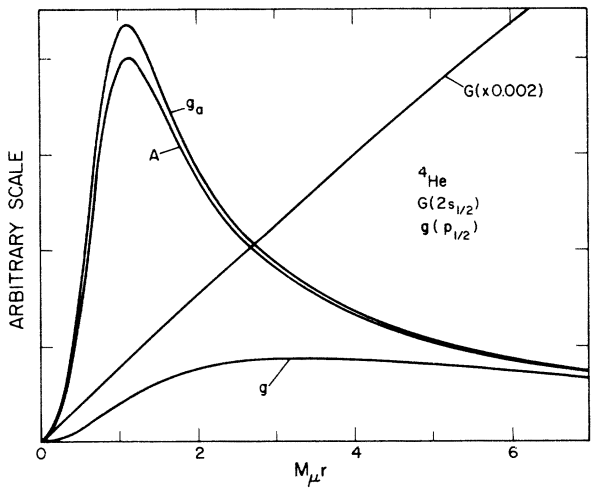

FIG. 2. Muon wave functions. G is the unperturbed $2s_{1/2}$ wave function (scaled smaller as shown), g is a typical first-order $p_{1/2}$ perturbation, A is this perturbation calculated in the closure approximation, and g_a is the same resulting from the approximation Eq. (35).

tive to the cutoff, since the integrand diverges as $r \to 0$. We approximate $G^{(n0)}(r)$ by the value of the $2s$ nonrelativistic point-nucleus radial wave function at $r=0$. Averaging and summing over initial and intermediate angular momenta, we obtain for $l \neq 0$

$$W_l(E) \approx -\frac{1}{E}\frac{2\pi\alpha^2}{(2l+1)^2(2l-1)}\frac{(Z\alpha M_\mu)^3}{R^{2l-1}}, \tag{37}$$

where R is the cutoff radius. Combining this with Eq. (34) gives

$$E_2^{(2s,l)} \approx -\frac{\alpha^4}{\pi^2 R^2}(ZM_\mu R)^3 \frac{l^2}{4^l}\frac{(2l-1)!\,(2l-2)!}{(l+1)!\,(l-1)!} \times \int_0^\infty dE\,\sigma(l,E)(RE)^{-2l}. \tag{38}$$

If we evaluate this expression for ^{4}He and $l=1$, using the known σ_{-2} sum[10-13]

$$\sigma_{-2} \equiv \int_0^\infty dE\,E^{-2}\sigma(E) = 0.007 \text{ fm}^2/\text{MeV} \tag{39}$$

and setting $R = 1.6$ fm (femtometers) we obtain $E_2^{(2s,1)} \approx 7$ meV, a value not far from the correct one. The approximate l dependence of the energy shift due to excitations in a small range about a given energy E may be noted from Eq. (38). This dependence is clearly quite strong, coming primarily from the factor $(RE)^{-2l}$. If $R = 1.6$ fm and $E = 25$ MeV, the explicitly l-dependent factors are about 100 times larger for $l=2$ than for $l=1$, implying that small admixtures of $l>1$ photons in the predominantly dipole cross sections can have large consequences for nuclear polarization. A careful numerical calculation of the energy shifts reduces the above factor of 100 by a considerable amount, presumably because the approximations leading to Eq. (38) become progressively worse as l increases.

B. Results

Figure 3 shows the numerically computed functions $W_l(E)$ for $l=1$, 2, and 3, for the $2s_{1/2}$ state in both ^{3}He and ^{4}He. The multiplicative factors included in the plots were chosen to absorb most of the l dependence in Eq. (34), as well as to reflect the fact that for higher multipoles, $\sigma(l,E)/\sigma(1,E)$ tends to be of order $(E/M_N)^{l-1}$, where M_N is the nucleon mass. The difference between the curves for ^{3}He and ^{4}He is due primarily to the different nuclear form factors, although there is a contribution of order 3–4% from the change in muon reduced mass. The ground-state form factors $\langle 0|\rho(\vec{r})|0\rangle$ used were two-parameter Fermi functions with parameters c and a adjusted to reproduce approximately both experimental rms radii and the 90–10% skin thickness of a Gaussian distribution with the same rms radius. The values of these parameters are given in Table I. The transition charge density $\langle k|\rho(\vec{r})|0\rangle$ was taken to be

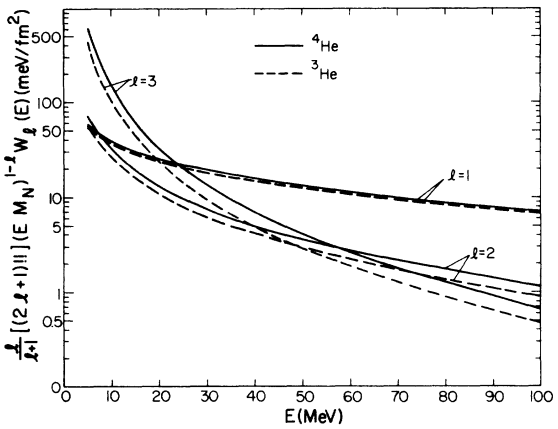

FIG. 3. Numerically computed functions $W_l(E)$. $M_N = 938$ MeV is the nucleon mass.

$$\langle k|\rho(\vec{r})|0\rangle \propto r\frac{d}{dr}\langle 0|\rho(\vec{r})|0\rangle. \tag{40}$$

Although these form factors are perhaps not the most appropriate to use here, the energy shifts they yield for the most important excitations (dipole) differ by less than 5% from those computed with similar Gaussian distributions. Numerical errors in the calculation were investigated carefully using both computational methods described in the appendixes in addition to the usual variation of numerical parameters. These internal checks suggest that such errors contribute less than 1% to the result. In addition, results from the same program have been compared with other nuclear polarization calculations appearing in the literature for other elements[7] with good agreement.

The photoabsorption cross sections of the He isotopes have been investigated extensively.[10-13] The total dipole, bremsstrahlung-weighted cross sections adopted here for the purpose of computing the energy shifts are shown in Fig. 4. Experimental uncertainties in these cross sections are typically of order 10%. Sums $\sigma_n \equiv \int_0^\infty dE\,\sigma(E)E^n$ are listed in Table II. In ^{4}He, there is currently a

TABLE I. Nuclear form-factor parameters.

	^{3}He	^{4}He
c (fm)	1.94	1.57
a (fm)	0.30	0.30
R (fm)	1.53	1.35
$\langle r^2\rangle^{1/2}$ (fm)	1.87	1.65

dispute over the relative size of the (γ,p) to the (γ,n) cross section.[10,11] Most of the present evidence indicates that these are equal,[11] which we have assumed here. In addition, we assume that the three- and four-body breakup processes are negligible. For ^{3}He, both the two- and three-body processes contribute.

Higher-multipole contributions to the cross sections are observed[12] at a level of 1–2%. The actual multipole composition is quite uncertain experimentally, as is the associated energy dependence. In order to estimate such contributions to the nuclear polarization in both ^{3}He and ^{4}He, we make use of a simple model calculation of Gibson and O'Connell[13] for ^{3}He. This model provides good agreement with the observed angular dependence of the ^{1}H$(d,^3$He$)\gamma$ cross section at 12.1 and 15.3-MeV excitation.[12] Expanding their result in powers of the photon energy E, we obtain for the (γ,p) cross sections

$$\sigma_{\gamma,p}(2,E)\approx\frac{5}{9}\frac{p^2}{\mu^2}\sigma_{\gamma,p}(1,E)$$

and (41)

$$\sigma_{\gamma,p}(3,E)\approx\frac{56}{2025}\frac{p^4}{\mu^4}\sigma_{\gamma,p}(1,E)\,,$$

where p is the final proton momentum and μ is its reduced mass in the proton-deuteron system. The higher-multipole contributions to the three-body breakup are more obscure. They are expected to be no larger, with a similar energy dependence which arises principally from the long-wavelength approximation. Thus, as an estimate of the total higher-multipole contributions, we apply Eqs. (41) to the total cross sections.

Although this prescription is not necessarily appropriate for any of the processes occurring in the ^{4}He breakup, it is nevertheless consistent with available experimental data[12] and has a reasonable energy dependence. One might expect the higher multipoles to contribute less in the more tightly bound nucleus, but the internal structure as manifested in the observed resonances in the cross section may produce significant higher-multipole components.

There is at present no well-established evidence for $l=0$ excitations in ^{3}He. In ^{4}He, a prominent $l=0$ resonance has been observed[14] at 20.1 MeV, with matrix element $\langle k\,|\int d^3r\,\rho(\vec{r})r^2\,|\,0\rangle = 1.1 \pm 0.16$ fm^2. We incorporate this state explicitly here, with the form factor

$$\langle k\,|\rho(\vec{r})\,|\,0\rangle \propto \frac{1}{r^2}\frac{d}{dr}r^4\frac{d}{dr}\langle 0\,|\rho(\vec{r})\,|\,0\rangle\,. \tag{42}$$

The connection between this and the actual form factor is probably purely imaginary. Fortunately, this state does not contribute much to the total result.

For the purpose of computing the energy shifts, the cross sections $\sigma(E)/E$ in Fig. 4 are fit linearly from threshold to 10 MeV for ^{3}He and to 23 MeV for ^{4}He. Beyond that, they are each represented by a Lorentzian,

$$\frac{\sigma(E)}{E}=\frac{E\sigma(E_m)}{E^2+(E^2-E_m^2)/\Gamma^2}\,. \tag{43}$$

The parameters E_m, Γ, and $\sigma(E_m)$ are given in Table II. These represent an approximate rather than a precision fit to the data. The functions $W_l(E)$ are also fit over appropriate ranges by analytic forms:

$$\frac{l}{l+1}[(2l+1)!!]^2 W_l(E)\approx K(E_0/E)^c\,. \tag{44}$$

The parameters K, c, and E_0 are given in Table III. These fits are all accurate to within a few percent over the range where the integrand in Eq. (34) is significant.

The energy shifts computed for the $2s_{1/2}$ state by means of the above prescriptions are given in Table IV. To within 1%, the computed $1s$ shifts

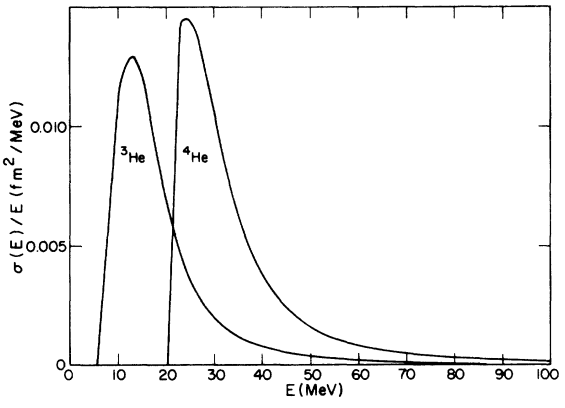

FIG. 4. Dipole, bremsstrahlung-weighted photoabsorption cross sections adopted in the present work.

TABLE II. Adopted photoabsorption cross-section parameters and the resulting sums.

	^{3}He	^{4}He
Photoabsorption threshold (MeV)	5.5	20.1
E_m (MeV)	15.0	26.0
$\sigma(E_m)$ (fm^2)	0.180	0.365
Γ (MeV)	15.8	19.3
σ_0 (fm^2 MeV)	3.9	8.0
σ_{-1} (fm^2)	0.20	0.23
σ_{-2} (fm^2/MeV)	0.013	0.0074

are larger by a factor of 8, while the $2p$ shifts are less than 0.01 meV. An overall uncertainty of the order of 10 to 20% should be assumed. This uncertainty comes primarily from uncertainties in the form factors and the multipole composition of the photoabsorption cross sections, and from the neglect of the transverse interaction. It is seen that the total shift (3.1 meV) for ^{4}He is in agreement with that of BJ, who obtain the same result. The primary physical differences between the two calculations are that BJ neglect the spatial variation of the muon $2s$ wave function, while it is treated exactly here; and that BJ include the transverse electromagnetic interaction, which is here neglected. They note that the first approximation makes their result too large by about 10%. They claim in a second paper[15] that neglect of the transverse interaction produces a result which is also too large, but by about 20%. Both of these estimates are consistent with the present calculation. However, one might expect a somewhat smaller contribution from the transverse interaction, since at most it should be of order v^2, with v some typical velocity in the excited nucleus [e.g., $v^2 \sim (2 \times 30\ \text{MeV})/(900\ \text{MeV}) \sim 0.07$]. The actual contribution of the transverse interaction depends in detail upon the behavior of the structure functions at small three-momentum transfer. One may as easily interpret the results of BJ [their Eqs. (18), (19), and (24), and Fig. 4] to estimate a 10% contribution from the transverse interaction. In fact, any such contribution between 0 and 20% is consistent with the present results.

The computational differences between the present calculation and that of BJ are significant. They avoid an explicit multipole expansion and the construction of transition charge densities, but essentially the same physical information must be specified in their structure functions. The good agreement which is obtained is convincing evidence that this information is accounted for adequately, both in the experiments which provide it and in the calculations. It should be emphasized that all the inherent ambiguities in either calculation are associated with only about 20% of the result; most of the energy shift is given unambiguously by the dipole photoabsorption cross sections.

The present results are in poor agreement with HKW. This is due primarily to the errors in their model, which are manifested in values of the photoabsorption sums σ_{-1} and σ_{-2}, which are too large (also discussed in Ref. 15). This may be seen most easily by noting that the transition strengths $B(El)$ for $l \neq 0$ are specified by the asymptotic forms of their transition potentials as $r \to \infty$ [see Eqs. (28) and (31)]. From their Eqs. (A11) and (A12) and their Table I, one may obtain for their state (11, 10)

$$B(E1) = 0.830\ \text{fm}^2,\ ^3\text{He}$$
$$= 0.866\ \text{fm}^2,\ ^4\text{He}\ .$$

Using Eq. (33) and their excitation energies 11.9 and 17.2 MeV, we see that these exhaust the TRK sum rule $\sigma_0 = 6NZ/A$ MeV fm^2, as required. However, the other sums σ_{-1} and σ_{-2} are much larger than the experimental ones, whose values as adopted here are noted in Table II. For example, we obtain from HKW for ^{4}He $\sigma_{-1} = 0.35$ fm^2 and $\sigma_{-2} = 0.02$ fm^2/MeV. In Table II, these sums are 0.23 fm^2 and 0.007 fm^2/MeV, respectively. In the approximation Eq. (38), the energy shift is proportional to σ_{-2} [σ_{-2} is proportional to the classical electric polarizability α(pol) and is numerically equal if α(pol) is in fm^3 and σ_{-2} is in mb/MeV]. In the exact numerical calculation, the energy shift is more nearly proportional to $\sigma_{-1.7}$; thus, we should *a priori* expect their result to be too large by a factor of 2 to 3. They allude to this problem by adjusting their excitation energies upward; however, they do not alter their model enough.

There is a further difficulty with HKW in comparison to the present work. In the above manner, the values of $B(El)$ may be extracted for each of their $l \neq 0$ excitations and used to normalize transition potentials. If this is done and shifts are computed by the present method, the results are all smaller than theirs by about 35%. If their monopole transition potentials are used as given, the results are also smaller by the same factor. The

TABLE III. Parameters for the approximate representation of the functions $l(l+1)^{-1}[(2l+1)!!]^2 W_l(E)$.

	^{3}He			^{4}He		
l	1	2	3	1	2	3
K (meV/fm^{2l})	53.9	7.41	6.54	25.7	6.53	7.89
E_0 (MeV)	5	5	5	20	20	20
c	0.61	0.27	0.14	0.70	0.40	0.27

TABLE IV. Nuclear-polarization energy shifts (meV).

l	^{3}He	^{4}He
0		−0.26
1	−2.91	−2.38
2	−1.55	−0.41
3	−0.39	−0.03
Total	−4.9	−3.1

origin of this discrepancy is not clear. Using their Gaussian form factors instead of Fermi functions decreases the discrepancy, but by only a few percent. In the most important (dipole) case it is possible to reproduce their term A, either with finite-nucleus relativistic wave functions or with point-nucleus nonrelativistic ones, which differ negligibly. It may be seen from Fig. 2 that HKW require a considerable degree of cancellation between their terms A and B in order to obtain the true perturbation accurately. If the present results are correct, the agreement for term A implies that they have calculated B to be too small, by about 35% in the dipole case.

III. OTHER CONTRIBUTIONS TO THE ENERGY LEVELS

Further corrections to the low-lying energy levels are summarized in Table V. Although it is not of primary interest here, the $1s_{1/2}$ state has been included for illustrative purposes. Slightly less refined (but essentially indentical) versions of these results have been published previously.[5] The calculational methods used here follow generally those outlined elsewhere.[16] It is helpful to recall that for the $1s$ state, the muon Bohr radius is about 130 fm, while for the $n=2$ states, it is about 500 fm. Thus, the radial scale of the problem is of the order of the electron reduced Compton wavelength $\lambda\!\!\!^{-}_e \approx 386$ fm. Except where noted otherwise, all of the corrections discussed below include nuclear finite-size effects calculated using Gaussian charge distributions $\rho(r) \propto e^{-(r/R)^2}$, whose parameters are given in Table I. None of these corrections varies by as much as 0.1 meV when the nuclear parameters are altered within a reasonable range.

A. Vacuum polarization

The Serber-Uehling[17] $[\alpha(Z\alpha)]$ and Källen-Sabry[18] $[\alpha^2(Z\alpha)]$ vacuum polarization corrections are by now straightforward to calculate.[19] The values listed in Table V include ladder iterations of the $\alpha(Z\alpha)$ electron diagram. The $\alpha(Z\alpha)^3$ vacuum polarization may be calculated in the point-nucleus approximation using prescriptions available in the literature.[20] Finite-nuclear-size corrections are insignificant.[21] For the present purpose, we have used Blomqvist's series[20] for $r/\lambda\!\!\!^{-}_e \leq 0.2$. For large r, we have approximated Vogel's tabulated function[20] by

$$V_3(r) \approx (Z\alpha)^3 \frac{56.1 - 5.65x}{x + 5.13x^3}, \qquad (45)$$

where the potential is in electron volts and $x = r/\lambda\!\!\!^{-}_e$. This function is used here where it is positive and $x > 0.2$, and is set to zero where it would otherwise be negative. It is accurate to better than 5% over the range $0.2 \leq x \leq 1$. It clearly becomes progressively worse for larger x; however, most of the contribution to any of the energy shifts comes from the region in which the function is accurately represented. With this estimate, the effect contributes about 0.02 meV to the $2s$-$2p$ differences. The vacuum polarization of order $\alpha^2(Z\alpha)^2$ is expected to be of comparable size,[22] although no attempt was made here to calculate it. Both of these effects should be calculated more carefully if the experiments are improved by an order of magnitude.

B. Vertex corrections

The vertex correction of order $\alpha(Z\alpha)$ is well known, the only uncertainty being in the evalua-

TABLE V. Corrections to the muonic helium energy levels (meV).

	^{3}He				^{4}He			
Level	$1s_{1/2}$	$2s_{1/2}$	$2p_{1/2}$	$2p_{3/2}$	$1s_{1/2}$	$2s_{1/2}$	$2p_{1/2}$	$2p_{3/2}$
Vacuum polarization								
$\alpha\,(Z\alpha)$	−18 475.6	−2042.0	−400.7	−400.4	−18 804.4	−2077.8	−412.0	−411.7
$\alpha^2\,(Z\alpha)$	−135.6	−14.9	−3.5	−3.5	−138.0	−15.2	−3.6	−3.6
$\alpha\,(Z\alpha)^3$	0.3	0.0	0.0	0.0	0.3	0.0	0.0	0.0
Muon $\alpha\,(Z\alpha)$	−2.6	−0.3	0.0	0.0	−2.6	−0.3	0.0	0.0
Vertex corrections								
$\alpha\,(Z\alpha)$	80.8	10.4	−0.2	0.1	83.3	10.7	−0.2	0.1
$\alpha\,(Z\alpha)^2$	1.6	0.2	0.0	0.0	1.6	0.2	0.0	0.0
Recoil	−23.8	−1.7	−1.3	−1.3	−18.0	−1.3	−1.0	−1.0
Nuclear polarization	−39.2	−4.9	0.0	0.0	−24.8	−3.1	0.0	0.0
Total	−18 594.1	−2053.2	−405.7	−405.1	−18 902.6	−2086.8	−416.8	−416.2

TABLE VI. Some special constants used in the energy-level calculations. 1 amu = 931.504 MeV.

α	1/137.036 04(11)
$\hbar c$ (MeV fm)	197.328 57(51)
M_μ (MeV)	105.659 48(35)
M_e (MeV)	0.511 0034(14)
$M(^3\mathrm{He})$ (amu)	3.016 03
$M(^4\mathrm{He})$ (amu)	4.002 60

tion of the Bethe sum.[23] For the $n=2$ states, the point-nucleus values are probably adequate, since the average excitation energy (17 Ry ~ 0.2 MeV) corresponds to a wavelength large compared to the nuclear radius. For the $1s$ state such an approximation may be nearly as good. It gives an energy shift within 5–10 meV of the mean of the Bethe-Negele sum-rule limits.[23] The entries in Table V make use of the point-nucleus sums[24] and include contributions from the anomalous moment (~5 meV for $1s$ and 0.6 meV for $2s$). The only other change from the electronic-atom treatment is in replacing[23] the value of the muon wave function at the origin by the expectation value of $\rho(r)$, the nuclear charge distribution. This reduces the contribution by about 10%. The $\alpha(Z\alpha)^2$ vertex correction has been calculated for electronic atoms in the point-nucleus approximation.[25] If we scale these results we get the entries in Table V, which are about half as large as an estimate which has been made previously for finite nuclei.[23] This prescription of one-half of the original finite-nucleus estimate has been adopted elsewhere for routine analysis.[16] The $\alpha^2(Z\alpha)$ vertex corrections amount to about 0.04% of the $\alpha(Z\alpha)$ vertex in electronic hydrogen,[25] and most are probably of comparable magnitude here (~0.004 meV), although the vacuum-polarization insertion may be substantially larger, as much as ~0.1 meV. As with the electronic vacuum polarization, these effects will bear further study if the experiments are improved substantially.

C. Other corrections

It is well established that the bound muon-helium nucleus exists in the $2s$ state with no atomic electrons present,[2,26] so that no correction to the energy levels due to such electrons is required. Indeed, any atomic electrons present would lead to immediate depopulation of the $2s$ state through monopole Auger-electron emission.

A further correction which must be applied is that due to nuclear recoil, which is not accounted for by using the muon reduced mass. This effect is calculated here from the Breit equation,[27] using the prescription of Friar and Negele[27] for finite nuclei. For the p states, the point-nucleus result [of order $(Z\alpha)^4 M_\mu/M_N$] suffices. For the s states there is an additional correction which may be calculated for a uniform nucleus to be

$$\Delta E(2s) = -\tfrac{9}{70}(Z\alpha)^5 R_0 M_\mu^3/M_N + O((Z\alpha)^6 R_0^2 M_\mu^4/M_N)\ , \tag{46}$$

where R_0 is the nuclear radius and $M_N \approx A \times 931$ MeV is here the nuclear mass. This amounts to about -0.28 meV in ^{4}He; a more refined numerical calculation gives -0.25 meV.

D. Uncertainties

The important constants used to calculate these energy levels are taken from the 1974 particle properties tables,[28] and from the 1971 mass tables.[29] They are listed in Table VI. The primary uncertainties in the predicted energy levels arising from the constants come from the muon mass and α. These together contribute about 5 ppm, which is presently insignificant in the $2s$-$2p$ energy differences. More important uncertainties in these energy differences arise from the nuclear polarization, to which we assign an overall uncertainty of 20%, and from the vertex corrections, which may be incorrect by as much as a few tenths of an meV. Thus the total theoretical un-

TABLE VII. Theoretical binding energies. For any nuclear rms radius r, $B = B_0 - c(r^2 - r_0^2)$. B_0 is given both with and without the corrections in Table V.

	^{3}He				^{4}He			
r_0^2 (fm)	3.51135				2.73375			
Level	$1s_{1/2}$	$2s_{1/2}$	$2p_{1/2}$	$2p_{3/2}$	$1s_{1/2}$	$2s_{1/2}$	$2p_{1/2}$	$2p_{3/2}$
B_0 (eV)								
No corrections	10 842.9086	2711.1146	2711.4662	2711.3218	10 941.2329	2735.6265	2735.9084	2735.7628
B_0 (eV)								
Corrected	10 861.5027	2713.1678	2711.8719	2711.7269	10 960.1355	2737.7133	2736.3252	2736.1790
c (eV/fm^2)	0.7904	0.0988	0.0	0.0	0.8154	0.1020	0.0	0.0

certainties in the $2s$-$2p$ energy differences are of order 1 meV.

Finally, it is necessary to account for the direct effect of the finite nuclear size upon the muon energy levels. This is accomplished here by solving the Dirac equation in the field of a Gaussian charge distribution, for several different values of the radial parameter R which are close to the electron scattering value.[2,4,5] The results are expressed in terms of the nuclear rms radius by the formula and parameters in Table VII. It should be noted that these values have been checked for numerical accuracy through the use of two separate and independent computer programs. They are not far from the results of other work[30] for the total $2s$-$2p$ splittings, although values of some of the corrections differ. The principal difference is in the $\alpha(Z\alpha)$ vacuum polarization, which was computed here (and in Ref. 5) including both finite nuclear size and higher-order iterations of the basic diagram. It appears that neglect of the latter in the other work[30] is the source of the current disagreement, as these higher-order iterations contribute about 2 meV to the $2s$-$2p$ energy differences.

ACKNOWLEDGMENTS

I would like to thank a number of people who have contributed useful information and discussion during the course of this work. These include B. L. Berman, J. P. Davidson, B. F. Gibson, E. M. Henley, A. Ikeda, M. Ikeda, S. Parter, R. M. Steffen, and L. Wilets.

APPENDIX A: ZERO-ORDER SOLUTIONS

The zero-order muon radial wave functions [solutions of Eqs. (13)] are found by standard shooting techniques using a convergent sequence of estimates for the eigenvalue E_μ. Asymptotic solutions at very small and at very large radii are constructed by assuming that the potential $V^{(0)}(r)$ is locally constant in both cases, so that $F(r)$ and $G(r)$ are Riccati-Bessel functions with appropriate boundary conditions. Starting from an initial estimate of the eigenvalue and these approximate solutions, Eqs. (13) are integrated outward and inward to a matching point R in the positive-kinetic-energy region. The equations are solved using the variable-order, variable-step Adams-method differential-equation-solving package ODE.[31] The potential $V^{(0)}(r)$ is provided at arbitrary radii by an analytic formula outside the nucleus and tabular (cubic) interpolation inside. At the matching point R, an improved estimate of the eigenvalue is made using the following formula from first-order perturbation theory:

$$E' = E + F_i(R)G_0(R) - F_0(R)G_i(R) \ . \tag{47}$$

In the above, E is the old eigenvalue, E' is the improved estimate, and $F_i(R), G_i(R)$ and $F_0(R)$, $G_0(R)$ are the interior and exterior solutions, respectively, evaluated at the matching point and normalized so that $F_i^2(R) + G_i^2(R) = F_0^2(R) + G_0^2(R)$, $F_i(R)F_0(R) + G_i(R)G_0(R) > 0$, and $\int_0^\infty [F^2(r) + G^2(r)]\,dr = 1$.

APPENDIX B: FIRST-ORDER PERTURBED SOLUTIONS

The first-order perturbations [solutions of Eqs. (24)] are found in the present work by two different methods. The first is similar to the shooting technique used to find the zero-order eigenvalue. Approximate solutions are assumed at small and at large radii, and the equations are solved (with ODE) outward and inward to a matching point. In addition to the desired solutions of the inhomogeneous Eqs. (24), there exist in general two spurious solutions of the homogeneous part, since $E_\mu^{(n0)} - E_N^{(k)} + E_N^{(0)}$ is not an eigenvalue. One of these is unbounded at $r = 0$ and the other is unbounded at $r = \infty$. Because of the rapid growth of these functions, numerical solutions of Eqs. (24) obtained by stepwise construction inevitably contain linear combinations of these solutions to the homogeneous equations. These solutions can be constructed and removed by matching at R, so long as they are not so large that the small perturbation is lost in the finite-accuracy arithmetic. This removal process makes the assumed boundary conditions for g and f arbitrary in principle. For moderate to large Z, the nuclear excitations are usually of comparable magnitude to the muon binding energies, so that the size of the spurious solutions is not a problem. In the case of helium however, the energy $E_\mu^{(n0)} - E_N^{(k)} + E_N^{(0)}$ is too far below any eigenvalue for the perturbation to survive unless arithmetic of extraordinary accuracy is used.

A second approach may be used in this case. This involves obtaining the desired solutions at a number of points simultaneously, with the appropriate boundary conditions built in. We approximate Eqs. (24) by the second-order difference equations

$$\begin{aligned}
\frac{2}{h}\left[f_i^{(nk)} - f_{i-1}^{(nk)}\right] &= \frac{K_k}{r}\left[f_i^{(nk)} + f_{i-1}^{(nk)}\right] + \left[M_\mu + V^{(0)}(r) - E_\mu^{(n0)} + E_N^{(k)} - E_N^{(0)}\right]\left[g_i^{(nk)} + g_{i-1}^{(nk)}\right] + \frac{2}{h} V_I(r) G^{(n0)}(r) \ , \\
\frac{2}{h}\left[g_i^{(nk)} - g_{i-1}^{(nk)}\right] &= -\frac{K_k}{r}\left[g_i^{(nk)} + g_{i-1}^{(nk)}\right] + \left[M_\mu - V^{(0)}(r) + E_\mu^{(n0)} - E_N^{(k)} + E_N^{(0)}\right]\left[f_i^{(nk)} + f_{i-1}^{(nk)}\right] - \frac{2}{h} V_I(r) F^{(n0)}(r) \ ,
\end{aligned} \tag{48}$$

where h is a radial increment, $f_i = f(r = ih)$, etc., and the functions involving r are to be evaluated at $r = (i - \frac{1}{2})h$. In the above, $V_1(r)$ is the surviving multipole potential from the matrix element $\langle k | V - V^{(0)}(r) - E_1^{(n0)} | 0 \rangle$. The above equations are rearranged and assembled as a (large) set of simultaneous linear equations for all f_i and g_i from $i = 0$ to some appropriately large value. If n radial points are chosen, the result is $2n - 2$ equations, so that the expected two boundary conditions may be specified. In order to eliminate the unbounded solutions, we specify one of the functions at $r = 0$ and one at the maximum radius. Since $f(0) = g(0) = 0$, we may specify either at $r = 0$; we prefer to choose g because it is the large component and thus contributes more to the energy shift. At large radii we also specify g through the extreme nonrelativistic reduction Eq. (35), the validity of which has been discussed earlier in the present paper.

The computational problems of inverting a large matrix are not particularly severe in this instance, since the matrix has band structure with no more than four nonzero elements per row. When the solution is obtained, it is found in general that $f(0) \neq 0$. The resulting magnitude of $f(0)$ is a measure of the accuracy of the finite-difference approximation and the large r value specified for g. In the present work the solutions have been obtained using as many as 1800 radial points between $r = 0$ and 300 fm, with a coarser mesh beyond. The accuracy obtained was sufficient that the nuclear-polarization energy shifts changed by less than 0.01% when the boundary conditions at $r = 0$ were varied, e.g., to $f(0) = 0$ or $f(0) = -g(0)$.

---

*Work supported by the U.S. Energy Research and Development Administration.

†Present address: Institut für Kernphysik, KFA, D-5170 Jülich, W. Germany.

[1]See, e.g., G. A. Rinker, Jr. and L. Wilets, Phys. Rev. D 7, 2629 (1973); S. Barshay, Phys. Rev. D 7, 2635 (1973); L. B. Okun and V. I. Zakharov, Nucl. Phys. B 57, 252 (1973); S. J. Brodsky and S. D. Drell, Ann. Rev. Nucl. Sci. 20, 147 (1970); J. Bailey and E. Picasso, Prog. Nucl. Phys. 12, 43 (1970); A. Rich and J. C. Wesley, Rev. Mod. Phys. 44, 250 (1972).

[2]A. Bertin, G. Carboni, J. Duclos, U. Gastaldi, G. Gorini, G. Neri, J. Picard, O. Pitzurra, A. Placci, E. Polacco, G. Torelli, A. Vitale, and E. Zavattini, Phys. Lett. 55B, 411 (1975); A. Bertin, G. Carboni, A. Placci, E. Zavattini, U. Gastaldi, G. Gorini, G. Neri, O. Pitzurra, E. Polacco, G. Torelli, A. Vitale, J. Duclos, and J. Picard, Nuovo Cimento B26, 433 (1975).

[3]C. Joachain, Nucl. Phys. 25, 317 (1961); T. E. O. Erickson and J. Hüfner, Nucl. Phys. B 47, 205 (1972).

[4]J. Bernabeau and C. Jarlskog, Nucl. Phys. B 75, 59 (1974).

[5]E. M. Henley, F. R. Krejs, and L. Wilets, Nucl. Phys. A 256, 349 (1976).

[6]G. Green (unpublished); A. Dalgarno and J. T. Lewis, Proc. R. Soc. A 233, 70 (1955).

[7]M.-Y. Chen, Phys. Rev. C 1, 1167 (1970); see also H. F. Skardhamar, Nucl. Phys. A 151, 154 (1970).

[8]J. M. McKinley, Phys. Rev. 183, 106 (1969).

[9]A good discussion of these points may be found in J. S. O'Connell, in *Proceedings of the International Conference on Photonuclear Reactions and Applications*, Asilomar, California, 1973 (Lawrence Livermore Laboratory, Livermore, Calif., 1973), p. 71.

[10]A. N. Gorbunov and V. M. Spiridonov, Zh. Eksp. Teor. Fiz. 33, 21 (1957) [Sov. Phys.-JETP 6, 16 (1958)]; V. N. Fetisov, A. N. Gorbunov, and A. T. Varfolomeev, Nucl. Phys. 71, 305 (1965); J. R. Stewart, R. C. Morrison, and J. S. O'Connell, Phys. Rev. 138, B372 (1965); B. L. Berman, S. C. Fultz, and P. F. Yergin, Phys. Rev. Lett. 24, 1494 (1970); B. L. Berman, S. C. Fultz, and M. A. Kelly, Phys. Rev. Lett. 25, 938 (1970); Phys. Rev. C 4, 723 (1971); B. L. Berman, F. W. K. Firk, and C.-P. Wu, Nucl. Phys. A 179, 791 (1972); J. D. Irish, B. L. Berman, R. G. Johnson, B. J. Thomas, K. G. McNeill, and J. W. Jury, in *Proceedings of the International Conference on Few Particle Problems in the Nuclear Interaction*, Los Angeles, California, 1972, p. 888 (unpublished); B. F. Gibson, in *Proceedings of the International Conference on Photonuclear Reactions and Applications*, Asilomar, California, 1973 (Lawrence Livermore Laboratory, Livermore, Calif., 1973), p. 373; J. D. Irish, R. G. Johnson, B. L. Berman, B. J. Thomas, K. G. McNeill, and J. W. Jury, Phys. Rev. C 8, 1211 (1973); B. L. Berman, S. C. Fultz, and P. F. Yergin, Phys. Rev. C 10, 2221 (1974); B. L. Berman and S. C. Fultz, Rev. Mod. Phys. 47, 713 (1975).

[11]W. R. Dodge and J. J. Murphy, II, Phys. Rev. Lett. 28, 839 (1972); J. D. Irish, R. G. Johnson, B. J. Thomas, B. L. Berman, K. G. McNeill, and J. W. Jury, in *Proceedings of the International Conference on Photonuclear Reactions and Applications*, Asilomar, California, 1973 (Lawrence Livermore Laboratory, Livermore, Calif., 1973), p. 147; Can. J. Phys. 53, 802 (1975); B. L. Berman (private communication).

[12]B. D. Belt, C. R. Bingham, M. L. Halbert, and A. van der Woude, Phys. Rev. Lett. 24, 1120 (1970); W. E. Meyerhof, M. Suffert, and W. Feldman, Nucl. Phys. A 148, 211 (1970).

[13]B. F. Gibson and J. S. O'Connell, Phys. Lett. 32B, 331 (1975). See also I. M. Barbour and J. E. Hendry, Phys. Lett. 38B, 151 (1972).

[14]R. Frosch, R. E. Rand, H. Crannell, J. S. McCarthy, L. R. Suelzle, and M. R. Yearian, Nucl. Phys. A 110, 657 (1967); Th. Walcher, Phys. Lett. 31B, 442 (1970); Z. Phys. 237, 368 (1970).

[15]J. Bernabeu and C. Jarlskog, CERN report TH-2088

(to be published).
[16]G. A. Rinker, Jr. and R. M. Steffen, Los Alamos Scientific Laboratory report LA-6073-MS (unpublished).
[17]R. Serber, Phys. Rev. 48, 49 (1935); E. A. Uehling, Phys. Rev. 48, 55 (1935).
[18]B. Källén and A. Sabry, K. Dan. Vidensk. Selsk. Mat.-Fys. Medd. 29, No. 17 (1955).
[19]L. W. Fullerton and G. A. Rinker, Jr., Phys. Rev. A 13, 1283 (1976).
[20]J. Blomqvist, Nucl. Phys. B 48, 95 (1972); P. Vogel, At. Data Nucl. Data Tables 14, 600 (1974).
[21]J. Arafune, Phys. Rev. Lett. 32, 560 (1974); L. S. Brown, R. N. Cahn, and D. L. McLerran, Phys. Rev. Lett. 32, 562 (1974); M. Gyulassy, Phys. Rev. Lett. 32, 1393 (1974); G. A. Rinker, Jr. and L. Wilets, Phys. Rev. A 12, 748 (1975).
[22]L. Wilets and G. A. Rinker, Jr., Phys. Rev. Lett. 34, 339 (1975); M. K. Sundaresan and P. J. S. Watson, in *Proceedings of the Sixth International Conference on High Energy Physics and Nuclear Structure*, Santa Fe, N.M., 1975 (unpublished); D. Fujimoto, Phys. Rev. Lett. 35, 341 (1975); E. Borie (to be published).
[23]R. C. Barrett, S. J. Brodsky, G. W. Ericson, and M. H. Goldhaber, Phys. Rev. 166, 1589 (1968); H. A. Bethe and J. W. Negele, Nucl. Phys. A 117, 575 (1968); R. C. Barrett, Phys. Lett. 28B, 93 (1968).
[24]S. Klarsfeld and A. Maquet, Phys. Lett. 43B, 201 (1973).
[25]T. Appelquist and S. J. Brodsky, Phys. Rev. A 2, 2293 (1970); G. W. Ericson and D. R. Yennie, Ann. Phys. (N.Y.) 35, 271 (1965); 35, 447 (1965).
[26]G. Carboni and O. Pitzurra, Nuovo Cimento B25, 367 (1975); J. E. Russell, Phys. Rev. A 1, 742 (1970); J. Bernabeu, T. E. O. Ericson, and C. Jarlskog, Phys. Lett. 50B, 467 (1974).
[27]G. Breit, Phys. Rev. 34, 553 (1929); J. L. Friar and J. W. Negele, Phys. Lett. 46B, 5 (1973). See also H. Grotch and D. R. Yennie, Rev. Mod. Phys. 41, 350 (1969); R. C. Barrett, D. A. Owen, J. Calmet, and H. Grotch, Phys. Lett. 47B, 297 (1973).
[28]N. Barash-Schmidt, A. Barbaro-Galtieri, C. Bricman, V. Chaloupka, D. M. Chew, R. L. Kelly, T. A. Lasinski, A. Rittenberg, M. Roos, A. H. Rosenfeld, P. Söding, T. G. Trippe, and F. Uchiyama, Phys. Lett. 50B, 1 (1974).
[29]A. H. Wapstra and N. B. Gove, Nucl. Data Tables 9, 265 (1971).
[30]E. Borie, SIN Report PR-75-020 (to be published).
[31]L. F. Shampine and M. K. Gordon, *Computer Solution of Ordinary Differential Equations: The Initial Value Problem* (Freeman, San Francisco, 1975).