Calculation of the vacuum-polarization potential*

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The lowest-order vacuum-polarization potential, known as Uehling potential, is expanded for a spherical charge distribution in a convergent form valid for all distances. The accuracy of this expansion is carefully examined at different distances. The ratios of the vacuum-polarization potentials of orders $\alpha(Z\alpha)$, $\alpha^2(Z\alpha)$, $\alpha(Z\alpha)^3$, $\alpha(Z\alpha)^5$, and $\alpha(Z\alpha)^7$ to the Coulomb potential for a point nucleus are also calculated and presented in figures for $r \le 3.5ƛ_e$. Several simple fitting curves are suggested.

INTRODUCTION

Vacuum polarization originates from the process involving the creation of virtual electron-positron pairs by the electromagnetic fields. It is one of the radiative corrections in quantum-electrodynamic theory. Although quantum electrodynamics is our most complete and best verified theory in physics, there are still good reasons for further tests of its validity.[1-6]

In recent years, the tests have been carried out in a large variety of experiments at high and low energies for atoms and individual particles; in some cases, highly accurate calculations of vacuum-polarization potential over large distances are necessary. The lowest-order vacuum-polarization potential, known as the Uehling potential, has been expanded in several forms.[7-11] One finds, however, that all those expansions are inaccurate for distances greater than 500 fm, where the Uehling potential still has a non-negligible contribution, about 24 ppm of the Coulomb potential.

In this paper the Uehling potential is expanded in a convergent form valid for all distances. The ratios of the vacuum-polarization potentials of orders $\alpha(Z\alpha)$, $\alpha^2(Z\alpha)$, $\alpha(Z\alpha)^3$, $\alpha(Z\alpha)^5$, and $\alpha(Z\alpha)^7$ to the Coulomb potential for a point nucleus are also calculated for $r \leqslant 3.5\ ƛ_e$ (1351.6 fm), where the Uehling potential falls less than 0.1 ppm of the Coulomb potential.

Note added. After submitting this article for publication, we learned of a recent calculation by Fullerton and Rinker,[22] who present rational-approximations for the vaccum-polarization potentials of the orders $\alpha(Z\alpha)$ and $\alpha^2(Z\alpha)$.

UEHLING POTENTIAL

The renormalized leading term of the vacuum-polarization potential is[12]

$$A_\mu^{\text{vp}}(\vec{q}) = \frac{\alpha}{2\pi}\vec{q}^2\int_0^1 dv\,\frac{2v^2(1-\frac{1}{3}v^2)}{4+\vec{q}^2(1-v^2)}A_\mu(\vec{q}), \tag{1}$$

where A_μ and A_μ^{vp} are the μth components of the four-vector potentials, and $\vec{q}$ is the three-vector momentum. Here, relativistic units, $\hbar=m=c=1$, are used. The electric part, which modifies the Coulomb potential, has been referred to as the Uehling potential.[13] For a charge distribution $\rho(\vec{r})$, the Uehling potential can more conveniently be given in positional space as

$$V_{11}(r) = -\frac{\alpha}{\pi}\alpha\int d^3r'\,\frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|}\left[\int_0^1 dv\,\frac{v^2(1-\frac{1}{3}v^2)}{1-v^2}\times\exp\left(\frac{-2|\vec{r}-\vec{r}'|}{(1-v^2)^{1/2}}\right)\right]. \tag{2}$$

For a spherical charge distribution, (2) can be reduced to the familiar form[8]

$$V_{11}(r) = -\frac{2\alpha^2}{3r}\int_0^\infty dr'\,\rho(r')r'\times[\chi_2(2|r-r'|)-\chi_2(2|r+r'|)], \tag{3}$$

where $\chi_n(2r)$ is defined as

$$\chi_n(2r) = \int_1^\infty dt\,\frac{1}{t^n}\left(1+\frac{1}{2t^2}\right)\left(1-\frac{1}{t^2}\right)^{1/2}e^{-2rt}. \tag{4}$$

The integral $\chi_2(2r)$ may be reduced to the form[9]

$$\chi_2(2r) = f_2(2r)E_1(2r) + g_2(2r)e^{-2r}, \tag{5}$$

where $f_2(2r)$ and $g_2(2r)$ are entire functions of r, and $E_1(2r)$ is the exponential integral defined as

$$E_1(2r) = \int_1^\infty dt\,\frac{e^{-2rt}}{t}.$$

We expand $f_2(2r)$ and $g_2(2r)$ in power series,

$$f_2(2r) = \sum_{k=0}^\infty C_{2,k}(2r)^{2k+1}, \tag{6}$$

$$g_2(2r) = \sum_{k=0}^\infty D_{2,k}(2r)^k. \tag{7}$$

The convergence of these power series and the

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calculation of their coefficients are presented in the Appendix. The first few coefficients are as follows:

$$C_{20}=-1,\qquad D_{20}=\tfrac{9}{32}\pi,$$
$$C_{21}=0,\qquad D_{21}=\tfrac{9}{32}\pi-\ln 2-\tfrac{1}{6},$$
$$C_{22}=\tfrac{1}{320},\qquad D_{22}=-\tfrac{3}{64}\pi-\ln 2+\tfrac{5}{6},$$
$$C_{23}=\tfrac{1}{40\,320},\qquad D_{23}=-\tfrac{9}{64}\pi-\tfrac{1}{2}\ln 2+\tfrac{19}{24},$$
$$C_{24}=\tfrac{1}{5\,160\,960},\qquad D_{24}=-\tfrac{71}{768}\pi-\tfrac{1}{6}\ln 2+\tfrac{29}{72},$$
$$C_{25}=\tfrac{1}{851\,558\,400},\qquad D_{25}=-\tfrac{151}{3840}\pi-\tfrac{37}{960}\ln 2+\tfrac{541}{3600}$$
$$C_{26}=\tfrac{1}{182\,184\,837\,120},\qquad D_{26}=-\tfrac{97}{7680}\pi-\tfrac{1}{192}\ln 2+\tfrac{623}{14\,400},$$
$$C_{27}=\tfrac{1}{49\,594\,761\,216\,000},\qquad D_{27}=-\tfrac{523}{161\,280}\pi+\tfrac{1}{5040}\ln 2+\tfrac{170\,195}{16\,934\,400},$$

where C_{20}, C_{21}, C_{22}, and D_{20}, $D_{21}, \ldots, D_{24}$ had been obtained by McKee[9] in a different fashion, following Glauber *et al.*[7] For large k, it is more convenient to use the following recurrence relation and rapidly convergent series:

$$C_{2,k}=\frac{(k-1)(2k-5)}{4k^2(2k+1)(k-2)}C_{2,(k-1)}=\frac{-b_k}{(2k+1)!}, \tag{8}$$

$$D_{2,k}=(-1)^k\sum_{2l\geq k}^{\infty}\frac{(2l-k)!}{(2l+1)!}b_l, \tag{9}$$

where b_l has the recurrence relation

$$b_{l+1}=\frac{l(2l-3)}{2(l^2-1)}b_l;\quad b_0=1,\ b_1=0,\ b_2=-\tfrac{3}{8}. \tag{10}$$

To compute $f_2(2r)$ to an accuracy of 1 ppm, the expansion (6) terminated with C_{20} is adequate for $r<0.0669\lambda\!\!\!^{-}_e$ (Compton wavelength of electron $\lambda\!\!\!^{-}_e = 386.159\,05$ fm $=\alpha$ a.u.), and the expansion terminated with C_{22} is adequate for $r<0.293\lambda\!\!\!^{-}_e$, etc. We present these in Table I. Expansion (7) for $g_2(2r)$ is also analyzed in the same way in Table I. The term $g_2(2r)e^{-2r}$ dominates the term $f_2(2r)E_1(2r)$ throughout the region $r<0.1\lambda\!\!\!^{-}_e$, and both have about the same order of magnitude in the outer region. Note that in (3) we seem to have the subtraction of two almost equal quantities whenever r or r' is small. This difficulty, however, only exists in an analytical sense since $\chi_2(2r)$ is a fast-varying function. In practice, even at a fairly small distance $r=0.1$ fm from the center of a charge dis-

TABLE I. Numerical analysis of the expansions of $f_2(r)$ and $g_2(r)$.

Expansion terminated with:	Applicable radius ($\lambda\!\!\!^{-}_e$) with the accuracy of: 1 ppm	0.1 ppm
C_{20}	0.066 9	0.037 6
C_{22}	0.293	0.199
C_{23}	0.614	0.460
C_{24}	0.982	0.780
C_{25}	1.37	1.13
C_{26}	1.77	1.50
C_{27}	2.18	1.89
C_{28}	2.58	2.27
C_{29}	2.99	2.67
$C_{2,10}$	3.40	3.06
$C_{2,11}$	3.80	3.46
$C_{2,12}$	4.21	3.85
D_{20}	0.000 018 6	0.000 001 86
D_{21}	0.005 59	0.001 77
D_{22}	0.032 2	0.014 9
D_{23}	0.064 6	0.036 3
D_{24}	0.254	0.160
D_{25}	0.286	0.195
D_{27}	0.603	0.452
D_{29}	0.969	0.769
$D_{2,11}$	1.36	1.12
$D_{2,13}$	1.76	1.49
$D_{2,15}$	2.16	1.87
$D_{2,17}$	2.57	2.26
$D_{2,19}$	2.97	2.65
$D_{2,21}$	3.38	3.04
$D_{2,23}$	3.78	3.44
$D_{2,25}$	4.19	3.83

TABLE II. Numerical analysis of the expansions of $f_1(r)$ and $g_1(r)$.

Expansion terminated with:	Applicable radius ($\lambda\!\!\!^{-}_e$) with the accuracy of: 10 ppm	1 ppm
C_{10}	0.079 5	0.044 7
C_{12}	0.311	0.212
C_{13}	0.622	0.466
C_{14}	0.972	0.772
C_{15}	1.34	1.11
C_{16}	1.72	1.46
C_{17}	2.11	1.82
C_{18}	2.49	2.20
C_{19}	2.88	2.57
$C_{1,10}$	3.27	2.95
$C_{1,11}$	3.66	3.33
$C_{1,12}$	4.05	3.71
D_{10}	0.000 018 5	0.000 001 85
D_{11}	0.004 54	0.001 44
D_{12}	0.022 2	0.010 3
D_{13}	0.147	0.082 8
D_{14}	0.190	0.120
D_{15}	0.478	0.326
D_{16}	0.484	0.348
D_{18}	0.841	0.651
$D_{1,10}$	1.23	0.995
$D_{1,12}$	1.63	1.36
$D_{1,14}$	2.03	1.74
$D_{1,16}$	2.44	2.13
$D_{1,18}$	2.85	2.52
$D_{1,20}$	3.26	2.92
$D_{1,22}$	3.66	3.32
$D_{1,24}$	4.07	3.71

tribution, we will not lose more than two significant digits. For small r', this seeming difficulty is further reduced by the weighting factor r' and does not constitute a problem at all. Nevertheless, when a great accuracy for the Uehling potential is desired, it is important to include enough terms of $\chi_2(2r)$; a test run is a simple and effective way to find out.

In the limiting case $r=0$, the Uehling potential has the form

$$V_{11}(0) = -\tfrac{8}{3}\alpha^2 \int_0^\infty dr'\, \rho(r') r' \chi_1(2r'), \tag{11}$$

where $\chi_1(2r)$ was defined in (4) and has an expression of the form[7]

$$\chi_1(2r) = f_1(2r)E_1(2r) + g_1(2r)e^{-2r}. \tag{12}$$

Here, the entire functions $f_1(2r)$ and $g_1(2r)$ may be written as the convergent power series

$$f_1(2r) = \sum_{k=0}^\infty C_{1,k}(2r)^{2k}, \tag{13}$$

$$g_1(2r) = \sum_{k=0}^\infty D_{1,k}(2r)^{k}. \tag{14}$$

The first few coefficients are as follows:

$C_{10}=1,\qquad D_{10}=\ln 2 - \frac{5}{6},$

$C_{11}=0,\qquad D_{11}=\frac{3}{8}\pi + \ln 2 - \frac{11}{6},$

$C_{12}=-\frac{1}{64},\qquad D_{12}=\frac{3}{8}\pi + \frac{1}{2}\ln 2 - \frac{37}{24},$

$C_{13}=-\frac{1}{5760},\qquad D_{13}=\frac{11}{48}\pi + \frac{1}{6}\ln 2 - \frac{59}{72},$

$C_{14}=-\frac{1}{573\,440},\qquad D_{14}=\frac{5}{48}\pi + \frac{5}{192}\ln 2 - \frac{199}{576},$

$C_{15}=-\frac{1}{77\,414\,400},\qquad D_{15}=\frac{7}{192}\pi - \frac{7}{960}\ln 2 - \frac{787}{7200},$

$C_{16}=-\frac{1}{14\,014\,218\,240},\qquad D_{16}=\frac{29}{2880}\pi - \frac{19}{2880}\ln 2 - \frac{9353}{345\,600},$

where C_{10}, C_{11}, C_{12}, and D_{10}, $D_{11}, \ldots, D_{14}$ had been obtained by Glauber *et al.*[7] in a different fashion. For large k, the recurrence relation and rapidly convergent series are useful:

$$C_{1,k} = \frac{(k-1)(2k-5)}{4k^2(2k-1)(k-2)} C_{1,(k-1)} = \frac{b_k}{(2k)!}, \tag{15}$$

$$D_{1,k} = (-1)^k \sum_{2l \ge k+1}^{\infty} \frac{(2l-k-1)!}{(2l)!} b_l. \tag{16}$$

At different distances, the numbers of terms to be included in the expansions (13) and (14) to achieve certain accuracies are summarized in Table II.

UEHLING POTENTIAL OF A HOMOGENEOUS CHARGE SPHERE

As a special case, consider a homogeneous charge sphere of radius R,

$$\rho(r) = \begin{cases} \frac{3}{4}Z/\pi R^3, & r \le R, \\ 0, & r > R. \end{cases}$$

The Uehling potential of this charge distribution is

$$V_{11}(r) = -\frac{Z\alpha}{r}\frac{\alpha}{2\pi R^3} \times \int_0^R r'dr' [\chi_2(2|r-r'|) - \chi_2(2|r+r'|)]. \tag{17}$$

By carrying out the r' integration, we find

$$V_{11}(r) = \begin{cases} -(Z\alpha/r)(\alpha/2\pi R^3)\{\frac{2}{5}r + \frac{1}{4}[\chi_4(2R+2r) - \chi_4(2R-2r)] + \frac{1}{2}R[\chi_3(2R+2r) - \chi_3(2R-2r)]\}, & r \le R, \\ -(Z\alpha/r)(\alpha/2\pi R^3)\{\frac{1}{4}[\chi_4(2R+2r) - \chi_4(2r-2R)] + \frac{1}{2}R[\chi_3(2R+2r) + \chi_3(2r-2R)]\}, & r > R, \end{cases} \tag{18}$$

where $\chi_3(2r)$ and $\chi_4(2r)$ were defined in (4). In general, $\chi_n(2r)$ may be expanded for any integer n as

$$\chi_n(2r) = E_1(2r) \sum_{2k \ge \max[0,(1-n)]}^{\infty} C_{nk}(2r)^{2k+n-1} + e^{-2r} \sum_{k=\min[0,(n-1)]}^{\infty} D_{nk}(2r)^k. \tag{19}$$

The expansion coefficients C_{nk} and D_{nk} may easily be obtained from the recurrence relations

$$C_{(n+1)k} = -C_{nk}/(n+2k), \tag{20}$$

$$D_{(n+1)(k+1)} = [1/(k+1)][D_{(n+1)k} - D_{nk} + C_{(n+1)[(k+1-n)/2]}], \tag{21}$$

with the exceptions $D_{n,(n-1)} = -nD_{(n+1),n}$, for $n<0$, and $D_{0,-1} = C_{1,0}$. Here $C_{(n+1)[(k+1-n)/2]} = 0$ if $\frac{1}{2}(k+1-n)$ is not a non-negative integer or $k \le -2$. In addition, we need

$$D_{n0} = \begin{cases} \dfrac{3(n+1)}{2(n+2)} \dfrac{(n-3)!!}{n!!}, & n \text{ odd}, \\[2ex] \dfrac{3(n+1)}{2(n+2)} \dfrac{(n-3)!!}{n!!} \dfrac{\pi}{2}, & n \text{ even}, \end{cases} \tag{22}$$

where we define $(0)!! = 1$. The recurrence relations (20) and (21) may be reversed to calculate the coefficients of $\chi_n(2r)$ for $n \le 0$, which appears in a Taylor-series expansion of (3).[22]

VACUUM-POLARIZATION POTENTIAL OF A POINT NUCLEUS

The vacuum-polarization potential of a point nucleus may be expanded in powers of α and $Z\alpha$.

The first few orders are listed below.

1. Order $\alpha(Z\alpha)$. For a point nucleus with charge Ze, the Uehling potential (3) may be further reduced to

$$V_{11}(r)=-(Z\alpha/r)R_{11}(r), \tag{23}$$

where

$$R_{11}(r)=(\tfrac{2}{3}\alpha/\pi)\chi_1(2r). \tag{24}$$

It is obvious that $\chi_1(2r)$ has a logarithmic divergence as $r\to 0$. Its asymptotic behavior may be obtained with the substitution $y=2r(t-1)$,

$$\chi_1(2r)\sim\frac{e^{-2r}}{(2r)^{3/2}}\int_0^\infty dy\,e^{-y}y^{1/2}[1+O(y/2r)],\quad \text{as } r\to\infty$$

$$=[e^{-2r}/(2r)^{3/2}](\tfrac{9}{8}\pi)^{1/2}[1+O(1/2r)], \tag{25}$$

2. Order $\alpha^2(Z\alpha)$. This order has been derived by Källén and Sabry.[14] Using their results, Blomqvist[10] obtained

$$V_{21}(r)=-(Z\alpha/r)R_{21}(r), \tag{26}$$

where

$$R_{21}(r)=-\frac{\alpha^2}{\pi^2}\int_1^\infty dt\,e^{-2tr}\Big((\tfrac{13}{54}t^{-2}+\tfrac{7}{108}t^{-4}+\tfrac{2}{9}t^{-6})(t^2-1)^{1/2}+(-\tfrac{44}{9}t^{-1}+\tfrac{2}{3}t^{-3}+\tfrac{5}{4}t^{-5}+\tfrac{2}{9}t^{-7})\ln[t+(t^2-1)^{1/2}]$$
$$+(\tfrac{4}{3}t^{-2}+\tfrac{2}{3}t^{-4})(t^2-1)^{1/2}\ln[8t(t^2-1)]+(-\tfrac{8}{3}t^{-1}+\tfrac{2}{3}t^{-5})\int_t^\infty dx\,f(x)\Big), \tag{27}$$

$$f(x)=\frac{3x^2-1}{x(x^2-1)}\ln[x+(x^2-1)^{1/2}]-\frac{1}{(x^2-1)^{1/2}}\ln[8x(x^2-1)].$$

The expansion for small r was given as

$$V_{21}(r)=\frac{\alpha^2Z\alpha}{\pi^2}\Big(-\frac{4}{9r}(\ln r+\gamma)^2-\frac{13}{54r}(\ln r+\gamma)-[\zeta(3)+\tfrac{1}{27}\pi^2+\tfrac{65}{648}]\frac{1}{r}+\tfrac{13}{9}\pi^2+\tfrac{32}{9}\pi\ln 2-\tfrac{766}{135}\pi+\tfrac{5}{3}r(\ln r+\gamma)$$
$$-\tfrac{65}{18}r+(\tfrac{14}{27}\pi^2-\tfrac{80}{81}\pi)r^2-\tfrac{5}{18}r^3(\ln r+\gamma)^2+\tfrac{323}{216}r^3(\ln r+\gamma)+(\tfrac{1}{6}\zeta(3)-\tfrac{5}{216}\pi^2-\tfrac{6509}{2592})r^3+O(r^4)\Big). \tag{28}$$

The asymptotic behavior of $R_{21}(r)$ may also be obtained with the substitution $y=2r(t-1)$,

$$R_{21}(r)\sim\left(\frac{\alpha}{\pi}\right)^2\frac{e^{-2r}}{2r}\int_0^\infty dy\,e^{-y}\left[-2\left(\frac{y}{r}\right)^{1/2}\ln\left(\frac{y}{2r}\right)+O\left(\frac{y}{2r}\right)^{1/2}+2\int_{1+y/2r}^\infty f(x)\,dx\right],\quad \text{as } r\to\infty$$

$$\int_{1+y/2r}^\infty f(x)\,dx\sim\int_1^\infty f(x)dx+\left(\frac{y}{r}\right)^{1/2}\ln\left(\frac{y}{2r}\right)+O\left(\frac{y}{2r}\right)^{1/2},\quad \text{as } r\to\infty$$

Hence we have

$$R_{21}(r)\sim\left(\frac{\alpha}{\pi}\right)^2\frac{e^{-2r}}{2r}\left[2\int_1^\infty f(x)\,dx+O\left(\frac{1}{2r}\right)^{1/2}\right],\quad \text{as } r\to\infty. \tag{29}$$

3. Orders $\alpha(Z\alpha)^3$, $\alpha(Z\alpha)^5$, and $\alpha(Z\alpha)^7$. These represent the major effect of the distortion of the electron and positron wave functions in a strong Coulomb field. The leading order has been given previously as[10,15,16]

$$V_{13}(r)=-(Z\alpha/r)R_{13}(r), \tag{30}$$

where

$$R_{13}(r)=-\frac{\alpha(Z\alpha)^2}{\pi}\int_0^\infty dt\,e^{-2tr}\frac{1}{t^4}\left[-\tfrac{1}{12}\pi^2(t^2-1)^{1/2}\theta(t-1)+\int_0^t dx\,(t^2-x^2)^{1/2}f(x)\right], \tag{31}$$

$$f(x)=-2x\psi(x^2)-x\ln^2(1-x^2)+\frac{1-x^2}{x^2}\ln(1-x^2)\ln\frac{1+x}{1-x}+\frac{1-x^2}{4x}\ln^2\frac{1+x}{1-x}+\frac{2-x^2}{x(1-x^2)}\ln(1-x^2)$$
$$+\frac{3-2x^2}{1-x^2}\ln\frac{1+x}{1-x}-3x,\quad \text{for } x<1,$$

$$f(x)=\frac{1}{x^2}\psi\left(\frac{1}{x^2}\right)-\frac{3x^2+1}{2x}\left[\psi\left(\frac{1}{x}\right)-\psi\left(-\frac{1}{x}\right)\right]-\frac{2x^2-1}{2x^2}\left[\ln^2\left(1-\frac{1}{x^2}\right)+\ln^2\frac{x+1}{x-1}\right]$$
$$-(2x-1)\ln\left(1-\frac{1}{x^2}\right)\ln\frac{x+1}{x-1}+\frac{3x^2+1}{4x}\ln^2\frac{x+1}{x-1}-2\ln x\ln\left(1-\frac{1}{x^2}\right)$$
$$-\frac{3x^2+1}{2x}\ln x\ln\frac{x+1}{x-1}+\left(5-\frac{x(3x^2-2)}{x^2-1}\right)\ln\left(1-\frac{1}{x^2}\right)+\left(\frac{3x^2+2}{x}-\frac{3x^2-2}{x^2-1}\right)\ln\frac{x+1}{x-1}$$
$$+3\ln x-3,\ \text{for } x>1\,,$$

$$\psi(x)=-\int_0^x dx'\,\frac{\ln(1-x')}{x'}=\sum_{n=1}^{\infty}\frac{x^n}{n^2},\quad -1\leq x\leq 1,$$

$$\theta(x)=\begin{cases}0, & x<0\\ 1, & x>0.\end{cases}$$

The expansion for small r was given as[10]

$$V_{13}(r)=[\alpha(Z\alpha)^3/\pi]\{[-\tfrac{2}{3}\zeta(3)+\tfrac{1}{6}\pi^2-\tfrac{7}{9}](1/r)+2\pi\zeta(3)-\tfrac{1}{4}\pi^3+[-6\zeta(3)+\tfrac{1}{16}\pi^4+\tfrac{1}{6}\pi^2]r$$
$$+\tfrac{2}{9}\pi r^2(\ln r+\gamma)+[\tfrac{2}{3}\pi\zeta(3)+\tfrac{4}{9}\pi\ln 2-\tfrac{31}{27}\pi]r^2+\tfrac{1}{12}(\ln r+\gamma)^2r^3$$
$$+(\tfrac{5}{54}\pi^2-\tfrac{19}{36})(\ln r+\gamma)r^3+(\tfrac{13}{18}\zeta(3)-\tfrac{109}{432}\pi^2+\tfrac{859}{864})r^3+O(r^4\ln r)\}\,. \tag{32}$$

The asymptotic behavior of $R_{13}(r)$ may be obtained by expanding the integrand of (31) in powers of t,

$$R_{13}(r)\sim-\frac{\alpha(Z\alpha)^2}{\pi}\frac{16}{675}\int_0^\infty dt\,e^{-2rt}\frac{1}{t^4}[t^7+O(t^9)],\quad \text{as } r\to\infty$$
$$=-\frac{\alpha(Z\alpha)^2}{\pi}\frac{32}{225}\frac{1}{(2r)^4}\left[1+O\left(\frac{1}{r^2}\right)\right], \tag{33}$$

Higher-order corrections $\alpha(Z\alpha)^5$ and $\alpha(Z\alpha)^7$, considered by Wichmann and Kroll,[15] can be written

$$V_{15}(r)=-(Z\alpha/r)R_{15}(r)\,, \tag{34}$$
$$V_{17}(r)=-(Z\alpha/r)R_{17}(r)\,, \tag{35}$$

and the leading terms are

$$R_{15}(r)=-[\alpha(Z\alpha)^4/\pi][\tfrac{2}{3}\zeta(5)-\tfrac{21}{4}\zeta(4)+\tfrac{71}{15}\zeta(3)-\tfrac{1}{4}\zeta^2(2)]+O(r)\,, \tag{36}$$
$$R_{17}(r)=-[\alpha(Z\alpha)^6/\pi][-\tfrac{2}{3}\zeta(7)+\tfrac{445}{24}\zeta(6)-\tfrac{286}{21}\zeta(5)-\tfrac{1}{4}\zeta(2)\zeta(4)-\tfrac{5}{2}\zeta^2(3)]+O(r)\,. \tag{37}$$

The corresponding vacuum-polarization potentials may be estimated[10] by scaling in the same ratios as the $1/r$ contribution to the full contribution with the $\alpha(Z\alpha)^3$ potential.

RESULTS AND DISCUSSIONS

We use the expansions (24), (28), and (32) to obtain the ratios of the vacuum-polarization potentials to the Coulomb potential for $r\leq 3.5\lambda\!\!\!^{-}_e$, where the Uehling potential falls less than 0.1 ppm of the Coulomb potential. By evaluating numerically the integrals (27) and (31), Vogel[17] calculated the corresponding potentials with better than 0.1% accuracy for $0.105\lambda\!\!\!^{-}_e\leq r\leq 1.0\lambda\!\!\!^{-}_e$. All these results are presented in Figs. 1 and 2. In the regions considered, the exponential behavior is very prominent for $R_{11}(r)$, $R_{21}(r)$, and $R_{13}(r)$. We find that conics can fit the calculated curves very nicely in a semilogarithmic plane. For example, by using for $V_{11}(r)$ the five points

$$R_{11}(0.01)=4.983\,273\times10^{-3}\,,$$
$$R_{11}(0.1)=1.724\,463\times10^{-3}\,,$$
$$R_{11}(0.5)=2.755\,383\times10^{-4}\,,$$
$$R_{11}(1.0)=5.564\,347\times10^{-5}\,,$$
$$R_{11}(2.0)=3.724\,597\times10^{-6}$$

we obtain the fitting curve

$$R'_{11}(r)=5.853\,199\times10^{-3}\times\exp[-0.405\,603\,9r-(3.315\,715r^2+14.872\,73r-0.124\,458\,4)^{1/2}]\,, \tag{38}$$

which deviates from the exact curve by less than 1% in most regions within $0.01\lambda\!\!\!^{-}_e<r<3.5\lambda\!\!\!^{-}_e$. The largest deviation is only about 3%. This kind of fitting curve could conveniently be used to give a good offhand estimation of the vacuum-polarization potential. The rational approximations recently made available by Fullerton and Rinker[22] is more accurate than the fitting curve (38) and may be used in actual computations. Similarly, for $V_{21}(r)$, the five points

$$R_{21}(0.01)=4.245\,367\times10^{-5}\,,$$
$$R_{21}(0.05)=1.866\,034\times10^{-5}\,,$$
$$R_{21}(0.1)=1.198\,744\times10^{-5}\,,$$
$$R_{21}(0.5)=2.2925\times10^{-6}\,,$$
$$R_{21}(1.0)=5.7730\times10^{-7}$$

give the curve

$$R_{21}'(r)=6.247\,314\times10^{-5}\times\exp[6.889\,829r-(85.078\,11r^2+49.171\,60r-0.292\,997\,1)^{1/2}]\,, \quad (39)$$

which has about 1% accuracy in the region $0.01\lambda_e<r<0.03\lambda_e$ and about 0.1% accuracy in the region $0.03\lambda_e<r<1.0\lambda_e$. For $V_{13}(r)$, the five points

$$R_{13}(0.01)=-1.483\,025\times10^{-4}(Z\alpha)^2\,,$$
$$R_{13}(0.1)=-1.153\,944\times10^{-4}(Z\alpha)^2\,,$$
$$R_{13}(0.3)=-7.0274\times10^{-5}(Z\alpha)^2\,,$$
$$R_{13}(0.6)=-3.6246\times10^{-5}(Z\alpha)^2\,,$$
$$R_{13}(1.0)=-1.6109\times10^{-5}(Z\alpha)^2$$

determine the curve

$$R_{13}'(r)=-(Z\alpha)^2 8.400\,763\times10^{-4}\times\exp[0.372\,807\,9r-(4.416\,798r^2+11.399\,11r+2.906\,096)^{1/2}]\,, \quad (40)$$

which has an accuracy of about 0.1% in the region $0.01\lambda_e<r<1.0\lambda_e$. Because of the simple forms and good accuracies, the fitting curves (39) and (40), or others of that kind, could be used along with the expansions (28) and (32) instead of the cumbersome integrals (27) and (31). These fitting curves may be used to generate the extrapolated potentials for $r>1.0\lambda_e$, which have not been calculated before. Because of the smallness of the po-

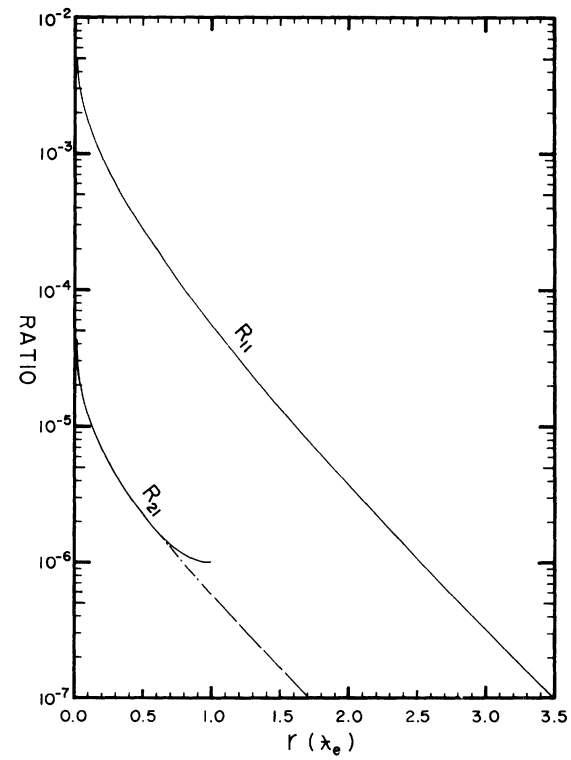

FIG. 1. Ratios of the vacuum-polarization potentials of orders $\alpha(Z\alpha)$ and $\alpha^2(Z\alpha)$ to the Coulomb potential for a point charge Ze. The solid lines are obtained by using the expansions (24) and (28). The dash-dotted line represents Vogel's numerical-integration result. The dashed line represents the fitting curve (39), which is indistinguishable from the more accurate curves in the region $0.01\lambda_e\le r\le1.0\lambda_e$. The fitting curve (38) is also indistinguishable from the solid curve R_{11}.

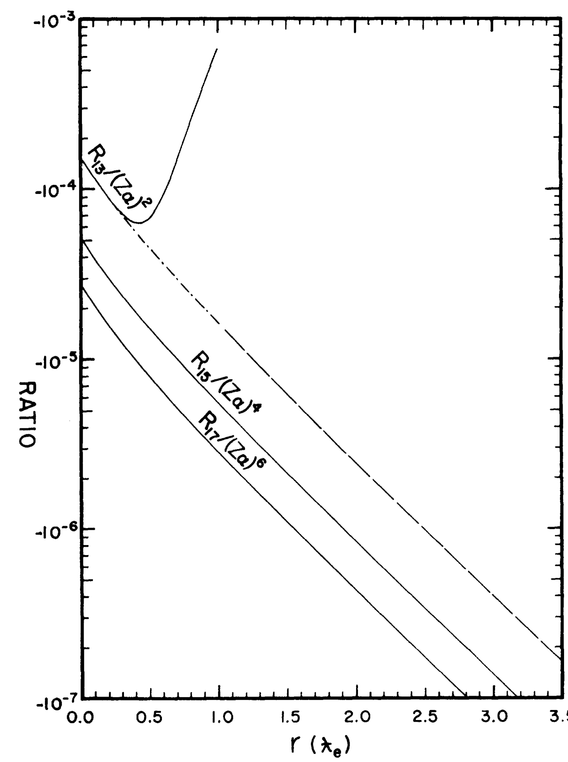

FIG. 2. Ratios of the vacuum-polarization potentials of orders $\alpha(Z\alpha)^3$, $\alpha(Z\alpha)^5$, and $\alpha(Z\alpha)^7$ to the Coulomb potential for a point charge Ze. The solid line $R_{13}/(Z\alpha)^2$ is obtained by using the expansion (32). The dash-dotted line represents Vogel's numerical-integration result. The dashed line represents the fitting curve (40), which is indistinguishable from the more accurate curves in the region $0.01\lambda_e\le r\le1.0\lambda_e$. The solid lines $R_{15}/(Z\alpha)^4$ and $R_{17}/(Z\alpha)^6$ are obtained by scaling.

tentials in this region, these extrapolations, though having nothing to do with the actual potentials, would not introduce very large errors. The minimax rational approximations with the right asymptotic forms, for example,

$$R'_{21}(r)=e^{-2r}\frac{\sum_{i=0}^{n} a_i r^i}{\sum_{i=0}^{n+1} b_i r^i}\,,\quad \text{for } r>0.5\lambda\!\!\!^{-}_e\,, \tag{41}$$

and

$$R'_{13}(r)=\frac{\sum_{i=0}^{n} c_i r^i}{\sum_{i=0}^{n+4} d_i r^i}\,, \tag{42}$$

would probably give good fits and generate better extrapolated potentials. Nevertheless, it would require many more parameters to attain the same accuracies as those of the fitting curves (39) and (40). All the fitting curves are also presented in Figs. 1 and 2.

As long as $Z\alpha \ll 1$, it is reasonable to assume that the expansion of the vacuum-polarization potential in powers of α and $Z\alpha$ is valid, and that the finite-nuclear-size effect is not important for the orders $\alpha^2(Z\alpha)$ and $\alpha(Z\alpha)^n$ with $n \geq 3$. Hence the expansions (3), (26), (30), (34), and (35) will give a rather accurate account of the vacuum-polarization potential. For high-Z nuclei, however, one is compelled to consider contributions from all orders of $\alpha(Z\alpha)^n$ and render a careful calculation of the finite-size effect.[18-21]

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APPENDIX: SERIES EXPANSION OF $\chi_2(z)$

Define

$$\chi_n(z)=\int_1^\infty dt\frac{1}{t^n}\left(1+\frac{1}{2t^2}\right)\left(1-\frac{1}{t^2}\right)^{1/2}e^{-zt}\,. \tag{A1}$$

We will consider only $\chi_2(z)$ here; the results for the general case $\chi_n(z)$ follow similarly. We expand $\chi_2(z)$ in terms of exponential integrals,

$$E_n(z)=\int_1^\infty dt\, t^{-n}e^{-zt}\,, \tag{A2}$$

and get

$$\chi_2(z)=\sum_{k=0}^{\infty} b_k E_{2k+2}(z)\,, \tag{A3}$$

where b_k were given in (10). By using the formula

$$E_n(z)=(-1)^n\frac{z^{n-1}}{(n-1)!}E_1(z)+e^{-z}\sum_{l=0}^{n-2}(-1)^l\frac{(n-l-2)!}{(n-1)!}z^l\,, \tag{A4}$$

we obtain the expansion (5) for $\chi_2(z)$ and its coefficients $C_{2,k}$ and $D_{2,k}$, presented in (8) and (9), respectively.

The sequence $\{C_{2,k}\}$ decreases rapidly and the power series (6) for $f_2(z)$ converges for all z, which can be proved, say, by the ratio test. The convergence of the series (9) for $D_{2,k}$ can be established by Gauss's test. Furthermore, we can see that the sequence $\{D_{2,k}\}$ starting from $k=1$ is alternating and monotonically decreasing. Consequently, for $z\leq 1$, the power series (7) for $g_2(z)$ converges by Leibniz's rule, and the error introduced by a truncated series of $g_2(z)$ is less than the absolute value of the first neglected term. Also, we note that the near equality in magnitude of successive coefficients (i.e., $D_{2,2k}\approx D_{2,2k+1}$) will not lead to numerical unstability, since $D_{2,2k}$ and $D_{2,2k+1}$ decrease very rapidly with increasing k. For $z>1$, we can always find a sufficiently large K such that

$$|D_{2,2m}|>|D_{2,2m+1}(z)|>|D_{2,2m+4}(z)^4| \tag{A5}$$

whenever $2m\geq K$; this may be proved by comparing the general terms in the summation expressions for $D_{2,k}$'s. Hence after the Kth term, we can rearrange the series $g_2(z)$ as

$$(T_{2m}+T_{2m+2})+(T_{2m+1}+T_{2m+3})+(T_{2m+4}+T_{2m+6})+\cdots\,, \tag{A6}$$

where $T_k=D_{2,k}(z)^k$. As a result, the new sequence

$$\{(T_{2m}+T_{2m+2}),(T_{2m+1}+T_{2m+3}),(T_{2m+4}+T_{2m+6}),\ldots\} \tag{A7}$$

is alternating and monotonically decreasing. Therefore $g_2(z)$ also converges for $z>1$, and a truncated series after the Kth term will introduce an error less than the absolute value of the first neglected term (T_i+T_{i+2}). This completes the proof of the convergencies of $f_2(z)$ and $g_2(z)$.

The summation of the series (9), i.e.,

$$D_{2,k}=(-1)^k\sum_{2l\geq k}^{\infty}\frac{(2l-k)!}{(2l+1)!}b_l=(-1)^k\sum_{2l\geq k}^{\infty}\frac{b_l}{(2l+1)(2l)\cdots(2l-k+1)}\,, \tag{A8}$$

for small k, may be carried out first by decomposing $1/(2l+1)(2l)\cdots(2l-k+1)$ into partial fractions. Then each summation involving a partial fraction may be calculated after expressing it as an integral of elementary functions; for example,

$$\sum_{l=m+1}^{\infty} \frac{b_l}{2l-2m-1} = \int_1^{\infty} dz\, z^{2m}\left[\left(1+\frac{1}{2z^2}\right)\left(1-\frac{1}{z^2}\right)^{1/2} - \sum_{l=0}^{m} b_l z^{-2}\right]. \quad \text{(A9)}$$

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