Parametric instabilities below the electron plasma frequency due to relativistic corrections

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We present a model which describes the interaction of a cold homogeneous plasma with a strong linearly polarized external electromagnetic wave, taking into account relativistic terms. Instabilities appear for transverse and longitudinal waves when the pump-field frequency is at the same time below the electron plasma frequency and above but very near the cutoff frequency obtained by Kaw and Dawson with a nonlinear relativistic theory.

Many authors[1-3] have shown by means of classical theories that when the pump-field frequency ω_0 is slightly greater than the plasma frequency ω_e, excitation of parametric instabilities growing as $(m_e/m_i)^{1/3}$ can lead to anomalous absorption. On the other hand, Tsintsadze,[4] using the Lorentz equation for a cold electron plasma, found parametric amplification for longitudinal and transverse waves with growth rates proportional to $\epsilon \equiv (V_{Ee}/c)^2 \equiv \nu_0^2$ with $V_{Ee} = eE_0/m_e\omega_0$. These growth rates vanish in the classical limit ($\epsilon \to 0$), which is in good agreement with Silin's work which shows that there is no parametric effect when ions are considered to be at rest. Relativistic theories have also been developed in the case of a circularly polarized pump field.[5,6]

In this work we start from Tsintsadze's theory (linearly polarized pump wave), taking ions into account in order to study their influence on instabilities owing to relativistic effects. We also find it necessary to rederive that theory because of a certain number of flaws in Ref. 4 which stem from a bad choice of the form of the driver and from the fact the dipole approximation cannot always be used. Here we set up a two-fluid hydrodynamic model from Maxwell's and Lorentz's equations using an expansion in powers of ϵ. We found parametric buildup for both potential and nonpotential high-frequency (hf) oscillations which take place further from the usual cutoff density in the overdense plasma.

We assume the driver's frequency ω_0 to be near the cutoff frequency; thus we may neglect its spatial dependence and its magnetic field H_0. We suppose too, that E_0 is strong enough to make electrons, but not ions, relativistic. However, to make the calculation easier, we use the Lorentz equation for both ions and electrons, and we neglect V_{Ei}/c with respect to V_{Ee}/c. We start with Maxwell's and Lorentz's equations:

$$\vec{\nabla}\times\vec{E} = -\frac{1}{c}\frac{\partial\vec{H}}{\partial t}, \tag{1a}$$

$$\vec{\nabla}\times\vec{H} = \frac{1}{c}\frac{\partial\vec{E}}{\partial t} + \frac{4\pi}{c}\sum_{\alpha=e,i} q_\alpha n_\alpha \vec{v}_\alpha, \tag{1b}$$

$$\vec{\nabla}\cdot\vec{E} = 4\pi\sum_\alpha q_\alpha n_\alpha, \tag{1c}$$

$$\vec{\nabla}\cdot\vec{H} = 0, \tag{1d}$$

$$\frac{\partial n_\alpha}{\partial t} + \vec{\nabla}\cdot n_\alpha\vec{v}_\alpha = 0, \tag{1e}$$

$$\left(\frac{\partial}{\partial t} + \vec{v}_\alpha\cdot\vec{\nabla}\right)\frac{\vec{v}_\alpha}{(1-v_\alpha^2/c^2)^{1/2}} = \frac{q_\alpha}{m_\alpha}\left(\vec{E} + \frac{\vec{v}_\alpha}{c}\times\vec{B}\right). \tag{1f}$$

If we linearize these equations by setting

$$\vec{E} = \vec{E}_0 + \vec{E}_1, \quad \vec{H} = \vec{H}_1,$$

$$n_\alpha = n_0 + n_{1\alpha}, \quad \vec{v}_\alpha = \vec{v}_{0\alpha} + \vec{v}_{1\alpha},$$

only Eqs. (1b) and (1f) remain to zeroth order:

$$\omega_0 \frac{d}{d\theta}\frac{\vec{v}_{0\alpha}}{(1-v_{0\alpha}^2/c^2)^{1/2}} = \frac{q_\alpha}{m_\alpha}\vec{E}_0\frac{d\phi}{d\theta}, \tag{2}$$

$$\omega_0\vec{E}_0\frac{d^2\phi}{d\theta^2} = -4\pi e n_0(\vec{v}_{0i} - \vec{v}_{0e}), \tag{3}$$

where $\vec{E}_0(t) \equiv \vec{E}_0\, d\phi/d\theta$ and $\theta = \omega_0 t$. Integrating Eq. (2), we find

$$\vec{v}_{0\alpha} = \frac{(q_\alpha m_e/q_e m_\alpha)\vec{V}_{Ee}\phi(\theta)}{[1+(m_e/m_\alpha)^2\epsilon\phi^2(\theta)]^{1/2}}. \tag{4}$$

Substituting Eq. (4) into Eq. (3) and using an expansion in ϵ, we find

$$\frac{d^2\phi}{d\theta^2} + (\omega_e^2+\omega_i^2)\frac{\phi(\theta)}{\omega_0^2} = \frac{\epsilon}{2\omega_0^2}\left[\omega_e^2 + \omega_i^2\left(\frac{m_e}{m_i}\right)^2\right]\phi^3.$$

This is the equation of an anharmonic oscillator, the solution of which yields $\vec{E}_0$ and $\vec{v}_{0\alpha}$:[7,8]

$$\vec{E}_0(t) = \vec{E}_0(\sin\omega_0 t + 3\epsilon\mu \sin 3\omega_0 t),$$

$$\vec{v}_{0\alpha}(t) = \vec{V}_{E\alpha}\{[\tfrac{3}{8}(m_e/m_\alpha)^2\epsilon - 1]\cos\omega_0 t + \epsilon[\tfrac{1}{8}(m_e/m_\alpha)^2 - \mu]\cos 3\omega_0 t\}, \quad (5)$$

where

$$\mu = \frac{1}{64}\frac{\omega_e^2 + \omega_i^2(m_e/m_i)^2}{\omega_e^2 - \omega_i^2}. \quad (6)$$

This calculation also leads to the dispersion relation

$$\omega_0^2 = \omega_e^{*2} + \omega_i^2,$$

where

$$\omega_e^* = \omega_e(1 - \tfrac{3}{16}\epsilon) \equiv \omega_e \alpha^{1/2} \quad (7)$$

is the electron plasma frequency reduced by relativistic effects, already found by Kaw and Dawson.[7]

As we use the dipole approximation, we are going to specify its domain of validity. We know[9] that this approximation is equivalent to the neglect of $v_{0\alpha}/\beta c$ relative to unity ($\beta = \omega_0/k_0 c$). It will be possible when $V_{Ee}/\beta c = \nu_0/\beta$ is negligible compared to unity. Here, in our theory, we neglect terms of higher order than ν_0^2. Thus the domain of validity of the dipole approximation can be defined by $\nu_0/\beta \lesssim \nu_0^3$ or $1/\beta \lesssim \epsilon$. This requirement might be stringent, but it is difficult without a more sophisticated theory to predict the influence of the terms we neglect using the dipole approximation in the dispersion relation. The dispersion relation is[8]

$$\frac{1}{\beta} = \frac{k_0 c}{\omega_0} = \left(1 - \frac{\omega_e^{*2} + \omega_i^2}{\omega_0^2}\right)^{1/2} \simeq \left(\frac{2\delta}{\omega_0} - \frac{m_e}{m_i}\right)^{1/2}, \quad (8)$$

with $\delta = \omega_0 - \omega_e^*$. The condition $1/\beta \lesssim \epsilon$ gives the maximum value of δ for which the dipole approximation is valid,

$$\delta_M = \tfrac{1}{2}(\epsilon^2 + m_e/m_i)\omega_0. \quad (9)$$

After linearizing the system (1) about equilibrium and performing a Fourier transformation on the space variables, we obtain the following set of equations for the nonequilibrium values:

$$i\vec{k}\times\vec{E}_1 = -\frac{1}{c}\frac{\partial \vec{H}_1}{\partial t}, \quad (10a)$$

$$i\vec{k}\times\vec{H}_1 = \frac{1}{c}\frac{\partial \vec{E}_1}{\partial t} + \frac{4\pi}{c}\sum_{\alpha=e,i} q_\alpha(n_0\vec{v}_{1\alpha} + n_{1\alpha}\vec{v}_{0\alpha}), \quad (10b)$$

$$i\vec{k}\cdot\vec{E}_1 = 4\pi\sum_{\alpha=e,i} q_\alpha n_{1\alpha}, \quad (10c)$$

$$\frac{\partial}{\partial t}n_{1\alpha} + i\vec{k}\cdot\vec{v}_{0\alpha}n_{1\alpha} + n_0 i\vec{k}\cdot\vec{v}_{1\alpha} = 0, \quad (10d)$$

$$D_\alpha\left(\frac{\vec{v}_{1\alpha}}{(1 - v_{0\alpha}^2/c^2)^{1/2}} + \frac{\vec{v}_{0\alpha}(\vec{v}_{0\alpha}\cdot\vec{v}_{1\alpha})}{c^2(1 - v_{0\alpha}^2/c^2)^{3/2}}\right) = \frac{q_\alpha}{m_\alpha}\left(\vec{E}_1 + \frac{\vec{v}_{0\alpha}\times\vec{H}_1}{c}\right). \quad (10e)$$

Here

$$D_\alpha \equiv \frac{\partial}{\partial t} + i\vec{k}\cdot\vec{v}_{0\alpha}. \quad (11)$$

We consider the $\vec{k}\parallel\vec{E}_0$ case only. Combining Eqs. (10) and performing the expansion in ϵ, we obtain an equation for transverse waves:

$$\left(\frac{\partial^2}{\partial t^2} + c^2k^2 + \omega_e'^2(1 - \tfrac{1}{4}\epsilon'\cos 2\omega_0 t)\right)\vec{H}_1 = 0, \quad (12)$$

where

$$\omega_e'^2 = \omega_e^2(1 - \tfrac{1}{4}\epsilon + m_e/m_i + c^2k^2/\omega_e^2), \quad (13)$$

$$\epsilon' = \frac{\epsilon}{1 - \frac{1}{4}\epsilon + m_e/m_i + c^2k^2/\omega_e^2}. \quad (14)$$

Eq. (12) is a Mathieu equation similar to the one obtained by Tsintsadze[4] for very large wavelength; we find the following buildup increment[10]:

$$\gamma = [(\epsilon\omega_e^2/16\omega_0)^2 - (\omega_e' - \omega_0)^2]^{1/2}. \quad (15)$$

This function has a maximum value for $\omega_0 = \omega_e'$ which corresponds to Tsintsadze's result. Figure 1 shows the evolution of the growth rate γ vs $D = \delta/\delta_0$ for $k = 0$ and for different values of ν_0, where $\delta_0 = \omega_0 m_e/2m_i$ represents the minimum difference between ω_0 and ω_e^* so that the driver may propagate ($\omega_0^2 = \omega_e^{*2} + \omega_i^2$). We can see on these curves that for a specified pump (ν_0 and ω_0 are fixed) γ increases with D and that it is reduced by the fact we take ions into account. This effect becomes stronger as ω_0 approaches the cutoff frequency. Finally we see that Tsintsadze's growth rate corresponds to a value D_T of D which is out of the domain of validity of the dipole approximation. The gap between $D_M = \delta_M/\delta_0$ and D_T is more important for smaller values of ν_0 (for $\nu_0 = 0.2$, $D_T = 2D_M$). Still, for the maximum value of the driver's intensity that we considered ($\nu_0 \simeq 0.3$), the gap becomes small and the growth rate given by Tsintsadze then has a physical meaning.

For longitudinal waves, in the same way as above, system (10) leads to

$$D_\alpha\left(\frac{D_\alpha(n_{1\alpha})}{(1 - v_{0\alpha}^2/c^2)^{3/2}}\right) = -\frac{q_\alpha}{m_\alpha}4\pi n_0\sum_{\beta=e,i} q_\beta n_{1\beta}. \quad (16)$$

If we let

$$n_{1\alpha} = y_\alpha(1 - v_{0\alpha}^2/c^2)^{3/4}\exp\left(-\int_0^t i\vec{k}\cdot\vec{v}_{0\alpha}\,dt\right) \quad (17)$$

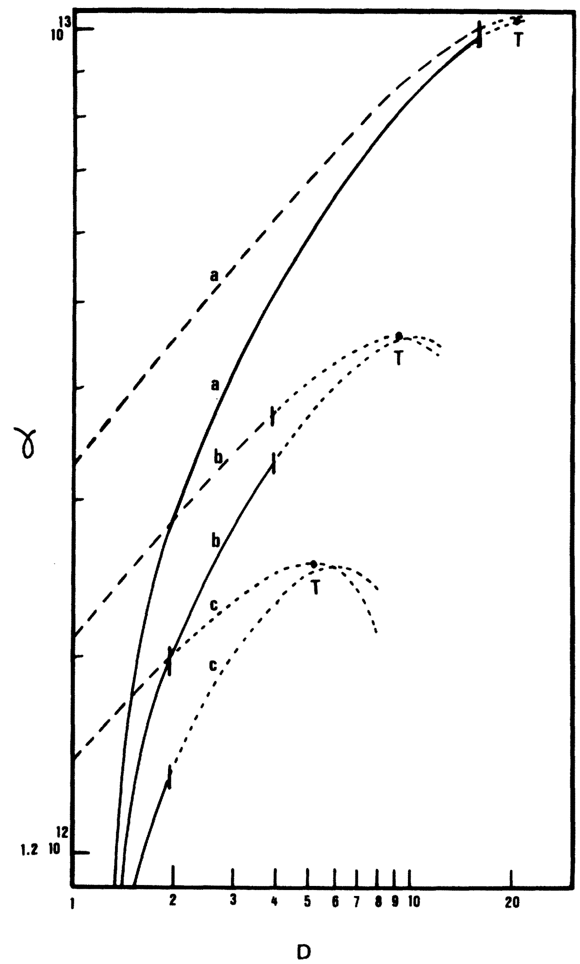

FIG. 1. Dependence of growth rate γ on the mismatch for (a) $\nu_0=0.3$, (b) $\nu_0=0.2$, and (c) $\nu_0=0.15$ (Nd laser and hydrogen plasma, $kc/\omega_0=0$). Dashed lines represent γ in the limit $m_i\to\infty$. We produce dotted curves in the domain where the dipole approximation is not valid. T represents Tsintsadze's results.

and perform a Laplace transformation on the time variable, we obtain a set of two equations:

$$(-\omega^2+\alpha^2\omega_e^2)y_e(\omega) - \tfrac{3}{8}\epsilon(\omega_e^2-2\omega_0^2)[y_e(\omega+2\omega_0)+y_e(\omega-2\omega_0)] - \omega_e^2\sum_n A_n y_i(\omega-n\omega_0)=0, \quad (18)$$

$$(-\omega^2+\omega_i^2)y_i(\omega)-\omega_i^2\sum_n A_n y_e(\omega+n\omega_0)=0, \quad (19)$$

with

$$A_n=J_n(a)\alpha-\tfrac{3}{16}\epsilon[J_{n-2}(a)+J_{n+2}(a)] +\tfrac{3}{16}a\epsilon[J_{n-1}(a)-J_{n+1}(a)] +\tfrac{1}{48}a\epsilon(1-8\mu)[J_{n-3}(a)-J_{n+3}(a)], \quad (20)$$

where J_n is the nth-order Bessel function and $a=\vec{k}\cdot\vec{V}_{Be}/\omega_0$.

Let us consider the high-frequency case $[\mathrm{Re}(\omega)\sim\pm\omega_e]$ and neglect the off-resonant terms. We then obtain a set of three equations between $y_i(\Omega)$, $y_e(\Omega-\omega_0)$, and $y_e(\Omega+\omega_0)$, with $\Omega=\omega\pm\omega_0$. Thus this system may have a nontrivial solution only if the determinant of the coefficient matrix is equal to zero, that is,

$$\begin{vmatrix} -\Omega^2+\omega_i^2 & \omega_i^2A_1 & -\omega_i^2A_1 \\ \omega_e^2A_1 & -(\Omega-\omega_0)^2+\omega_e^2\alpha^2 & -\tfrac{3}{8}\epsilon(\omega_e^2-2\omega_0^2) \\ -\omega_e^2A_1 & -\tfrac{3}{8}\epsilon(\omega_e^2-2\omega_0^2) & -(\Omega+\omega_0)^2+\omega_e^2\alpha^2 \end{vmatrix}=0\,. \quad (21)$$

Let us set $\Omega=x+i\gamma$, where $x\pm\omega_0$ and γ are, respectively, the frequency and the growth rate of this mode. A calculation similar to the one performed by Nishikawa[2] shows that when $x\neq 0$ we must have $\delta>0$, and when $x=0$, we must have $\delta<0$. This last case ($x=0$ or $\omega=\pm\omega_0+i\gamma$) might be less interesting, since $\delta<0$ corresponds to a driver whose frequency is smaller than the cutoff frequency, that is, to a region which is roughly the skin depth and where the intensity of the driver decreases rapidly. Still, if we perform the calculation corresponding to this case, we find that when $m_i\to\infty$,

$$\gamma^2=-\Lambda\equiv-[\delta^2+\tfrac{3}{8}\epsilon\omega_e^*\delta]. \quad (22)$$

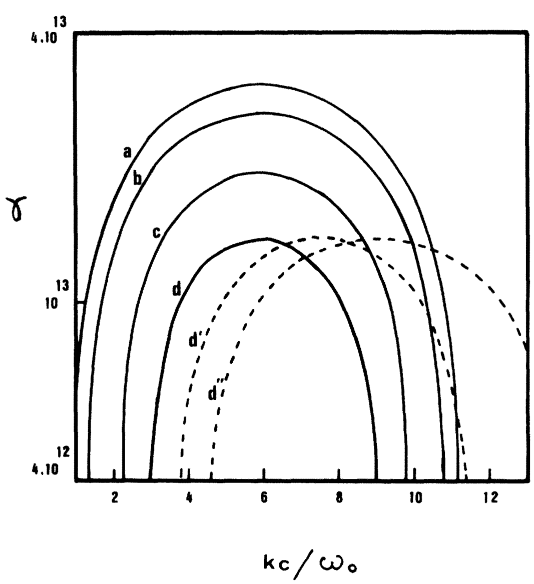

FIG. 2. Dependence of growth rate γ on the wave vector k for $\nu_0=0.3$ and (a) $D=15$, (b) $D=10$, (c) $D=5$, (d) $D=3$; $\nu_0=0.24$ and $D=3$ (d'); $\nu_0=0.2$ and $D=3$ (d''). (Nd laser and hydrogen plasma.)

The maximum value of this growth occurs when $\delta_T = -\frac{3}{16}\epsilon\omega_e^*$ and is $\gamma_T = \frac{3}{16}\epsilon\omega_e^*$. This growth rate is the one obtained by Tsintsadze and corresponds to an instability confined in a region which is the skin depth.

Now we study the $x \neq 0$ case and find the growth rate

$$\gamma = \tfrac{1}{2}[-\omega_i^2 - \Lambda + 2(\omega_i^2\Lambda + K\delta/\omega_0)^{1/2}]^{1/2}, \qquad (23)$$

where

$$K = A_1^2\omega_i^2\omega_e^2\alpha^{-1/2}. \qquad (24)$$

We can now make two remarks: (i) When $\epsilon \to 0$ in Eq. (21), we find Nishikawa's matrix coefficient in which the thermal velocity is neglected; and (ii) if $m_i \to \infty$ in Eq. (23), we have

$$4\gamma^2 = -(\delta^2 + \tfrac{3}{8}\epsilon\omega_e^*\delta). \qquad (25)$$

We shall have a real root only if $\delta < 0$, which is in contradiction with the $\delta > 0$ necessary condition of this $x \neq 0$ case.

Figure 2 shows that for a fixed value of kc/ω_0 and ν_0, γ increases with D. Furthermore, for one ν_0 there is one value k_M of the wave vector for which the growth rate has a maximum value γ_M. On the other hand, for a fixed value of D, γ_M remains constant and k_M increases when ν_0 decreases. The calculation of the threshold of these instabilities ($\gamma = 0$) allows us to ascertain that they correspond to present laser technology.

To conclude, only a model taking ions into account can show the instabilities for potential waves which were predicted by Tsintsadze.

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