Eikonal approximations in quantum statistical mechanics

H. M. Fried

Service de Physique Théorique, Centre d'Etudes Nucléaires de Saclay, BP No.2-91190 Gif-sur-Yvette, France
and Physics Department, Brown University, Providence, Rhode Island 02912

(Received 1 March 1976)

Eikonal approximations to thermodynamic Green's functions are introduced into the functional representations of quantum statistical mechanics. In terms of summations and quadratures over an instantaneous interparticle potential, one obtains explicit representations at reasonably large temperatures for quantum corrections to classical cluster integrals.

I. INTRODUCTION

The functional formalism of many-body problems in quantum statistical physics has a history as old as that of conventional relativistic field theory. Over the years, in the latter subject there has appeared a collection of techniques and approximations collectively described by the word "eikonal," in which the soft components of real and virtual meson excitations are explicitly summed, producing coherent and important effects in scattering and production amplitudes; the word "soft" here means meson frequency components k small (in some appropriate relativistic sense) compared to the 4-momentum P of a set of distinct scattering particles.

The purpose of these remarks is to attempt a similar analysis for problems of nonrelativistic many-body physics. Here, the word "soft" shall refer to the size of Fourier components k of the transform $\tilde{V}(k)$ of an instantaneous interparticle potential $V(r_1 - r_2)$ compared to the typical thermal momentum P of a particle in the medium. For simplicity, the present discussion is restricted to nonrelativistic situations, although this is not crucial. Eikonal approximations in this context lead to the explicit summation of all such soft degrees of freedom, for appropriate thermodynamic functions; one is able, e.g., to exhibit quantum cluster integrals in terms of summations and quadratures over the soft portion of elementary interparticle potentials. Since soft excitations, in this sense, correspond to long-range potentials, and since the latter are attractive in gases, one might intuitively expect these techniques to have a certain relevance to condensation phenomena; however, at temperatures large enough for these approximations to be valid, one is close to the classical domain, and the sum of all cluster integrals is necessary. To my knowledge the eikonal methods used in this context are new, while only the barest indication of possible application is described in this paper. These methods provide, in the limit of high temperature, an independent derivation of the classical cluster expansion[1]; at finite (but sufficiently large) temperature, one obtains explicit quantum corrections to the classical expressions.

The plan of presentation is as follows: In Sec. II there appears a brief review of basic formalism, wherein the generating functional of the grand canonical partition function is compactly expressed in terms of certain functional operations which are themselves quite familiar from field theory usage. In particular, cluster decomposition properties follow easily from this analysis, as do familiar Hartree-Fock results based upon the simplest of approximations. Section III describes explicit representations for $G_{\rm th}(x, y\,|A)$, the thermal Green's function defined in the presence of an arbitrary source or potential $A(z, z_0)$; it is this material which is at the heart of the paper. Beginning with a standard form for the real-time imaginary-temperature free-particle thermodynamic Green's function $G_{\rm th}^{(0)}(x-y)$, a representation is defined for $G_{\rm th}(A)$ and explicitly obtained in the soft or Block-Nordsieck (BN) approximation. Application of $G_{\rm th}^{\rm BN}[A]$ to the calculation of the cluster coefficients is outlined in Sec. IV, with detailed results written only for the simplest nontrivial cluster integral. Compact expressions for all of the classical cluster integrals are easily obtained. Finally, a brief summary is given in Sec. V.

II. REVIEW OF THE BASIC FORMALISM

A brief review of the basic functional formalism will first be presented to fix the notation and establish continuity with existing BN approximations. Essentially every item in this section was devised years ago in the work of many authors, in particular that of Martin and Schwinger[2] and of Fradkin,[3] which papers contain numerous references to other, original efforts. The functional techniques employed here are quite elementary, were invented years ago by these and other au-

thors, and have been conveniently summarized elsewhere.[4] The functional notation of Ref. 4 will be followed everywhere in this paper.

The essential quantity of interest is the partition function

$$Z[\beta,\mu]=\mathrm{Tr}e^{-\beta(H-\mu N)}, \tag{1}$$

where β and μ are the standard inverse temperature and chemical potential, respectively, while H and N represent the complete (Heisenberg) Hamiltonian and number operator for the system of interest. Attention is here restricted to the nonrelativistic many-body system specified by the Lagrangian density $\mathfrak{L}=\mathfrak{L}_0+\mathfrak{L}'$,

$$\mathfrak{L}_0=i\psi^\dagger\partial_0\psi-(1/2m)\vec{\nabla}\psi^\dagger\cdot\vec{\nabla}\psi, \tag{2}$$

$$\mathfrak{L}'=-\frac{1}{2}\int d^3x\int d^3x'\psi^\dagger(\vec{\mathbf{x}},t)\psi^\dagger(\vec{\mathbf{x}}',t)\times V(\vec{\mathbf{x}}-\vec{\mathbf{x}}')\psi(\vec{\mathbf{x}}',t)\psi(\vec{\mathbf{x}},t), \tag{3}$$

where $V(\vec{\mathbf{r}})$ represents an instantaneous interparticle potential. Standard equal-time commutation relations for bosons and anticommutation relations for fermions will be assumed,

$$[\psi(\vec{\mathbf{x}},t),\psi^\dagger(\vec{\mathbf{x}}',t)]_\mp=\delta(\vec{\mathbf{x}}-\vec{\mathbf{x}}'). \tag{4}$$

The formal calculation of Z may be begun by first rewriting (1) in the equivalent form

$$Z=\sum_{(\alpha)}\langle n_\alpha,0|e^{-\beta\bar{H}}|n_\alpha,0\rangle, \tag{5}$$

where $\bar{H}=H-\mu N$, the index (α) and its summation refers to all states of the system containing n_α particles, and the same real-time ($t=0$) index has been appended to each state. With the imaginary temperature continuation $\beta\to i\tau$, and the identification of the matrix element $\langle b,0|e^{-it\bar{H}}|a,0\rangle$ as the probability amplitude $\langle b,t|a,0\rangle$ for the development of the system described by the Hamiltonian $\bar{H}$, one then has the continued partition function expressed as

$$Z[i\tau,\mu]=\sum_{(\alpha)}\langle n_\alpha,\tau|n_\alpha,0\rangle, \tag{6}$$

which is in the appropriate form to apply elementary action principle considerations.

The system now considered is specified by the Lagrangian density

$$\mathfrak{L}=\mathfrak{L}_0+\mathfrak{L}'+\mu\psi^\dagger\psi+j^*\psi+\psi^\dagger j, \tag{7}$$

where the introduction of the chemical potential leads to a boson system described by $\bar{H}$, and where boson sources $j(\vec{\mathbf{r}},t)$ and $j^*(\vec{\mathbf{r}},t)$ have been introduced in order to provide the basic calculational tools. These boson sources satisfy the commutation relation

$$[\delta/\delta j(x),j(y)]=\delta(x-y)=\delta(\vec{\mathbf{x}}-\vec{\mathbf{y}})\delta(x_0-y_0), \tag{8}$$

and their inclusion in (7) generates the source dependence of the states in (6). If quanta of the ψ field are fermions, (7) is to be replaced by

$$\mathfrak{L}=\mathfrak{L}_0+\mathfrak{L}'+\mu\psi^\dagger\psi+\eta^\dagger\psi+\psi^\dagger\eta, \tag{9}$$

where $\eta,\eta^\dagger$ are anticommuting c-number sources, anticommuting with themselves and with all other fermion quantities,

$$\{\delta/\delta\eta(x),\eta(y)\}=\delta(x-y). \tag{10}$$

Spin indices have been completely suppressed. For convenience of notation, explicit manipulations are performed for the case of bosons, and corresponding fermion results are then appended.

The methods of conventional field theory may now be invoked to obtain (6). One defines a thermodynamic generating functional

$$\mathfrak{z}_{\mathrm{th}}\{j^*,j\}=\sum_{(\alpha)}\langle n_\alpha,\tau|n_\alpha,0\rangle,\quad \mathfrak{z}_{\mathrm{th}}\{0,0\}=Z[i\tau,\mu], \tag{11}$$

and employs the action principle to write

$$\frac{1}{i}\frac{\delta}{\delta j(x)}\mathfrak{z}_{\mathrm{th}}\{j^*,j\}=\sum_{(\alpha)}\langle n_\alpha,\tau|\psi^\dagger(x)|n_\alpha,0\rangle, \tag{12}$$

$$\frac{1}{i}\frac{\delta}{\delta j^*(x)}\mathfrak{z}_{\mathrm{th}}\{j^*,j\}=\sum_{(\alpha)}\langle n_\alpha,\tau|\psi(x)|n_\alpha,0\rangle, \tag{13}$$

where it is understood that the time coordinates x_0 of (12) and (13) must lie in the range $\tau\geq x_0\geq 0$ if the right-hand side of each expression is not to vanish identically. Again, the states of (12) and (13) depend on the sources. Repeated functional differentiation may be carried out, generating a Taylor expansion which serves to demonstrate that

$$\mathfrak{z}_{\mathrm{th}}\{j^*,j\}=\sum_{(\alpha)}\langle n_\alpha,\tau|\left(\exp i\int_0^\tau[j^*\psi+\psi^\dagger j]\right)_+|n_\alpha,0\rangle, \tag{14}$$

with

$$\int_0^\tau d^4x\equiv\int d^3x\int_0^\tau dx_0.$$

The symbol $(\ )_+$ denotes conventional time ordering and the states and operators of (14) are now independent of the sources. The range of integration over all subsequent time coordinates will be understood to be limited, as in (14).

The action principle may be used to calculate $\mathfrak{z}_{\mathrm{th}}\{j^*,j\}$ directly, by replacing $\mathfrak{L}'$ by $g\mathfrak{L}'$, obtaining a differential equation for the g dependence of $\mathfrak{z}_{\mathrm{th}}$, and expressing the latter as a particular functional operation upon $\mathfrak{z}_{\mathrm{th}}^{(0)}\equiv\mathfrak{z}_{\mathrm{th}}|_{g=0}$. Or one may calculate

differential equations for the quantities of (12) and (13) and their solutions, using an alternate but equivalent set of manipulations. Either method quickly leads to[4]

$$\mathfrak{z}_{\rm th}\{j^*,j\}=\exp\left(i\int_0^\tau \mathfrak{L}'\left\{\frac{1}{i}\frac{\delta}{\delta j},\frac{1}{i}\frac{\delta}{\delta j^*}\right\}\right)\times\mathfrak{z}_{\rm th}^{(0)}\{j^*,j\}\,, \tag{15}$$

where g has been replaced by unity and the dependence of $\mathfrak{L}'\{\psi^\dagger,\psi\}$ on its arguments is given by (3). Had fermion fields $\psi^\dagger,\psi$ and sources $\eta,\eta^\dagger$ been involved, (15) would be replaced by

$$\mathfrak{z}_{\rm th}\{\eta^\dagger,\eta\}=\exp\left(i\int_0^\tau \mathfrak{L}'\left\{-\frac{1}{i}\frac{\delta}{\delta\eta},\frac{1}{i}\frac{\delta}{\delta\eta^\dagger}\right\}\right)\times\mathfrak{z}_{\rm th}^{(0)}\{\eta^\dagger,\eta\}\,. \tag{16}$$

Such formal solutions are quite well known, but there is one further point that sould be discussed. The functional operations of (12) and (13) are quite clear as they stand, but subsequent differentiation generates an ambiguity concerning the order of differentiation at equal space-time coordinates. The point is that it is not simply sufficient to replace, in the derivation of (15), thermodynamic averages of the products $\psi^\dagger(x)\psi(x)$ by equivalent functional differentiation

$$\frac{1}{i}\frac{\delta}{\delta j(x)}\,\frac{1}{i}\frac{\delta}{\delta j^*(x)}\,,$$

for the latter operations commute while the operator fields do not. A prescription must be defined, analogous to the use of normal-ordered fields in the Lagrangians of conventional relativistic field theory. In the present case, for the explicit $\mathfrak{L}'$ of (3), the following procedure will be adopted: First, $\mathfrak{L}'$ is rewritten in terms of the combination $\psi^\dagger(x)\psi(x)$, as is always possible using the relations (4); one then rewrites this combination in terms of its symmetric and antisymmetric parts,

$$\psi^\dagger\psi\equiv\tfrac{1}{2}\{\psi^\dagger,\psi\}+\tfrac{1}{2}[\psi^\dagger,\psi]. \tag{17}$$

Only the symmetric part of (17) is identified with the functional differentiation operations $(-i\delta/\delta j)\times(-i\delta/\delta j^*)$. The remainder, $-\frac{1}{2}[\psi,\psi^\dagger]$, is by (4) just the infinite c number $-\frac{1}{2}\delta(\vec{0})$, which may be identified in a variety of ways, e.g., as $-\frac{1}{2}\langle 0|\psi_0\psi_0^\dagger|0\rangle$, or as $\lim_{x\to y\to 0}G_c^{(0)}(x-y)$, where $G_c^{(0)}$ denotes the conventional free-field particle propagator

$$G_c^{(0)}(x-y)=-i\langle 0|(\psi_0(x)\psi_0^\dagger(y))_+|0\rangle. \tag{18}$$

In this nonrelativistic context, one may write

$$G_c^{(0)}(x-y)=-i\,\theta(x_0-y_0)\langle 0|\psi_0(x)\psi_0^\dagger(y)|0\rangle, \tag{19}$$

and use that definition of the step function $\theta(\epsilon)$ (as the limit of a sequence) in which $\theta(0)\equiv\frac{1}{2}$, so that $G_c^{(0)}(0)=-\frac{1}{2}\delta(\vec{0})$.

It is in this sense that (15) is to be understood: using (4) one first rewrites the interaction part of the action operator as

$$\int_0^\tau\mathfrak{L}'=\frac{V(0)}{2}\int_0^\tau d^4x\,\psi^\dagger(x)\psi(x)-\frac{1}{2}\int_0^\tau d^4x\int_0^\tau d^4x'\,\psi^\dagger(x)\psi(x)U(x-x')\times\psi^\dagger(x')\psi(x'), \tag{20}$$

where $U(x)\equiv V(\vec{x})\,\delta(x_0)$, and each equal-space time product $\psi^\dagger\psi$ is then replaced, in the formal solution (15), by $-(\delta/\delta j)\delta/\delta j^*-iG_c^{(0)}(0)$. These extra infinite factors will always cancel in the final, explicit solutions; they are a nuisance to retain throughout a calculation, but it is really necessary for an understanding of their origin.

Similar considerations may be carried through for fermions, where only the antisymmetric part of $\psi^\dagger\psi$ is to be replaced by $(\delta/\delta\eta)\delta/\delta\eta^\dagger$, while the remaining symmetric quantity $\frac{1}{2}\{\psi^\dagger,\psi\}$ is the infinite c number $\frac{1}{2}\langle 0|\{\psi^\dagger,\psi\}|0\rangle=\frac{1}{2}\langle 0|\psi\psi^\dagger|0\rangle$. This last identification is correct only if the vacuum state $|0\rangle$ corresponds to the state of zero fermions, rather than that state representing a completely filled Fermi sea, and for present purposes this shall be assumed. Using the appropriate anticommutation relations, those parts of the action operator analogous to (20) have the same signs as in (20), and it is to be understood that in the formal solution (16) each $\psi^\dagger\psi$ combination is replaced by $(\delta/\delta\eta)\delta/\delta\eta^\dagger+iG_c^{(0)}(0)$.

Before exhibiting a functional construction of the remaining portion of the formal solution (15)

$$\mathfrak{z}_{\rm th}^{(0)}\{j^*,j\}=\exp\left(-i\int j^*G_{\rm th}^{(0)}j\right)\mathfrak{z}_{\rm th}^{(0)}\{0,0\}, \tag{21}$$

where $Z_{\rm th}^{(0)}\{0,0\}=Z_0[i\tau,\mu]$, the free-particle partition function, it is most convenient to extract all μ dependence from these forms and insert it at a later stage. Direct application of the action principle is the simplest way, for it leads to the differential equation

$$\frac{\delta}{\delta\mu}\mathfrak{z}_{\rm th}\{j^*,j\}=i\sum_{(\alpha)}\left\langle n_\alpha,\tau\left|\int_0^\tau d^4x\,\psi^\dagger(x)\psi(x)\right|n_\alpha,0\right\rangle_{j^*,j\neq 0}=-i\int_0^\tau d^4x\left(\frac{\delta}{\delta j(x)}\frac{\delta}{\delta j^*(x)}+iG_c^{(0)}(0)\right)\mathfrak{z}_{\rm th}\{j^*,j\}, \tag{22}$$

with solution

$$\mathfrak{z}_{\rm th}\{j^*,j\} = \exp\left(-i\mu\int_0^\tau \frac{\delta}{\delta j}\frac{\delta}{\delta j^*}\right) \times \exp[\mu\tau\Omega G_c^{(0)}(0)]\,\mathfrak{z}_{\rm th}\{j^*,j\}\big|_{\mu=0}, \quad (23)$$

where Ω represents the total volume $\int d^3x$. Removal of the infinite $G_c(0)$ phase factor will occur automatically later on. For the moment, we just consider the $\mu = 0$ functionals, which will turn out to be related as given by (21),

$$\mathfrak{z}^{(0)}_{{\rm th},\mu=0}\{j^*,j\} = \exp\left(-i\int j^* G^{(0)}_{{\rm th},\mu=0} j\right)\mathfrak{z}^{(0)}_{{\rm th},\mu=0}\{0,0\}, \quad (24)$$

where $\mathfrak{z}^{(0)}_{{\rm th},\mu=0}\{0,0\} = Z_0[i\tau,0]$.

Calculation of $\mathfrak{z}^{(0)}_{{\rm th},\mu=0}\{j^*,j\}$ proceeds most simply from the observation that

$$\mathfrak{z}^{(0)}_{{\rm th},\mu=0}\{j^*,j\} = \sum_{(\alpha)}\left\langle n_\alpha,\tau\left|\left(\exp\left(i\int_0^\tau [j^*\psi_0+\psi_0^\dagger j]\right)\right)_+\right|n_\alpha,0\right\rangle, \quad (25)$$

with all states and operators of (25) represented by free-field quantities. An alternate expression is

$$\mathfrak{z}^{(0)}_{{\rm th},\mu=0}\{j^*,j\} = {\rm Tr}\left[\left(\exp\left(i\int_0^\tau [j^*\psi_0+\psi_0^\dagger j]\right)\right)_+ e^{-\beta H_0}\right], \quad (26)$$

using the continuation $\beta \rightarrow i\tau$, as convenient. The free-field operators satisfy simple differential equations,

$$[i\partial_0 + (1/2m)\nabla^2]\psi(\vec{x},t) = 0,$$

while their corresponding causal (and in this nonrelativistic situation, retarded) propagators satisfy the related inhomogeneous equation, with solution

$$G_c^{(0)}(z) = \int \frac{d^4p}{(2\pi)^4}\frac{e^{ip\cdot z}}{p_0-\epsilon(p)+i\epsilon} = -i\,\theta(z_0)\int \frac{d^3p}{(2\pi)^3}e^{i\vec{p}\cdot\vec{z}-i\epsilon(p)z_0}, \quad (27)$$

with $p\cdot z \equiv \vec{p}\cdot\vec{z} - p_0 z_0$ and $\epsilon(p) = \vec{p}^2/2m$.

Because the field operators of (26) are free, the time-ordered bracket there may be rewritten as the product[4]

$$\left(\exp\left(i\int(j^*\psi_0+\psi_0^\dagger j)\right)\right)_+ = \exp\left(-i\int j^* G_c^{(0)} j\right):\exp\left(i\int(j^*\psi_0+\psi_0^\dagger j)\right):, \quad (28)$$

with all time variables ranging between 0 and τ, viz:

$$\int j^* G_c^{(0)} j \equiv \int_0^\tau d^4x\int_0^\tau d^4y\, j^*(x)G_c^{(0)}(x-y)\,j(y).$$

Extracting this factor, it is convenient to define

$$Q[j^*,j] \equiv \exp\left(+i\int j^* G_c^{(0)} j\right)\mathfrak{z}^{(0)}_{{\rm th},\mu=0}\{j^*,j\} = {\rm Tr}\left[\exp\left(i\int\psi_0^\dagger j\right)\exp\left(i\int j^*\psi_0\right)e^{-\beta H_0}\right], \quad (29)$$

and to determine the differential equations satisfied by Q. The quantity

$$\frac{1}{i}\frac{\delta}{\delta j^*(x)}Q = {\rm Tr}\left[\exp\left(i\int\psi_0^{(\dagger)}j\right)\psi_0(x)\exp\left(i\int j^*\psi_0\right)e^{-\beta H_0}\right]$$

may be transformed by passing the $\psi_0(x)$ operator through the factor $\exp(i\int\psi_0^{(\dagger)}j)$, using the cyclic property of the trace and the time-translation property of H_0, to yield

$$[1-\exp(i\beta\partial_{x_0})]\frac{1}{i}\frac{\delta}{\delta j^*(x)}Q = \left(\int G^{(+)}(x-u)j(u)\right)Q, \quad (30)$$

where

$$G^{(+)}(x-y) = i[\psi_0^\dagger(y),\psi_0(x)] = -i(2\pi)^{-3}\int d^3p\, e^{ip\cdot(x-y)}.$$

Equation (30) [together with a related equation for $\delta Q/\delta j(x)$] has the immediate solution

$$Q[j^*,j] = Q[0,0]\exp\left(-i\int j^*(u)[\exp(i\beta\partial_{u_0})-1]^{-1}\times G^{(+)}(u-v)j(v)\right),$$

so that

$$\mathfrak{z}^{(0)}_{{\rm th},\mu=0}\{j^*,j\} = Q[0,0]\times\exp\left(-i\int_0^\tau d^4u\int_0^\tau d^4v\, j^*(u)\times G^{(0)}_{{\rm th},\mu=0}(u-v)j(v)\right), \quad (31)$$

with

$$G^{(0)}_{{\rm th},\mu=0}(z) = -i\int\frac{d^3p}{(2\pi)^3}e^{i\vec{p}\cdot\vec{z}-i\epsilon(p)z_0}\times\{\theta(z_0)+1/(e^{\beta\epsilon(p)}-1)\}, \quad (32)$$

a well-known result[5] in which the nonground-state or finite-temperature part of this thermal Green's function has Fourier components on the energy shell only.[6]

Finally, we may pass to the $\mu \neq 0$ forms by calculating the interaction-free version of (23),

$$\mathfrak{z}_{\rm th}^{(0)}\{j^*,j\} = e^{\mu\tau\Omega G_c^{(0)}(0)} \exp\left(-i\mu \int \frac{\delta}{\delta j}\,\frac{\delta}{\delta j^*}\right)\exp\left(-i\int j^* G^{(0)}_{{\rm th},\mu=0} j\right) Q[0,0]$$

$$= Q[0,0] e^{\mu\tau\Omega G_c^{(0)}(0)} \exp\left(-i\int j^* G^{(0)}_{\rm th} j\right) \exp[-\,{\rm Tr}\ln(1+\mu G^{(0)}_{{\rm th},\mu=0})], \tag{33}$$

using the well-known Gaussian combinatoric[4]

$$\exp\left(-i\int \frac{\delta}{\delta j} A \frac{\delta}{\delta j^*}\right)\exp\left(i\int j^* B j\right)$$

$$= \exp\left(i\int j^* B(1-AB)^{-1} j\right)\exp[-\,{\rm Tr}\ln(1-AB)].$$

Here, the Green's function $G^{(0)}_{\rm th} \equiv G^{(0)}_{{\rm th},\mu=0}[1 + \mu G^{(0)}_{{\rm th},\mu=0}]^{-1}$ satisfies the same differential equation as does $G^{(0)}_{{\rm th},\mu=0}$, except that the latter's $\epsilon(p)$ is everywhere replaced by $\epsilon(p) - \mu \equiv Q(p)$,

$$G^{(0)}_{\rm th}(z) = -i\int \frac{d^3p}{(2\pi)^3}\, e^{ip\cdot z - iQ(p)z_0} \times\{\theta(z_0) + 1/(e^{\beta Q(p)} - 1)\}. \tag{34}$$

The determinantal factor of (33) is

$$-{\rm Tr}\int_0^\mu d\mu'\, G^{(0)}_{{\rm th},\mu=0}[1+\mu' G^{(0)}_{{\rm th},\mu=0}]^{-1}$$

$$= -\int_0^\tau d^4x \int_0^\mu d\mu'\, G_{{\rm th},\mu'}(x-x), \tag{35}$$

explicitly indicating, in (35), the μ dependence in the Green's function of (34) over which one is integrating. Equation (35) splits into two terms,

$$-\tau\Omega G_c^{(0)}(0) + i\tau\Omega\int \frac{d^3p}{(2\pi)^3}\int_0^\mu d\mu'(e^{\beta(\epsilon(p)-\mu')} - 1)^{-1},$$

the first of which just cancels the infinite phase of (33), while the second yields the difference

$$i\tau\Omega\int \frac{d^3p}{(2\pi)^3}\left\{-\frac{1}{\beta}\ln(1-e^{-\beta(\epsilon-\mu)}) + \frac{1}{\beta}\ln(1-e^{-\beta\epsilon})\right\}.$$

Comparing normalization factors in (31) and (33), there follows the identification

$$\ln Z_0[\beta,\mu] = \Omega\int \frac{d^3p}{(2\pi)^3}\ln(1-e^{-\beta(\epsilon(p)-\mu)})^{-1},$$

the familiar ideal-Bose-gas result. Combining (15) and (33), one obtains the generic result

$$\frac{Z[\beta,\mu]}{Z_0[\beta,\mu]} = \exp\left(i\int \mathfrak{L}'\left\{\frac{1}{i}\,\frac{\delta}{\delta j}, \frac{1}{i}\,\frac{\delta}{\delta j^*}\right\}\right) \times \exp\left(-i\int j^* G^{(0)}_{\rm th} j\right)\Bigg|_{\substack{j,j^*\to 0\\ \tau\to -i\beta\mu}}. \tag{36}$$

For the particular interaction (20), this ratio may be written as

$$\exp\left[-\frac{i}{2}\int\left(\frac{\delta}{\delta j}\,\frac{\delta}{\delta j^*}\right) U\left(\frac{\delta}{\delta j}\,\frac{\delta}{\delta j^*}\right)\right] \times \exp\left[-\frac{i}{2}V(0)\int\left(\frac{\delta}{\delta j}\,\frac{\delta}{\delta j^*}\right)\right] \times \exp\left(-i\int j^* G^{(0)}_{\rm th} j\right)\Bigg|_{j,j^*\to 0}, \tag{37}$$

if only one remembers to replace each $(\delta/\delta j)\delta/\delta j^*$ combination in (37) by $(\delta/\delta j)\delta/\delta j^* + iG_c^{(0)}(0)$. An immediate example of the cancellation of the associated phase factors is provided by the action of the last operator of (37)

$$\exp\left(-\frac{i}{2}V(0)\int \frac{\delta}{\delta j}\,\frac{\delta}{\delta j^*}\right)\exp\left(-i\int j^* G^{(0)}_{\rm th} j\right)$$

$$= \exp\left(-i\int j^* G^{(0)}_{\rm th}[\mu + \tfrac{1}{2}V(0)] j\right) \times \exp[{\rm Tr}\ln(1+\tfrac{1}{2}V(0)G^{(0)}_{\rm th})]. \tag{38}$$

Here the Green's function denoted by $G^{(0)}_{\rm th}[\mu + \frac{1}{2}V(0)]$ is just that of (34) with μ replaced by $\mu + \frac{1}{2}V(0)$, while the trace-log term again splits into

$$-{\rm Tr}\ln(1+\tfrac{1}{2}V(0)G^{(0)}_{\rm th}) = -\tfrac{1}{2}V(0)\int_0^1 d\lambda \int_0^\tau d^4x\, G^{(0)}_{\rm th}[x-x;\ \mu+\tfrac{1}{2}V(0)]$$

$$= -\tfrac{1}{2}V(0)\tau\Omega G_c^{(0)} + \frac{i}{2}V(0)\tau\Omega\int\frac{d^3p}{(2\pi)^3}\int_0^1 d\lambda (e^{\beta[\epsilon(p)-\mu-\lambda/2V(0)]} - 1)^{-1}. \tag{39}$$

The first term of (39) cancels against the $+iG_c^{(0)}(0)$ factor, which should have been included on the left-hand side of (38); and elementary integration of the remaining part of (39) just produces the difference

$$\ln Z_0[i\tau, \mu+\tfrac{1}{2}V(0)] - \ln Z_0[i\tau, \mu],$$

leading to

$$\frac{Z[i\tau, \mu]}{Z_0[i\tau, \mu+\frac{1}{2}V(0)]} = \exp\left[-\frac{i}{2}\int\left(\frac{\delta}{\delta j}\frac{\delta}{\delta j^*}\right)U\left(\frac{\delta}{\delta j}\frac{\delta}{\delta j^*}\right)\right] \times \exp\left(-i\int j^* G_{\rm th}^{(0)}[\mu+\tfrac{1}{2}V(0)]j\right)\Big|_{j,j^*\to 0}. \quad (40)$$

It is most convenient to shift μ in (40) by the amount $-\frac{1}{2}V(0)$, and so calculate

$$\frac{Z[i\tau, \mu-\frac{1}{2}V(0)]}{Z_0[i\tau, \mu]} = \exp\left[-\frac{i}{2}\int\left(\frac{\delta}{\delta j}\frac{\delta}{\delta j^*}\right)U\left(\frac{\delta}{\delta j}\frac{\delta}{\delta j^*}\right)\right] \times \exp\left(-i\int j^* G_{\rm th}^{(0)} j\right)\Big|_0, \quad (41)$$

where the thermodynamic Green's function appearing in (41) is that of (34). A subsequent shift or renormalization of the chemical potential, $\mu \to \mu+\frac{1}{2}V(0)$, is to be made at the end of any computation. The $+iG_c^{(0)}(0)$ additions to the combinations $(\delta/\delta j)/(\delta/\delta j^*)$ still remain to be performed.

As a final preliminary, (41) may be rewritten using the method of equivalent boson excitations,[7]

$$\frac{Z[i\tau, \mu-\frac{1}{2}V(0)]}{Z_0[i\tau, \mu]} = \exp\left(\frac{i}{2}\int\frac{\delta}{\delta A}U\frac{\delta}{\delta A}\right) \times\exp\left(i\int\frac{\delta}{\delta j}A\frac{\delta}{\delta j^*}\right) \times \exp\left(-i\int j^* G_{\rm th}^{(0)} j\right)\Big|_{\substack{j,j^*\to 0\\ A\to 0}}. \quad (42)$$

Carrying through the j, j^* operations first yields

$$\frac{Z[i\tau, \mu-\frac{1}{2}V(0)]}{Z_0[i\tau, \mu]} = \exp\left(\frac{i}{2}\int\frac{\delta}{\delta A}U\frac{\delta}{\delta A}\right) \times\exp[-\mathrm{Tr}\ln(1-AG_{\rm th}^{(0)})]\big|_{A\to 0}. \quad (43)$$

With the exception of an infinite quantity linear in A, contained in the Tr ln term of (43) and canceling against the omitted $iG_c^{(0)}(0)$ dependence of (41), Eq. (43) represents the essential formal solution to this problem. One may explicitly show the necessary cancellation by defining the boson loop function

$$L_{\rm th}^{(b)}[A] = -\mathrm{Tr}\ln(1-AG_{\rm th}^{(0)}) - G_c^{(0)}(0)\int_0^\tau A,$$

or

$$L_{\rm th}^{(b)}[A] = \int_0^1 d\lambda \int_0^\tau d^4x A(x)[G_{\rm th}(x, x|\lambda A) - G_c^{(b)}(0)], \quad (44)$$

and so obtain the divergence-free result

$$\frac{Z[i\tau, \mu-\frac{1}{2}V(0)]}{Z_0[i\tau, \mu]} = \exp\left(\frac{i}{2}\int\frac{\delta}{\delta A}U\frac{\delta}{\delta A}\right)e^{L_{\rm th}^{(b)}[A]}\big|_{A\to 0}. \quad (45)$$

For spin-$\frac{1}{2}$ fermions, an exactly analogous discussion goes through, yielding a relation of form (45), except that the fermion loop function is now given by

$$L_{\rm th}^{(f)}[A] = \mathrm{Tr}\ln(1+AG_{{\rm th},f}^{(0)}) - 2G_c^{(0)}(0)\int_0^\tau A, \quad (46)$$

where $G_{{\rm th},f}^{(0)}$ includes dependence on spin indices.

In a conventional, relativistic-field-theory calculation of the sum of all vacuum-to-vacuum amplitudes, $L_{\rm th}[A]$ would represent the sum of all closed loops over arbitrary numbers of source insertions A, while the effects of radiative corrections are given by the "linkage" operations induced by the functional differential operator. Statistically, the combinatoric situation is the same; the only differences are that the loops are built out of thermodynamic Green's functions, rather than causal propagators, and that each "radiative correction" here corresponds to a potential interaction between a pair of particles, or the same particle, in the medium.

Many applications within this class of statistical problems may be obtained by suitable modeling of (44) or (46). The heart of any such method is the appropriate approximation of

$$G_{\rm th}(x, y|A) = \langle x|G_{\rm th}^{(0)}[1 - AG_{\rm th}^{(0)}]^{-1}|y\rangle, \quad (47)$$

the thermal Green's function defined in the presence of an external field $A(z)$. The material of Sec. III represents one attempt to do this for a particular physical situation.

Results of the so-called Hartree-Fock approximation are easily obtained from this formalism, by retaining only linear and quadratic A dependence of $L_{\rm th}[A]$. For example, if in the boson situation one approximates $L_{\rm th}[A]$ by $L_{\rm th}^{(1)}[A] + L_{\rm th}^{(2)}[A]$, where

$$L_{\rm th}^{(1)}[A] = -i\bar{N}\int_0^\tau A, \quad L_{\rm th}^{(2)}[A] = \frac{i}{2}\int_0^\tau\int_0^\tau A K_{\rm th}^{(2)} A,$$

with

$$\bar{N} \equiv \frac{1}{\Omega}\int\frac{d^3p}{(2\pi)^3}(e^{\beta(\epsilon(p)-\mu)}-1)^{-1},$$

$$K_{\rm th}^{(2)}(x, y) = -iG_{\rm th}^{(0)}(x-y)G_{\rm th}^{(0)}(y-x), \quad (48)$$

the functional operations of (45) correspond to the summation of all tadpoles and simple bubbles,

$$\frac{Z[i\tau,\mu-\frac{1}{2}V(0)]}{Z_0[i\tau,\mu]} \simeq \exp\left(\frac{i}{2}\int \frac{\delta}{\delta A}U\frac{\delta}{\delta A}\right)\exp\left(-i\bar{N}\int A+\frac{i}{2}\int AK_{\rm th}^{(2)}A\right)\Big|_{A\to 0}$$

$$=\exp[-\tfrac{1}{2}\mathrm{Tr}\ln(1+UK_{\rm th}^{(2)})]\exp\left(-\frac{i}{2}\bar{N}^2\int\int U(1+K_{\rm th}^{(2)}U)^{-1}\right), \tag{49}$$

again using the Gaussian combinatoric. With a real-time imaginary-temperature Fourier-series representation for $G_{\rm th}^{(0)}(z)$, given in Sec. III, it is not difficult to show that the logarithm of (48) may be rewritten as

$$-\frac{\Omega}{2}\sum_{n=-\infty}^{+\infty}\int\frac{d^3p}{(2\pi)^3}\ln[1+K_n(p)\tilde{V}(p)]$$

$$-\frac{\beta}{2}\Omega\bar{N}^2\tilde{V}(0)\left(1+\tilde{V}(0)\frac{\partial}{\partial\mu}\bar{N}\right)^{-1}, \tag{50}$$

where

$$K_n(p)=\frac{1}{i\tau}\int\frac{d^3q}{(2\pi)^3}\sum_{m=-\infty}^{+\infty}[\omega_m-Q(p-q)]^{-1}$$

$$\times[\omega_m+\omega_n-Q(q)]^{-1},$$

and $\omega l\equiv 2\pi l/\tau$. The first term of (50) represents the complete sum of "ring diagrams" (pictured on the book jacket of Ref. 5), while the second may be interpreted as a kind of "unitarized" Van der Waals potential. These results, quoted here for a boson system, are the essence of the Hartree-Fock approximation.

More generally, it is a straightforward matter to employ the "connectedness property" of field theory in order to display the cluster decomposition properties possessed by the logarithm of (45). With

$$\mathfrak{D}_A\equiv+\frac{i}{2}\int\frac{\delta}{\delta A}U\frac{\delta}{\delta A},$$

one may write[4]

$$C_{\rm th}[\beta,\mu-\tfrac{1}{2}V(0)]\equiv\ln\left[\frac{Z[i\tau,\mu-\frac{1}{2}V(0)]}{Z_0[i\tau,\mu]}\right]$$

$$=e^{\mathfrak{D}_A}e^{L_{\rm th}(A)}\Big|_{A\to 0}^{\rm conn}-1, \tag{51}$$

where the subcript "conn" is an instruction to retain only the connected parts of all terms, in particular those that arise in the expansion of the $L_{\rm th}[A]$ exponential. If the symbol $\bar{L}_{\rm th}[A]$ is used to represent that closed loop which contains all "self-linkages,"

$$\bar{L}_{\rm th}(A)=e^{\mathfrak{D}_A}L_{\rm th}[A],$$

then one can write

$$C_{\rm th}[\beta,\mu-\tfrac{1}{2}V(0)]=\sum_{n=1}^{\infty}C_n,$$

where

$$C_n=\frac{1}{n!}\exp\left(i\sum_{l>m=1}^{n}\int\frac{\delta}{\delta A_l}U\frac{\delta}{\delta A_m}\right)$$

$$\times\bar{L}_{\rm th}[A_1]\cdots\bar{L}_{\rm th}[A_n]\Big|_{A_i\to 0}^{\rm conn}. \tag{52}$$

The operations shown in (52) refer to connected linkages between different loops, and provide a functional representation of the quantum cluster integrals. In Sec. IV a calculation of C_2 will be performed by summing over all appropriate soft linkages, and it will then be apparent that the $T\to\infty$ and/or $\hbar\to 0$ limit of that quantity is just the simplest classical cluster integral. In fact, the entire sum of classical cluster integrals follows easily from (52). From an operational point of view, however, the only possibility of actually performing the functional operations of (52) is restricted to those situations where $\bar{L}_{\rm th}[A]$ is not more complicated that the exponential of a linear or quadratic functional of A; and it is precisely the simpler of these possibilities that appears when the restriction to soft linkages is made.

III. REPRESENTATION OF $G_{\rm th}[A]$

In this section, a fairly conventional Fourier-series representation is first given for $G_{\rm th}^{(0)}$, the free-particle thermodynamic propagator. This is followed by a similar representation for $G_{\rm th}[A]$, which suggests an appropriate version of the no-recoil BN approximation.

A. Fourier representation of $G_{\rm th}^{(0)}$

Because the time coordinates of $G_{\rm th}^{(0)}$ are limited to the range $0-\tau$, with a familiar periodicity over the range τ, it is useful to introduce a Fourier-series representation for these variables. Following Matsubara,[8] we introduce the complete set $\langle x_0|n\rangle=\tau^{-1/2}e^{-i\omega_n x_0}$, and expand

$$G_{\rm th}^{(0)}(\vec{x},\vec{y};x_0,y_0)=\sum_{n,l=-\infty}^{\infty}\langle x_0|n\rangle G_{\rm th}^{(0)}(\vec{x},\vec{y}|n,l)$$

$$\times\langle l|y_0\rangle. \tag{53}$$

Of course, the fact that $G_{\rm th}^{(0)}$ depends on coordinate differences simplifies matters considerably, and one easily inverts (53) to obtain a representation

for $G_{\rm th}^{(0)}(\vec{\mathbf{z}}|n,l)$, or for (53) itself,

$$G_{\rm th}^{(0)}(z)=-\frac{i}{\tau}\sum_n\int\frac{d^3p}{(2\pi)^3}\,e^{i\vec{\mathbf{p}}\cdot\vec{\mathbf{z}}-i\omega_n z_0}G_n(p),$$

with

$$G_n(p)=[\omega_n-\epsilon(p)+\mu+i\eta]^{-1}. \tag{54}$$

The periodicity condition $G_{\rm th}^{(0)}(\vec{\mathbf{z}},z_0)|_{z_0<0}=G_{\rm th}^{(0)}(\vec{\mathbf{z}},z_0+\tau)|_{z_0+\tau>0}$ follows directly from (34), and has as its consequence the restriction $\omega_n=2\pi n/\tau$, with $n=0,\pm1,\pm2,\ldots$. With contour integration around an infinite number of poles lying on the real axis, the validity of (54) is easily verified; in fact, $G_n(p)^{-1}$ can have either a positive or negative imaginary part. The use of this representation in (48) and (49) leads directly to (50).

B. Exact representation for $G_{\rm th}[A]$

The central object of concern in this functional formalism is the thermodynamic Green's function in the presence of a source $A(z)$. If A is independent of time, $G_{\rm th}[A]$ may be written down immediately, by repeating every step of the calculation leading to (31) and (34), just as one found (for the boson case) (32), or

$$G_{\rm th}^{(0)}(x-y)=G_c^{(0)}(x-y)+[\exp(i\beta\partial_{x_0})-1]^{-1}\times[G_c^{(0)}(x-y)-G_{\bar{c}}^{(0)}(x-y)], \tag{55}$$

where G_c and $G_{\bar{c}}$ represent causal and anticausal (or, in this nonrelativistic situation, retarded and advanced) propagators; thus one may construct

$$G_{\rm th}(x,y|A)=G_c(x,y)A+[\exp(i\beta\partial_{x_0})-1]^{-1}\times[G_c(x,y|A)-G_{\bar{c}}(x,y|A)], \tag{56}$$

if only $A(z)=A(\vec{\mathbf{z}})$. If A depends on z_0, however, (56) is wrong and there does not appear to be any simple way of relating $G_{\rm th}[A]$ to $G_c[A]$ and $G_{\bar{c}}[A]$. This is discussed further in the Appendix.

The relation (47) satisfied by $G_{\rm th}[A]$ is similar to that satisfied by $G_c[A]$ (except for the time-coordinate restrictions),

$$G_{\rm th}(x,y|A)=G_{\rm th}^{(0)}(x-y)+\int_0^\tau d^4z\,G_{\rm th}^{(0)}(x-z)A(z)G_{\rm th}(z,y|A), \tag{57}$$

where the limitation on the range of the dummy z_0 variable, following the derivation leading to (43), is explicitly shown in (57); it is understood, also, that the x_0, y_0 variables are to exist in the same range. We now introduce a double Fourier representation, analogous to that of (53),

$$G_{\rm th}(x,y|A)=\sum_{n,l}\langle x_0|n\rangle\langle\vec{\mathbf{x}},n|\,G_{\rm th}[A]\,|\vec{\mathbf{y}},l\rangle\langle l|y_0\rangle, \tag{58}$$

using an obvious notation. With $\langle\vec{\mathbf{x}}|\vec{\mathbf{p}}\rangle=(2\pi)^{-3/2}\times e^{i\vec{\mathbf{p}}\cdot\vec{\mathbf{x}}}$, (57) becomes

$$\langle\vec{\mathbf{p}}_1,n|\,G_{\rm th}[A]\,|\vec{\mathbf{p}}_2,l\rangle=\delta_{nl}\langle\vec{\mathbf{p}}_1|\vec{\mathbf{p}}_2\rangle G_n(p_1)+\sum_m\int\frac{d^3Q}{(2\pi)^3}\,G_n(p_1)\tilde{A}_{n-m}(\vec{\mathbf{p}}_1-\vec{\mathbf{Q}})\langle\vec{\mathbf{Q}},m|G_{\rm th}[A]|\vec{\mathbf{p}}_2,l\rangle, \tag{59}$$

where

$$\tilde{A}_l(q)=\frac{1}{\tau}\int_0^\tau dz_0\int\frac{d^3z}{(2\pi)^3}\,e^{-i\vec{\mathbf{q}}\cdot\vec{\mathbf{z}}+i\omega_l z_0}A(\vec{\mathbf{z}},z_0).$$

In momentum space, $G_{\rm th}^{(0)}$ is diagonal, while A is Hermitian. Equation (59) is a nontrivial integral equation, which one must solve or approximate in some tractable way.

Previous experience with eikonal approximations suggests a particular form of representation for

$$\langle\vec{\mathbf{p}}_1,n|G_{\rm th}[A]\,|\vec{\mathbf{p}}_2,l\rangle=\langle\vec{\mathbf{p}}_1,n|\,[(G_{\rm th}^{(0)})^{-1}-A\,]^{-1}|\vec{\mathbf{p}}_2,l\rangle\,. \tag{60}$$

Since $G_{\rm th}^{(0)}$ is diagonal, so is $(G_{\rm th}^{(0)})^{-1}$,

$$\langle\vec{\mathbf{p}},n|\,(G_{\rm th}^{(0)})^{-1}|\vec{\mathbf{p}}',l\rangle=\langle\vec{\mathbf{p}}|\vec{\mathbf{p}}'\rangle\delta_{nl}(G_n(p))^{-1},$$

where, as in (54), we choose the small positive imaginary term, and write the representation

$$\langle\vec{\mathbf{p}}_1,n|\,G_{\rm th}[A]\,|\vec{\mathbf{p}}_2,l\rangle=-i\langle\vec{\mathbf{p}}_1,n|\int_0^\infty ds\,e^{is[(G_{\rm th}^{(0)})^{-1}-A]}|\vec{\mathbf{p}}_2,l\rangle=-i\int_0^\infty ds\,e^{isG_n^{-1}(p_1)}\langle\vec{\mathbf{p}}_1,n|\,\mathfrak{F}(s)|\vec{\mathbf{p}}_2,l\rangle\,, \tag{61}$$

where

$$\mathfrak{F}(s)\equiv e^{-is(G_{\rm th}^{(0)})^{-1}}e^{is[(G_{\rm th}^{(0)})^{-1}-A]}. \tag{62}$$

It is slightly more convenient to multiply through by $\sum_n\sum_{p_2}\langle\vec{\mathbf{p}}_2,n|\vec{\mathbf{y}},y_0\rangle$ and consider

$$\langle \vec{p}, n|G_{\rm th}[A]|\vec{y}, y_0\rangle = -i\int_0^\infty ds\, e^{isG_n^{-1}(p)}\langle \vec{p}, n|\,\mathfrak{F}(s)|\vec{y}, y_0\rangle. \tag{63}$$

In order to determine $\mathfrak{F}(s)$, one constructs the equation

$$\frac{\partial}{\partial s}\langle \vec{p}, n|\,\mathfrak{F}(s)|\,y\rangle = -i\langle \vec{p}, n|\,e^{-isG_n^{-1}(p)}Ae^{is(G_{\rm th}^{(0)})^{-1}}\mathfrak{F}(s)|\,y\rangle$$

$$= -i\sum_l\sum_{\vec{k}}\tilde{A}_l(\vec{k})\exp\left[-is\left(\omega_l+\frac{\vec{k}^2}{2m}-\frac{\vec{p}\cdot\vec{k}}{m}\right)\right]\langle \vec{p}-\vec{k}, n-l|\,\mathfrak{F}(s)|\,y\rangle, \tag{64}$$

obtained by inserting a complete set of $(\vec{k}, l)$ states. It is useful to guarantee proper normalization at $s=0$ by defining

$$\langle \vec{p}, n|\,\mathfrak{F}(s)|\,y\rangle \equiv \langle \vec{p}|\,\vec{y}\rangle\langle n|\,y_0\rangle f(s;\vec{p}, n; y), \tag{65}$$

and so obtain the corresponding equation for f,

$$\frac{\partial f}{\partial s}(s;\vec{p}, n; y) = -i\sum_{\vec{k},l}\tilde{A}_l(\vec{k})\exp\left[i\vec{k}\cdot\vec{y}-i\omega_l y_0-is\left(\omega_l+\frac{\vec{k}^2}{2m}-\frac{\vec{p}\cdot\vec{k}}{m}\right)\right] f(s;\vec{p}-\vec{k}, n-l; y), \tag{66}$$

with $f(0;\vec{p}, n; y)=1$. So far, all is exact.

C. Eikonal approximation

We now define the soft excitation, or eikonal approximation relevant to this statistical situation, by recalling that the source dependence A in $L_{\rm th}[A]$ is subsequently to be acted upon by the operator

$$\frac{i}{2}\int\frac{\delta}{\delta A}U\frac{\delta}{\delta A} = \frac{i}{2\tau}\sum_n\int\frac{d^3k}{(2\pi)^3}\frac{\delta}{\delta\tilde{A}_n(\vec{k})}\tilde{V}(\vec{k})\frac{\delta}{\delta\tilde{A}_{-n}(-\vec{k})},$$

where

$$\frac{\delta}{\delta\tilde{A}_n(k)} \equiv \int d^3x\, e^{i\vec{k}\cdot\vec{x}}\int_0^\tau dx_0\, e^{-i\omega_n x_0}\frac{\delta}{\delta A(\vec{x}, x_0)}.$$

If one is interested in an approximation where the effects of all soft k are correctly extracted—and by soft one means $|\vec{k}|<|\vec{p}|$, where $\vec{p}$ represents a typical thermal momentum of a particle in the medium—then it will be permissible to neglect in (66) all $\vec{k}$ compared to $\vec{p}$, for the particle momenta $\vec{p}$ appearing in these combinations are always to be subsequently averaged over typical thermal distributions. Thus (66) is replaced by

$$\frac{\partial}{\partial s}f_{\rm BN}(s;\vec{p}, n; y)$$

$$= -i\sum_l\sum_{\vec{k}}\tilde{A}_l(\vec{k})\exp\left[i\vec{k}\cdot\left(\vec{y}+s\frac{\vec{p}}{m}\right)-i\omega_l(y_0+s)\right]$$

$$\times f_{\rm BN}(s;\vec{p}, n-l; y)$$

$$= -i\sum_l A_l\left(\vec{y}+s\frac{\vec{p}}{m}\right)e^{-i\omega_l(y_0+s)}f_{\rm BN}(s;\vec{p};n-l;y). \tag{67}$$

Incorporating the $s=0$ boundary condition, (67) may be written

$$f_{\rm BN}(s;\vec{p}, n; y)$$

$$= 1-i\int_0^s ds'\sum_l A_l\left(\vec{y}+s'\frac{\vec{p}}{m}\right)e^{-i\omega_l(y_0+s')}$$

$$\times f_{\rm BN}(s';\vec{p}, n-l; y), \tag{68}$$

with the iterative solution

$$f_{\rm BN} = 1-i\int_0^s ds'\, B(y+s'v(p))$$

$$+(-i)^2\int_0^s ds'\, B(y+s'v(p))$$

$$\times\int_0^{s'} ds''B(y+s''v(p))+\cdots$$

or

$$f_{\rm BN}(s;\vec{p}, n; y) = \exp\left(-i\int_0^s ds'\, B(y+s'v(p))\right), \tag{69}$$

where $v_\mu(p)=(\vec{p}/m, 1)$ and

$$B(z) = \sum_l A_l(\vec{z})e^{-i\omega_l z_0}.$$

Note that $B(z)\equiv A(z)$ only if $|z_0|<\tau$; and that $f_{\rm BN}$ is independent of n.

Substitution of (69) into (63), or its configuration-space equivalent, then yields the soft-excitation thermodynamic propagator

$$G_{\rm th}^{\rm BN}(x, y|A) = -\frac{i}{\tau}\sum_n \int \frac{d^3p}{(2\pi)^3} e^{i\vec{p}\cdot(\vec{x}-\vec{y})-i\omega_n(x_0-y_0)} \int_0^\infty ds\, e^{is[\omega_n - Q(p)]} \exp\left(-i\int_0^s ds'\, B(y+s'v(p))\right), \tag{70}$$

where again $Q(p)\equiv\epsilon(p)-\mu$; and in which the typically simple eikonal source dependence, the exponential of a linear form, again appears. The A dependence may be made more explicit by rewriting the s integration, using $s=l\tau+\vartheta$, with $0<\vartheta<\tau$, and summing over all positive integers l,

$$\int_0^\infty ds\, F(s) = \sum_{l=0}^\infty \int_0^\tau d\vartheta\, F(l\tau+\vartheta),$$

$$G_{\rm th}^{\rm BN}(x, y|A) = -\frac{i}{\tau}\int \frac{d^3p}{(2\pi)^3} e^{i\vec{p}\cdot(\vec{x}-\vec{y})} \sum_{l=0}^\infty \int_0^\tau d\vartheta \sum_n e^{-i\omega_n(x_0-y_0-\vartheta)} e^{-iQ(p)(l\tau+\vartheta)} f_{\rm BN}(l\tau+\vartheta). \tag{71}$$

If $z_0 = x_0 - y_0 > 0$, then $\tau > z_0$ and $|z_0-\vartheta|<\tau$, so that the closure sum over the states n may be performed,

$$\sum_n e^{-i\omega_n(z_0-\vartheta)} = \tau\delta(z_0-\vartheta).$$

If $z_0<0$, $|\vartheta - z_0 - \tau|<\tau$, and one must use

$$\sum_n e^{-i\omega_n(z_0+\tau-\vartheta)} = \tau\delta(z_0+\tau-\vartheta).$$

In this way,

$$G_{\rm th}^{\rm BN}(x, y|A) = -i\int \frac{d^3p}{(2\pi)^3} e^{i\vec{p}\cdot\vec{z}} \sum_{l=0}^\infty e^{-il\tau Q}\left[\theta(z_0)e^{-iQz_0} f_{\rm BN}(l\tau+z_0) + \theta(-z_0)e^{-iQ(\tau+z_0)} f_{\rm BN}((l+1)\tau+z_0)\right]. \tag{72}$$

Rewriting $\theta(-z_0)$ in (72) as $1-\theta(z_0)$ and combining terms, one obtains

$$G_{\rm th}^{\rm BN}(x, y|A) = G_c^{(0)}(x-y) f_{\rm BN}(z_0) - i\int \frac{d^3p}{(2\pi)^3} e^{i\vec{p}\cdot\vec{z}-iQz_0} \sum_{l=0}^\infty e^{-i(l+1)\tau Q} f_{\rm BN}((l+1)\tau+z_0), \tag{73}$$

where for $z_0>0$

$$\ln f_{\rm BN}((l+1)\tau+z_0) = -i\sum_\lambda \int \frac{d^3k}{(2\pi)^{3/2}} \bar{A}_n(\vec{k}) e^{i\vec{k}\cdot\vec{y}-i\omega_n y_0}\left(\sum_{m=0}^l \int_0^\tau d\vartheta\, e^{i(\vec{k}\cdot\vec{v}-\omega_n)(m\tau+\vartheta)} + \int_0^{z_0} d\vartheta\, e^{i(\vec{k}\cdot\vec{v}-\omega_n)((l+1)\tau+\vartheta)}\right). \tag{74}$$

Note that in the limit $A\to 0$, (73) reproduces the free thermodynamic propagator $G_{\rm th}^{(0)}$, while for constant A one obtains $G_{\rm th}^{(0)}$ with a shifted chemical potential.

The terms in large parentheses in Eq. (74) are just

$$R_l(\vec{k}\cdot\vec{v})\int_0^\tau d\vartheta\, e^{i\vartheta(\vec{k}\cdot\vec{v}-\omega_n)} + e^{i\tau(l+1)\vec{k}\cdot\vec{v}}\int_0^{z_0} d\vartheta\, e^{i\vartheta(\vec{k}\cdot\vec{v}-\omega_n)},$$

with

$$R_l(\vec{k}\cdot\vec{v}) = (1-e^{i\tau(l+1)\vec{k}\cdot\vec{v}})(1-e^{i\tau\vec{k}\cdot\vec{v}})^{-1};$$

for $z_0\to 0$, the situation of subsequent usage,

$$\Delta G_{\rm th}^{\rm BN}(x, x|A) \equiv G_{\rm th}^{\rm BN}(x, x|A) - G_c^{(0)}(0) = -i\int \frac{d^3p}{(2\pi)^3} \sum_{l=0}^\infty e^{-i(l+1)\tau Q(p)} \exp\left(-i\sum_n \int \frac{d^3k}{(2\pi)^{3/2}} \bar{A}_n(\vec{k}) f_{x,l}(\vec{k}, n)\right), \tag{75}$$

where

$$f_{x,l}(\vec{k}, n) = R_l(\vec{k}\cdot\vec{v}) e^{i(\vec{k}\cdot\vec{x}-\omega_n x_0)} \int_0^\tau d\vartheta\, e^{i\vartheta(\vec{k}\cdot\vec{v}-\omega_n)}.$$

From (44), one sees that the combination $\Delta G_{\rm th}^{\rm BN}$ is precisely the function that enters into the finite-loop expansion; for this boson case,

$$L_{\rm th}^{\rm BN}[A]=\int_0^1 d\lambda \int_0^\tau d^4x\, A(x)\Delta G_{\rm th}^{\rm BN}(x,x|\lambda A). \quad (76)$$

Substitution of (76) into (45) shows that the momentum argument of $\tilde{A}_n(\vec{k})$ in (75) is to be set equal (in magnitude) to the $\vec{k}$ coordinate of $\vec{V}(\vec{k})$; if the restriction $|k|<|p|$ is kept, the corresponding condition in (75) is that $(l+1)(\beta/m)|\vec{k}\cdot\vec{p}|<1$, or $R_l(\vec{k}\cdot\vec{v})\sim l+1$. This permits a slight simplification of (75),

$$\Delta G_{\rm th}^{\rm BN}(x,x|A)=-i\int\frac{d^3p}{(2\pi)^3}(e^{i\tau[\mathcal{Q}(p)+\mathcal{A}]}-1)^{-1}, \quad (77)$$

where

$$\mathcal{A}=\frac{1}{\tau}\int_0^\tau d\vartheta\sum_n\int\frac{d^3k}{(2\pi)^{3/2}}\tilde{A}_n(\vec{k})\times e^{i(\vec{k}\cdot\vec{x}-\omega_n x_0)}e^{i\vartheta(\vec{k}\cdot\vec{v}-\omega_n)},$$

or

$$\mathcal{A}=\frac{1}{\tau}\int_0^\tau d\vartheta\, A(\vec{x}+\vartheta\vec{v}(p),x_0+\vartheta-\tau).$$

IV. EIKONAL APPROXIMATIONS TO CLUSTER INTEGRALS

Examination of the representations (75) and (76), and the way in which (52) indicates that they are to be employed, shows that the functional operations defining the nth cluster integral can, in the eikonal approximation, be carried through. After first exhibiting the complete classical cluster expansion as a special limiting case, we shall calculate in detail C_1 and C_2, although there is little difficulty in extending the analysis to arbitrary C_n.

A. Classical cluster expansion

As a preliminary computation, we first extract the complete partition function in the classical limit, where the only factors of $\hbar$ permitted appear in the definition of the available phase space and later enter into the definition of thermodynamic wavelength $\lambda\!\!\!^{-}=(2\pi\hbar^2/mKT)^{1/2}$; the classical (CL) expansion is given in ascending powers of $\zeta/\lambda\!\!\!^{-3}$, where ζ represents the fugacity, $\zeta=e^{\beta\mu}$. This limit may be reached from the approximate BN functions of Sec. III by retaining only the $l=0$ Maxwell-Boltzmann-like contributions of (75), and by discarding the $i\vartheta\vec{k}\cdot\vec{v}(p)$ term in the definition there of $f_{x,l}(\vec{k},n)$:

$$\Delta G_{\rm th}^{\rm BN}(x,x|A)\to -i\int\frac{d^3p}{(2\pi)^3}e^{-i\tau\mathcal{Q}(p)}\times\exp\left(-i\sum_n\int\frac{d^3k}{(2\pi)^{3/2}}\tilde{A}_n(\vec{k})\times f_{x,0}^{\rm CL}(\vec{k},n)\right), \quad (78)$$

with

$$f_{x,0}^{\rm CL}(\vec{k},n)=e^{i(\vec{k}\cdot\vec{x}-\omega_n x_0)}\int_0^\tau d\vartheta\, e^{-i\vartheta\omega_n} \quad (79)$$

The reason for the latter approximation is that such terms always correspond to lengths $\hbar\vartheta\vec{v}\sim O(\hbar\beta\vec{v})\sim O(\lambda)$, which are then combined with dummy variable length parameters $\vec{x}$ themselves of range of the potential; and these $\hbar$ factors may be discarded in the classical limit. (Alternatively, they can be dropped in the high-temperature limit.) In fact, remembering that corrections to the BN approximation always involve[9] further shifts of these configuration coordinates by amounts proportional to $\hbar$, one sees that (78) and (79) provide a simple representation for the classical thermodynamic Green's function. A glance at (79) shows that it reduces to $\tau\delta_{n,0}e^{i\vec{k}\cdot\vec{x}}$, so that

$$\Delta G_{\rm th}^{\rm CL}(x,x|A)=-i\int\frac{d^3p}{(2\pi)^3}e^{-\beta\mathcal{Q}(p)}e^{-\beta A_0(\vec{x})}=-i\left(\frac{\rho}{\lambda\!\!\!^{-3}}\right)e^{-\beta A_0(\vec{x})}, \quad (80)$$

exhibiting the dependence of $\Delta G_{\rm th}^{\rm CL}$ upon

$$A_0(\vec{x})=\frac{1}{\tau}\int_0^\tau dx_0\, A(\vec{x},x_0)$$

only. The corresponding closed-loop functional (76) is then

$$L_{\rm th}^{\rm CL}[A]=-\frac{\beta\rho}{\lambda\!\!\!^{-3}}\int d^3x\, A_0(\vec{x})\int_0^1 d\lambda\, e^{-\beta\lambda A_0(\vec{x})}=\left(\frac{\rho}{\lambda\!\!\!^{-3}}\right)\int_0^1 d\lambda\frac{\partial}{\partial\lambda}\int d^3x\, e^{-\beta A_0(\vec{x})}, \quad (81)$$

and it is interesting to note that in this classical limit $L_{\rm th}$ has the eikonal form of a four-dimensional problem.

Substitution of (81) into (82) then generates all of the classical cluster integrals directly, after the necessary replacement of μ by $\mu+\frac{1}{2}V(0)$. Note that only the $n=0$ dependence of the linkage operators is now relevant,

$$\mathfrak{D}_A\to\frac{i}{2\tau}\int\frac{d^3k}{(2\pi)^3}\frac{\delta}{\delta\tilde{A}_0(\vec{k})}\tilde{V}(k)\frac{\delta}{\delta\tilde{A}_0(-\vec{k})}=-\frac{1}{2\beta}\int d^3x\int d^3y\frac{\delta}{\delta A_0(\vec{x})}V(\vec{x}-\vec{y})\frac{\delta}{\delta A_0(\vec{y})},$$

while the non-self-linkage operator to be used in (52) is just

$$\exp\left(-\frac{1}{\beta}\sum_{l>m=1}^{n}\int d^3x_l\int d^3x_m\frac{\delta}{\delta A_l(\vec{x}_l)}\times V(\vec{x}_l-\vec{x}_m)\frac{\delta}{\delta A_m(\vec{x}_m)}\right),$$

with the now-useless subscript 0 [of $A_0(\vec{x})$] suppressed.

We first calculate C_1^{CL}, involving self-linkages only:

$$C_1^{CL}(\beta, \mu - \tfrac{1}{2}V(0)) = \left(\frac{\rho}{\lambda^3}\right)\int_0^1 d\lambda \frac{\partial}{\partial\lambda}\int d^3x\, e^{-\lambda^2\beta V(0)/2},$$

or

$$C_1^{CL}(\beta, \mu) = (\Omega\rho/\lambda^3)(1 - e^{-\beta V(0)/2}).$$

Since (51) generates

$$C_{th}(\beta, \mu) = \ln\left(\frac{Z(\beta, \mu)}{Z_0(\beta, \mu)}\right) + \ln\left(\frac{Z_0(\beta, \mu)}{Z_0(\beta, \mu + \frac{1}{2}V(0))}\right), \quad (82)$$

and the second term of (82) may be recognized in this classical limit as $C_1^{CL}(\beta, \mu)$, there follows

$$\ln\left(\frac{Z^{CL}(\beta, \mu)}{Z_0^{CL}(\beta, \mu)}\right) = \sum_{n=2}^{\infty} C_n^{CL}(\beta, \mu). \quad (83)$$

The only effect of C_1^{CL} is to provide the necessary shift of chemical potential. More generally, nontrivial quantum corrections are contained in C_1, such as the so-called exchange terms corresponding to the simplest particle self-energy graph and its generalizations.

The simplest nontrivial classical cluster integral is obtained from

$$C_2^{CL}(\beta, \mu - \tfrac{1}{2}V(0)) = \frac{1}{2!}\exp\left(-\frac{1}{2\beta}\int \frac{\delta}{\delta A} V \frac{\delta}{\delta A}\right)\left(\frac{\zeta}{\lambda^3}\right)^2 \int d^3x_1 \int d^3x_2 \int_0^1 d\lambda_1 \frac{\partial}{\partial\lambda_1}\int_0^1 d\lambda_2 \frac{\partial}{\partial\lambda_2}$$

$$\times e^{-\lambda_1\beta A(\vec{x}_1)} e^{-\lambda_2\beta A(\vec{x}_2)}\Big|_{0,\,conn}$$

$$= \frac{1}{2!}\left(\frac{\zeta}{\lambda^3}\right)^2 \Omega \int d^3x_{12}\int_0^1 d\lambda_1 \frac{\partial}{\partial\lambda_1}\int_0^1 d\lambda_2 \frac{\partial}{\partial\lambda_2} e^{-\lambda_1^2\beta V(0)/2} e^{-\lambda_2^2\beta V(0)/2} e^{-\lambda_1\lambda_2\beta V(\vec{x}_{12})}\Big|_{conn}$$

$$= \frac{1}{2!}\left(\frac{\zeta}{\lambda^3}\right)^2 \Omega \int d^3x_{12}[e^{-\beta V(0)}(e^{-\beta V(\vec{x}_{12})} - 1) + (e^{-\beta V(0)/2} - 1)^2]\Big|_{conn}, \quad (84)$$

writing out each step explicitly. But that last term of (84) containing no $V(\vec{x}_1 - \vec{x}_2)$ dependence is disconnected and must be discarded, yielding the canonical result

$$C_2^{CL}(\beta, \mu) = \frac{1}{2!}\left(\frac{\zeta}{\lambda^3}\right)^2 \Omega \int d^3x_{12} f_{12}, \quad (85)$$

with $f_{12} \equiv e^{-\beta V(\vec{x}_1 - \vec{x}_2)} - 1$.

In this way, one may calculate all of the higher classical cluster integrals, $C_l^{CL}(\beta, \mu) = (1/l!)(\zeta/\lambda^3)^l \times b_l$, finding that the proper chemical potential renormalization always multiplies the totally connected parts, which themselves may be expressed using the pleasantly compact representation

$$b_l = \int d^3x_1 \cdots \int d^3x_l \int_0^1 d\lambda_1 \frac{\partial}{\partial\lambda_1} \cdots \int_0^1 d\lambda_l \frac{\partial}{\partial\lambda_l} \exp\left(-\beta \sum_{i>j=1}^{l} \lambda_i\lambda_j V(\vec{x}_i - \vec{x}_j)\right). \quad (86)$$

For example, from (86) there immediately follows

$$b_3 = \int d^3x_1 \int d^3x_2 \int d^3x_3 (f_{12}f_{23}f_{31} + 3f_{12}f_{23}).$$

B. Quantum cluster integrals

We now consider the eikonal approximation to the quantum cluster integrals describing methods which permit the calculation of all C_n. Somewhat different formulations of this approximation are possible, however, generating somewhat different results, and it is useful to distinguish these situations.

The simple substitution of (76) into (52), with the understanding that each potential $\tilde{V}(\vec{k})$ is to be replaced by an appropriate $\tilde{V}_s(\vec{k})$, where all frequency components k are treated as soft, $k<p$, is the obvious way of approximating all of the C_n. One here imagines that the fundamental relations (45)–(52) are rewritten with $\tilde{V} \equiv \tilde{V}_s + \tilde{V}_H$, and an expansion is then performed in powers of $\tilde{V}_H$, so that the leading terms in the eikonal expansion have effectively V replaced by $\tilde{V}_s$. A typical and convenient choice, previously used in a nonthermodynamic eikonal formulation of potential scattering,[9] is

$$\tilde{V}(k) = e^{-a\vec{k}^2}\tilde{V}(k) + (1 - e^{-a\vec{k}^2})\tilde{V}(k)$$

$$\equiv \tilde{V}_s(k) + \tilde{V}_H(k), \quad (87)$$

with the constant a here represented by $\langle p^2\rangle^{-1} \sim \beta/m$. For all reasonable potentials $\tilde{V}$, only soft frequency components effectively appear in $\tilde{V}_s$, with the converse true for $\tilde{V}_H$.

It is interesting to note that the Fourier transform of $\tilde{V}_s$ corresponds to a potential

$$V_s(\vec{r}) = (\pi\lambda^2)^{-3/2} \int d^3z\, V(\vec{z})\, e^{-\lambda^{-2}(\vec{r}-\vec{z})^2}, \tag{88}$$

if one uses $a = \frac{1}{2}\pi\beta/m$. This has the qualitative features

$$V_s(\vec{r}) \sim V(\vec{r}), \quad r \gg \lambda\!\!\!^{-}, \tag{89}$$

$$V_s(\vec{r}) \sim \langle V\rangle_\lambda, \quad r \ll \lambda\!\!\!^{-}, \tag{90}$$

where the average of (90) is performed essentially over the volume of radius $\lambda\!\!\!^{-}$. A physical interpretation of this eikonal approximation, with its replacement of V by V_s, then follows. The statistical output of the method should be reasonable if V and V_s do not differ in essential form. In the typical case of strong short-range repulsion for $r < r_0$, and weak long-range attraction for $r > r_0$, this situation is realized for sufficiently large temperatures, such that $r_0 \gg \lambda\!\!\!^{-}$. For actual substances, this is essentially the classical regime, in which quantum corrections may be expected to be small. Hence the output of this eikonal approximation is terms which differ by small quantum corrections—although not in the form of a perturbation expansion in $\hbar$—from the classical results. As pointed out by Lieb,[10] "small" can mean up to 30%. For lower temperatures, the expansion of cluster coefficients in powers of $\tilde{V}_H$ presumably becomes important.

Another method of calculation, for technical reasons somewhat simpler, is to begin with the exact forms (52), write a simple parametric differential equation for each C_n, and introduce the approximate BN forms only at a later stage. This method effectively generates a modified perturbation expansion, wherein a certain finite number of potential interactions have both hard and soft components, while all of the remaining potential interactions are soft; essentially, one takes a simple skeleton diagram, built out of a few $V(k)$ exchanges, and laces it with an additional, infinite number of soft V_s exchanges. This latter method will be used to compute C_2, and to sketch the construction of C_3, while the more direct substitution operation will be used for C_1. Simple and obvious generalizations exist for the calculation of all higher C_n.

The direct substituting of (76) into the $n=1$ term of (52) requires the evaluation of the one functional operation simpler than Gaussian, and characteristic of each eikonal computation,

$$\exp\left(\frac{i}{2}\int \frac{\delta}{\delta A} V \frac{\delta}{\delta A}\right)\exp\left(i\int fA\right) = \exp\left(i\int fA - \frac{i}{2}\int fVf\right).$$

In the present case, one readily obtains

$$C_1[i\tau, \mu - \tfrac{1}{2}V(0)] = -i\int \frac{d^3p}{(2\pi)^3}\int_0^\tau d^4x \int_0^1 d\lambda \sum_n \int \frac{d^3q}{(2\pi)^3} e^{i\vec{q}\cdot\vec{x} - i\omega_n x_0}\tilde{V}(-\vec{q}) \times \sum_{l=0}^{\infty} e^{-i\tau(l+1)Q(p)}\lambda f_{x,l}(-\vec{q}, -n)\exp\left(-\frac{i\lambda^2}{2\tau}\sum_m \int \frac{d^3k}{(2\pi)^3} f_{x,l}(\vec{k}, m)\tilde{V}(k) \times f_{x,l}(-\vec{k}, -m)\right),$$

or

$$C_1[\beta, \mu - \tfrac{1}{2}V(0)] = -\beta\int \frac{d^3p}{(2\pi)^3}\,\Omega \sum_{l=0}^{\infty}\int_0^1 \lambda\, d\lambda\, e^{-\beta(l+1)Q(p)} \int \frac{d^3q}{(2\pi)^3}\tilde{V}(q) R_l(\vec{q}\cdot\vec{v}) \times \exp\left(-\frac{\beta\lambda^2}{2}\int \frac{d^3k}{(2\pi)^3}\tilde{V}(k) R_l(\vec{k}\cdot\vec{v}) R_l(-\vec{k}\cdot\vec{v})\right), \tag{91}$$

where again $\vec{v} = \vec{v}(p) = \vec{p}/m$. Each potential $\tilde{V}$ entering into (91) is to be replaced by $\tilde{V}_s$, in accordance with the above discussion, as in the $\frac{1}{2}V(0)$ terms used in the subsequent "renormalization" of the chemical potential. If each $R_l(\vec{k}\cdot\vec{v})$ is simplified by the replacement leading from (75) to (76), one finds

$$C_1[\beta, \mu] \simeq -\Omega\int \frac{d^3p}{(2\pi)^3}\sum_{l=0}^{\infty}\frac{e^{-\beta(l+1)[\epsilon(p)-\mu]}}{l+1} \times e^{+\beta(l+1)V(0)/2} \times (1 - e^{-\beta(l+1)^2 V(0)/2}).$$

From (51) and the definition of $Z_0[\beta,\mu]$,

$$\ln Z[\beta,\mu]=\ln Z_0[\beta,\mu+\tfrac{1}{2}V(0)]+\sum_{n=1}^{\infty}C_n[\beta,\mu],$$

which may, in this approximation, be written

$$\ln Z[\beta,\mu]=\sum_{n=2}^{\infty}C_n[\beta,\mu]+C_1'[\beta,\mu],$$

with

$$C_1'[\beta,\mu]=\Omega\int\frac{d^3p}{(2\pi)^3}\sum_{l=0}^{\infty}\frac{e^{-\beta(l+1)Q(p)}}{l+1}\times e^{-\beta l(l+1)V(0)/2}. \tag{92}$$

Only the $l=0$ term, the Maxwell-Boltzmann (MB) contribution of (92), is independent of $V(0)$; for higher l, there is a residual $V(0)$ dependence.

At first glance, this latter result is perhaps surprising, for one might have expected the $V(0)$ dependence to cancel for every l, and C_1' to reproduce simply the complete quantum-mechanical (in this case, boson) $\ln Z_0$; this would be the case if the $V(0)$ terms of (92) were neglected. Appearing here, however, are residual "self-energy" effects, which are interwoven into the specifically quantum portion of these distributions. Such quantities correspond to an incoherent self-energy averaging process: if, as in the method of interaction, any particle may emit "excitations" which have a statistical probability of being absorbed by other particles in the medium, there is no reason why such excitations cannot be reabsorbed by the same particle. Of course, one here is not referring to any self-energy effect of the specifically causal ($T=0$) part of the thermodynamic propagator, since that was long ago removed in passing to the definition (44).

Taking such statistical self-effects seriously has an interesting consequence for the particular case of hard-core potentials, where $V(0)$ is large and positive. [It is always possible to choose a short-range repulsive potential which is constrained to vanish at the origin, although one then runs the risk of introducing spurious bound states. In this case, however, $V(0)\to V_s(0)$, which, by (90), does not vanish.] Specifically, one sees that C_1' receives a sizable contribution only from the MB $l=0$ term, with all higher-l (quantum symmetry) terms vanishing exponentially as $V(0)$ increases. Similar effects will be seen to occur for the remaining C_n. That the "exchange" terms represented by the quantum portion of C_1' vanish in the limit $V(0)\to\infty$ is perhaps reasonable in view of the work of Lieb,[10] who showed that these contributions to the second virial coefficient are bounded by an exponentially decreasing function of T, and hence may be neglected in any numerical computation.

The second method, corresponding to a modified perturbation expansion, will now be illustrated for C_2. Without any approximation, from (52) one has

$$C_2[\lambda U']\equiv C_2^{(\lambda)}[\beta,\mu-\tfrac{1}{2}V(0)]=\frac{1}{2!}\exp\left(i\lambda\int\frac{\delta}{\delta A_1}U'\frac{\delta}{\delta A_2}\right)\times\bar{L}_{\rm th}[A_1]\bar{L}_{\rm th}[A_2]\Big|_{0,\,\rm conn}, \tag{93}$$

where a prime has been appended to the U (cross-linkage) term explicitly appearing in (93), and a factor of λ inserted as well in order to generate a differential equation; at the end of the computation, $U'=U$ and $\lambda=1$. One observes that

$$\frac{\delta C_2[\lambda U']}{\delta U'(z)}=\frac{i\lambda}{2}\int_0^\tau d^4x\int_0^\tau d^4y\,\delta(x-y-z)\times\exp\left(i\lambda\int\frac{\delta}{\delta A_1}U'\frac{\delta}{\delta A_2}\right)\times\frac{\delta\bar{L}_{\rm th}[A_1]}{\delta A_1(x)}\frac{\delta\bar{L}_{\rm th}[A_2]}{\delta A_2(y)}\Big|_{0,\,\rm conn},$$

and hence that the relation

$$\lambda\frac{\partial C_2[\lambda U']}{\partial\lambda}=\int_0^\tau d^4z\,U'(z)\frac{\delta C_2[\lambda U']}{\delta U'(z)} \tag{94}$$

may be used to construct C_2,

$$C_2[U]=\int_0^1\frac{d\lambda}{\lambda}\int_0^\tau d^4z\,U'(z)\frac{\delta C_2[\lambda U']}{\delta U'(z)}\Big|_{U'=U}, \tag{95}$$

where the BN forms are inserted into the forms of (95),

$$\delta L_{\rm th}[A]/\delta A(x)=\Delta G_{\rm th}(x,x|A)\to\Delta G_{\rm th}^{\rm BN}(x,x|A).$$

The self-energy terms are now independent of the λ integration, and one finds

$$C_2[U]=-\beta\Omega\int_0^1 d\lambda\int\frac{d^3p_1}{(2\pi)^3}\sum_{l_1=0}^{\infty}\exp[-\beta(l_1+1)Q(p_1)-\tfrac{1}{2}\beta V_{l_1}^2]\int\frac{d^3p_2}{(2\pi)^3}\sum_{l_2=0}^{\infty}\exp[-\beta(l_2+1)Q(p_2)-\tfrac{1}{2}\beta V_{l_2}^2]\times\int d^3z\,V(z)\exp\left(-\beta\lambda\int\frac{d^3k}{(2\pi)^3}\tilde{V}_s(k)\int_0^1 d\xi\,e^{i\vec{k}\cdot[\vec{z}+\xi r(\vec{p}_1-\vec{p}_2)/m]}R_{l_1}(\vec{k}\cdot\vec{v}_1)R_{l_2}(-\vec{k}\cdot\vec{v}_2)\right), \tag{96}$$

where

$$V_{l^2} = \int \frac{d^3k}{(2\pi)^3} \tilde{V}(\vec{k}) R_l(\vec{k}\cdot\vec{v}) R_l(-\vec{k}\cdot\vec{v}) .$$

Because of the explicit factor of $U'(z) = V'(z)\,\delta(z_0) \to V(z)\,\delta(z_0)$ in (95), which contains both hard and soft components, the time coordinates of both Green's functions in $\partial C_2/\partial\lambda$ are the same; and it is this feature which permits some technical simplifications alluded to previously. Renormalizing the chemical potential, and retaining only the MB $l_1 = l_2 = 0$ terms of (96), one obtains (explicitly inserting $\hbar$ factors)

$$C_2[\beta,\mu] \simeq -\beta\Omega\xi^2 \int \frac{d^3p_1}{(2\pi\hbar)^3} e^{-\beta\epsilon(p_1)} \int \frac{d^3p_2}{(2\pi\hbar)^3} e^{-\beta\epsilon(p_2)} \int_0^1 d\lambda \int d^3z\, V(z) \exp\left(-\beta\lambda \int_0^1 d\xi\, V_s(\vec{z} + \xi\tau\hbar(\vec{p}_1 - \vec{p}_2)/m)\right) \tag{97}$$

as the simplest eikonal approximation to the second cluster integral. In the limit $T \to \infty$ (or $\hbar \to 0$ everywhere, except in the phase-space normalization), one immediately recovers the classical expression (85). Equation (97) should provide a good representation of the second virial integral for a hard-sphere gas near its classical limit.

This second technique may be easily to extended to the construction of all of the higher C_n, although the actual computations become more tedious as n increases. For example, to construct that approximation to C_3 which involves two $\tilde{V}$ factors containing both hard and soft components plus all possible soft $\tilde{V}_s$ interactions, one may proceed from the definition (suppressing chemical potential renormalization)

$$C_3[U_{12}, U_{23}, U_{31}] = \exp\left(i \int \frac{\delta}{\delta A_1} U_{12} \frac{\delta}{\delta A_2} + i \int \frac{\delta}{\delta A_2} U_{23} \frac{\delta}{\delta A_3} + i \int \frac{\delta}{\delta A_3} U_{31} \frac{\delta}{\delta A_1}\right) \bar{L}_{\rm th}[A_1]\,\bar{L}_{\rm th}[A_2]\,\bar{L}_{\rm th}[A_3]\Big|_{0,\,\rm conn} ,$$

and build the quantity

$$D_3[\lambda U_{12}, \mu U_{23}, U_{31}] \equiv \int d^4z\, U_{12}(z) \int d^4w U_{23}(w) \frac{\delta^2 C_3[\lambda U_{12}, \mu U_{23}, U_{31}]}{\delta U_{12}(z)\delta U_{23}(w)} = \lambda \frac{\partial}{\partial\lambda}\, \mu \frac{\partial}{\partial\mu}\, C_3[\lambda U_{12}, \mu U_{23}, U_{31}] . \tag{98}$$

Integrating the λ, μ dependence of (98), and retaining only connected terms, one finds

$$C_3[U,U,U] = \int_0^1 \frac{d\lambda}{\lambda} \int_0^1 \frac{d\mu}{\mu} (D_3[\lambda U, \mu U, U] + 2D_3[\lambda U, \mu U, 0]) . \tag{99}$$

Substitution of the BN forms (75) into (99) then generates the stated approximation.

V. SUMMARY

The main thrust of this paper has been to develop a method of eikonal approximation to the functional equations and representations of quantum statistical mechanics. By essentially kinematic restrictions on the supposed relevance of relatively low-frequency excitations introduced into the basic Green's functions of the theory, one is able to provide explicit representations for quantum corrections to the classical cluster integrals in that semiclassical domain where such corrections are anticipated to be small. Possible extension of these techniques to phase-transition phenomena remains an open question. It is surely a tantalizing question, for much recent work[11] on the latter topics applies approximation methods of functional integration to the explicit extraction of low-frequency effects. Other approximations to the functional formalism used here are of course possible; e.g., the application of special kinematical restrictions, different from eikonal, to the basic $G_c[A]$ of (57), followed by the functional operations of (45), can be used to study associated statistical models.

ACKNOWLEDGMENTS

It is a pleasure to thank J. D. Walecka for extending the hospitality of the Physics Department, Stanford University, where this paper was begun; and to thank the staff of the Service de Physique Théorique, Centre d'Etudes Nucléaires de Saclay, for their hospitality, where it was completed.

APPENDIX

Equation (56) of the text will no longer be valid if A depends on z_0, and no simple method of relating $G_{\rm th}[A]$ to $G_c[A]$ and $G_{\bar c}[A]$ presents itself. One possible conjecture, generating $G_{\rm th}[A]$ in terms of $G_c[A]$ and $G_{\bar c}[A]$ in the nonrelativistic situation (or in terms of the four independent $G_\alpha[A]$, with $\alpha = c, \bar c, R, A$, in the relativistic case) is suggested by an alternate way of building the BN Green's function (75).

Define

$$G'_{\rm th}(x, y|A) = G_c(x, y|A) + [\exp(-\tau\partial_{x_0}) - 1]^{-1} \times [G_c(x, y|A) - G_{\bar c}(x, y|A)],$$

and ask under what circumstances $G'_{\rm th}(x, y|A)$ and $G_{\rm th}(x, y|A)$ can satisfy the appropriate differential equation, that of $G_c[A]$, in either variable. The crucial question of boundary conditions is suppressed for the next few sentences.

Expand the operator $[\exp(-\tau\partial_{x_0}) - 1]^{-1}$ as

$$\sum_{l=0}^{\infty} (-1)^l \exp[-\tau(l+1)\partial_{x_0}],$$

and test to see if $G'_{\rm th}$ satisfies the differential equation from the left-hand side,

$$[i\partial_{x_0} + \nabla_x^2/2m - A(x)]G'_{\rm th}(x, y|A) = \delta^{(4)}(x - y).$$

Clearly, this will be true only if $A(x, x_0)$ is independent of x_0, or if it is periodic in x_0 with period τ, $A(\vec{\mathbf{x}}, x_0 \pm \tau) = A(\vec{\mathbf{x}}, x_0)$. This latter case is quite outside the spirit of the analysis leading to solutions of (57), where the range of the time coordinate of A lies between 0 and τ. Here, one places no restriction on that variable, nor indeed none (yet) upon those of $G_c[A]$ and $G_{\bar c}[A]$.

With $A(x)$ chosen periodic in x_0, one has a solution of the necessary equation to which boundary conditions must be appended, as well as choosing the particular integer N, in any quantity $A(\vec{\mathbf{z}}, z_0 \pm N\tau)$, buried in the A dependence of $G_c[A]$ and $G_{\bar c}[A]$. A prescription without further justification can be found for the special case of BN solutions in nonrelativistic field theory,

$$G_{\mathfrak{C}}^{\rm BN}(x, y|A) = \mp i \int_0^\infty ds \int \frac{d^4p}{(2\pi)^4} e^{ip\cdot(x-y)} \exp\left(\pm is(p_0 - \epsilon(p)) \mp i \int_0^s ds' A(y \pm s'v(p)) \right),$$

where $\mathfrak{C} = c$ ($\bar c$) for the upper (lower) sign. The prescription is as follows: Write $G'_{\rm th}$, as above, in terms of $G_c^{\rm BN}[A]$ and $G_{\bar c}^{\rm BN}[A]$; replace the parametric s integrals, using

$$\int_0^\infty ds\, F(s) = \sum_{l=0}^{\infty} \int_0^\tau d\theta\, F(l\tau + \theta),$$

and transform the time coordinate of A, using the assumed periodicity, so that this total argument satisfies $|z_0| \leq \tau$. The resulting expression for $\Delta G'_{\rm th}[A]$ is then identical to the $\Delta G_{\rm th}^{\rm BN}[A]$ of (75).

It would be most interesting to know if there exists a more general method for the construction of $G_{\rm th}[A]$ in terms of the $G_\alpha[A]$; surely the conjecture that works for the BN situation—choose A periodic, and adjust its time coordinate to be $\leq \tau$—is too simple to be true in the general case. Knowledge of such a prescription could be most useful for the construction of statistical models based upon soluble or semisoluble models of field theory.

[1] H. D. Ursell, Proc. Camb. Philos. Soc. 23, (1927), 685; J. E. Mayer and M. G. Mayer, *Statistical Mechanics* (Wiley, New York, 1940); T. W. Janssen, Physica (Utr.) Suppl. 26, 575 (1960); C. Bloch and C. de Dominicis, Nucl. Phys. 10 (1959), 181; R. A. Smith, Ann. Phys. (N.Y.) 81 (1973), 402.

[2] P. C. Martin and J. Schwinger, Phys. Rev. 115, 1342 (1959).

[3] E. S. Fradkin, Zh. Eksp. Teor. Fiz. 36, 951 (1959) [Sov. Phys.-JETP 9, 912 (1959)].

[4] H. M. Fried, *Functional Methods and Models in Quantum Field Theory* (MIT, Cambridge, Mass., 1972).

[5] A. Fetter and J. D. Walecka, *Quantum Theory of Many-Particle Systems* (McGraw-Hill, New York, 1971).

[6] This point has been emphasized recently by L. Dolan and R. Jackiw, Phys. Rev. D 9, 3337 (1974).

[7] I. Bialynicki-Birula, J. Math. Phys. 3, 1094 (1962). That Gaussian combinatoric leading from (42) to (43) is discussed in Ref. 4.

[8] T. Matsubara, Prog. Theor. Phys. 14, 351 (1955).

[9] H. M. Fried and Y. Hahn, Phys. Rev. D 10, 1908 (1974).

[10] E. H. Lieb, J. Math. Phys. 8, 43 (1967).

[11] K. G. Wilson and J. Kogut, Phys. Rep. 12, 75 (1974).