Strong-signal laser operation. I. General theory*

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A Lamb semiclassical theory is developed for multimode, mode-locked laser operation. The equality of frequency spacings between adjacent modes causes population pulsations generated in the nonlinear atomic response to be integer multiples of the adjacent-mode spacing. This allows the atomic response to be written in terms of Fourier series, thereby reducing coupled differential equations of motion to algebraic equations ultimately expressible in terms of a continued fraction. A unidirectional ring configuration is assumed so that the more complicated spatial dependence of standing waves is avoided. Reductions are made to the rate-equation limit and to the special two-mode cases of bidirectional ring operation and single-mode, standing-wave, gas-laser operation. Numerical analysis of the equations is motivated. Specific results are given in the following (companion) paper.

I. INTRODUCTION

A rigorous laser theory particularly suited to Brewster-window gas lasers has been given by Lamb,[1] who described the laser by a classical electromagnetic field and assumed the active medium was made up of thermally moving atoms whose response to the field was governed by the laws of quantum mechanics. The field was taken to have the very general form

$$E(z,t)=\frac{1}{2}\sum_{n=1}^{N}E_n(t)\exp\{-i[\nu_n t+\phi_n(t)]\}U_n(z)+\text{c.c.}, \tag{1}$$

where the N amplitudes $E_n(t)$ and phases $\phi_n(t)$ are slowly varying functions of time, ν_n is the optical frequency of the nth longitudinal cavity mode, the wave number $K_n=(n+n_0)\pi/L$, and L is the length of the cavity. The self-consistency requirement that a quasistationary field should be sustained by the induced polarization led to equations which determine the E_n, ϕ_n, and ν_n in terms of the laser parameters. Partly because of the generality, it was not obvious how to integrate the atomic equations of motion (Schrödinger's equation) exactly. Lamb used third-order perturbation theory and considered up to three-mode operation explicitly. His approach was subsequently written in a computer-oriented version by Sargent, Lamb, and Fork[2] which allows the analysis of many-mode operation.[3]

An important special case of the multimode field (1) is that for the mode-locked operation which we define by the conditions

$$\dot{E}_n=0\,, \tag{2}$$

$$(\nu_{n+1}+\dot{\phi}_{n+1})-(\nu_n+\dot{\phi}_n)=(\nu_n+\dot{\phi}_n)-(\nu_{n-1}+\dot{\phi}_{n-1})=\Delta\,, \tag{3}$$

that is, all beat frequencies between adjacent modes equal the same constant, $\Delta\cong\frac{1}{2}c/L$. [Condition (2) could be relaxed, allowing a pulse train to build up and decay away, but condition (3) is essential for mode-locking.] This condition represents a considerable simplification of (1), for all combination tones produced by frequency beating in the nonlinear medium coincide with the mode frequencies or are placed some integral multiple of Δ away. This allows one to expand the atomic polarization and the population difference of the medium in Fourier series. These series reduce the atomic equations of motion to sets of difference equations which can, in turn, be written as infinite continued fractions. Because the medium has finite bandwidth, the fractions can be truncated numerically on a computer for the two-mode case. For higher-mode operation we have found it more convenient to work directly with the population-pulsation recursion relations. We follow this procedure considering first the unidirectional ring laser, for which the mode functions are given by

$$U_n(z)=e^{iK_n z}\,. \tag{4}$$

The success of the Fourier-analysis continued-fraction strong-signal theory as applied to the two-mirror standing-wave laser by Stenholm and Lamb[4] and Feldman and Feld[5] prompted the application of that approach to the bidirectional ring laser. This was done by Menegozzi and Lamb,[6] although they did not obtain strong-signal numerical results.

In Sec. II, we give self-consistency equations for the field amplitudes and phases for the general unidirectional multimode case. The stable, stationary solutions of these equations yield the mode-locked fields predicted by the theory. In Sec. III, we calculate the polarization of the (possibly) inhomogeneously broadened medium in terms of the Fourier coefficients for the density matrix of the medium. In Sec. IV, these coefficients are determined from the atomic equations of motion. In Sec. V, the rate-equation approximation (REA) solution is given. This contains no population pulsations and hence cannot predict mode-locking (phase information is lost).[7] Nevertheless, it provides starting values for numerical analysis and casts some light on free-running operation. In Sec. VI, one- and two-mode operations are considered. The transformations allowing the standing-wave and bidirectional ring-laser configurations to be included as special cases are presented. In Sec. VII, we motivate the numerical analysis required to obtain solutions of the field self-consistency equations. This includes discussion of truncation of the continued fraction for the polarization of the medium, an iterative solution of the recursion relation for the Fourier coefficients for the general multimode case, a many-variable Newton-Raphson zeroing procedure (for calculating $\dot{E}=0$, etc.), and an iterative solution of the mode-locking conditions Eq. (3). Specific results for two- and higher-mode ring lasers are given in the following (companion) paper.[8]

We note that Risken and Nummedal[9] have studied the mode-locked unidirectional ring laser in the time rather than the frequency domain, and obtained pulsed-field solutions. Our specific numerical studies have dealt with only a few modes in the frequency domain, while well-defined pulsed solutions would require many modes. In principle, such a numerical analysis based on the present paper should agree with their results, but it would require large computational facilities. Aspects of the amplifications of spontaneous emission in laser amplifiers (e.g., noise amplifiers) can also be understood on the basis of our equations.[10] Previous motivation for the present work has been given by Sargent.[11]

II. FIELD SELF-CONSISTENCY EQUATIONS

From Maxwell's equations, the self-consistency equations for the general field (1) are found to be[1,12]

$$\dot{E}_n = -\tfrac{1}{2}(\nu/Q_n)E_n - \tfrac{1}{2}(\nu/\epsilon_0)\,\mathrm{Im}(\mathcal{P}_n)\,, \tag{5}$$

$$\nu_n + \dot{\phi}_n = \Omega_n - \tfrac{1}{2}(\nu/\epsilon_0)\,\mathrm{Re}(\mathcal{P}_n)/E_n\,, \tag{6}$$

where the complex polarization coefficient $\mathcal{P}_n$ is defined in terms of the polarization of the medium $P(z,t)$ by

$$P(z,t) = \frac{1}{2}\sum_n \mathcal{P}_n(t)\exp[-i(\nu_n t + \phi_n)]U_n(z) + \mathrm{c.c.} \tag{7}$$

and the passive cavity frequency $\Omega_n = K_n c$. We are concerned with the unidirectional ring laser with mode functions (4) for which the electric field (1) is

$$E(z,t) = \frac{1}{2}\sum_{n=1}^{N} E_n \exp[-i(\nu_n t + \phi_n - K_n z)] + \mathrm{c.c.} \tag{8}$$

We are interested in solutions satisfying the mode-locking condition (3). This can be written conveniently in terms of the relative phase angles

$$\psi_n = 2(\nu_n t + \phi_n) - (\nu_{n+1}t + \phi_{n+1}) - (\nu_{n-1}t + \phi_{n-1}) \tag{9}$$

as

$$\dot{\psi}_n = 0, \quad \text{for } n=2 \text{ through } n=N-1. \tag{10}$$

From (6), we see that

$$\dot{\psi}_n = -\frac{1}{2}\frac{\nu}{\epsilon_0}\mathrm{Re}\left(\frac{2\mathcal{P}_n}{E_n} - \frac{\mathcal{P}_{n+1}}{E_{n+1}} - \frac{\mathcal{P}_{n-1}}{E_{n-1}}\right). \tag{11}$$

Our problem thus reduces to finding the zeros of Eqs. (5) and (11).

Without loss of generality, we choose the ν_n to contain the complete frequency dependence of mode n and set $\dot{\phi}_n = 0$. Then (3) reduces to

$$\nu_n = \nu_q + (n-q)\Delta\,, \tag{12}$$

and the relative phase angles (9) become

$$\psi_n = 2\phi_n - \phi_{n+1} - \phi_{n-1}\,. \tag{13}$$

There are many possible phase relationships. We give two here by way of illustration. For this, it is convenient to write the electric field (8) as a complex envelope multiplied by a carrier wave,

$$E(z,t) = \tfrac{1}{2}\{\exp[-i(\nu_q t + \phi_q - K_q z)]\}\left\{\sum_n E_n \exp[-i(\phi_n - \phi_q)]\exp[-i(n-q)(\Delta t - \pi z/L)]\right\} + \mathrm{c.c.}, \tag{14}$$

where the first term in braces represents the carrier wave, and the second the complex envelope. Consider first the relationship

$$\psi_n = 0\,, \tag{15}$$

or equivalently

$$\phi_n = \phi_q + (n-q)\delta\,, \tag{16}$$

for some phase factor δ. The field (14) reduces to

$$E(z,t)=\tfrac{1}{2}\exp[-i(\nu_q t+\phi_q-K_q z)] \times \sum_n E_n \exp[-i(n-q)(\Delta t+\delta-\pi z/L)]+\text{c.c.} \tag{17}$$

Here we can choose our time origin such that the δ cancels. Thus without loss of generality (15) is equivalent to equal phases,

$$\phi_n=\phi_q, \quad \text{for all } n \text{ and } q. \tag{18}$$

In particular, at the position $z=0$, the field is

$$E(0,t)=\tfrac{1}{2}\exp[-i(\nu_q t+\phi_q)] \sum_n E_n e^{-i(n-q)\Delta t}+\text{c.c.} \tag{19}$$

At times equal to integral multiples of $2\pi/\Delta$, the exponentials $\exp[-i(n-q)\Delta t]$ are all unity, yielding a large value for the field envelope. At other times, the phasors $E_n\exp[-i(n-q)\Delta t]$ tend to cancel and the magnitude of the envelope is small. This gives a sequence of pulses in time. For general z, the exponentials are all unity at points spaced by L which propagate along z as a train of pulses. Because of the large amplitude variations, this phase relationship (15) or (18) is sometimes called AM.

In contrast, consider a field given by

$$E_n=E_0|J_{n-q}(\Gamma)|\,, \tag{20}$$

$$\phi_n=\phi_q=\begin{cases}0, & n\geq q\\ [1-(-)^{q-n}]\pi, & n<q\end{cases} \tag{21}$$

where $J_{n-q}(\Gamma)$ is a Bessel function. In terms of the ψ_n,

$$\psi_n=\begin{cases}0, & n>q\\ \pi, & n=q\\ (-1)^n 2\pi, & n<q\end{cases} \tag{22}$$

Eq. (8) becomes

$$E(z,t)=\tfrac{1}{2}\exp[-i(\nu_q t+\phi_q-K_q z)] \times \sum_k J_k(\Gamma)\exp[-ik(\Delta t-\pi z/L)]+\text{c.c.} \tag{23}$$

Using the Bessel function identity

$$\exp(-i\Gamma\sin\theta)=\sum_{k=-\infty}^{\infty} J_k(\Gamma)e^{-ik\theta}\,, \tag{24}$$

we find, for (23),

$$E(z,t)=E_0\cos[\nu_q t+\phi_q-K_q z+\Gamma\sin(\Delta t-\pi z/L)]\,, \tag{25}$$

which has constant amplitude E_0 and variable frequency $\nu_q+\Delta\Gamma\cos(\Delta t-\pi z/L)$. This field is sometimes called FM and phases given by (21) are termed FM.[13] In self-locking problems full FM operation has not been observed, although FM phases do occur without the amplitude restriction (20). This results in a field with amplitude variations which are less pronounced than those of the AM case.

III. POLARIZATION OF THE MEDIUM

We consider a medium of two-level atoms with line center ω as depicted in Fig. 1. We describe the medium by a population matrix

$$\rho(z,\omega,t)=\sum_{\alpha=a,b}\int_{-\infty}^{t} dt_0\lambda_\alpha(z,\omega,t_0)\rho(\alpha,z,\omega,t,t_0)\,, \tag{26}$$

where $\rho(\alpha,z,\omega,t,t_0)$ is the single-atom density matrix for an atom excited at time t_0 in state $\alpha=a$ or b at the place z. $\lambda_\alpha(t_0,\omega,z)$ is the pumping rate to the αth state and varies slowly enough to be factored outside the t_0 integration. Using the Schrödinger equation for the single-atom density matrix, one can show[12] by differentiating (26) with respect to t that the corresponding equations for the components of $\rho(z,\omega,t)$ are

$$\dot\rho_{ab}(z,\omega,t)=-(i\omega+\gamma)\rho_{ab}+i\hbar^{-1}\mathcal{V}_{ab}(\rho_{aa}-\rho_{bb})\,, \tag{27}$$

$$\dot\rho_{aa}(z,\omega,t)=\lambda_a-\gamma_a\rho_{aa}-i\hbar^{-1}(\mathcal{V}_{ab}\rho_{ba}-\text{c.c.})\,, \tag{28}$$

$$\dot\rho_{bb}(z,\omega,t)=\lambda_b-\gamma_b\rho_{bb}+i\hbar^{-1}(\mathcal{V}_{ab}\rho_{ba}-\text{c.c.})\,, \tag{29}$$

$$\rho_{ba}(z,\omega,t)=\rho_{ab}^*(z,\omega,t)\,. \tag{30}$$

Here the $\mathcal{V}_{ab}$ is the electric-dipole perturbation energy for the multimode field (8) written in the rotating-wave approximation,

$$\mathcal{V}_{ab}=-\tfrac{1}{2}\wp\sum_n E_n\exp[-i(\nu_n t+\phi_n-K_n z)]\,, \tag{31}$$

where $\wp$ is the electric-dipole matrix element between the upper and lower levels.

We define a population difference

$$D(z,\omega,t)\equiv\rho_{aa}(z,\omega,t)-\rho_{bb}(z,\omega,t) \tag{32}$$

and sum

$$M(z,\omega,t)\equiv\rho_{aa}(z,\omega,t)+\rho_{bb}(z,\omega,t) \tag{33}$$

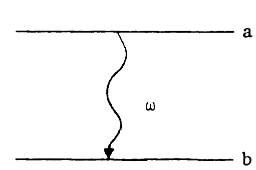

FIG. 1. Level diagram of atomic medium having a line center ω.

with the equations of motion [from (28) and (29)]

$$\dot{D} = \lambda_a - \lambda_b - \gamma_{ab} D - \tfrac{1}{2}(\gamma_a - \gamma_b)M - 2i\hbar^{-1}(\mathcal{V}_{ab}\rho_{ba} - \text{c.c.}), \quad (34)$$

$$\dot{M} = \lambda_a + \lambda_b - \gamma_{ab} M - \tfrac{1}{2}(\gamma_a - \gamma_b)D. \quad (35)$$

Here the average decay rate γ_{ab} is defined by

$$\gamma_{ab} = \tfrac{1}{2}(\gamma_a + \gamma_b) \quad (36)$$

and is smaller than γ, which includes effects of collisions.

We eliminate the population-sum variable M in (34) by substituting the formal integral of (35),

$$M(z,\omega,t) = -\tfrac{1}{2}(\gamma_a - \gamma_b)\int_{-\infty}^{t} dt'\, D(z,\omega,t')\exp[-\gamma_{ab}(t-t')] + (\lambda_a + \lambda_b)/\gamma_{ab} \quad (37)$$

into (34). One finds

$$\dot{D} = -\gamma_{ab} D + \tfrac{1}{4}(\gamma_a - \gamma_b)^2 \int_{-\infty}^{t} dt'\, D(z,\omega,t')\exp[-\gamma_{ab}(t-t')] + \frac{\gamma_a\gamma_b}{\gamma_{ab}} N(z,\omega,t) - 2i\hbar^{-1}(V_{ab}\rho_{ba} - \text{c.c.}), \quad (38)$$

where the population inversion

$$N(z,\omega,t) = \lambda_a/\gamma_a - \lambda_b/\gamma_b = N(z,t)W(\omega), \quad (39)$$

and where $W(\omega)$ gives the inhomogeneity distribution. The inversion $N(z,t)$ is assumed to vary little during the lifetimes of the atoms and of the field (Q/ν).

Taking a clue from the interaction process used in perturbation theory, one can see that in the presence of a field satisfying (2), the polarization element ρ_{ab} can be written as the Fourier series

$$\rho_{ab}(z,\omega,t) = N(z,\omega,t)\sum_m p_m(\omega)\exp[-i(\nu_m t - K_m z)]. \quad (40)$$

Similarly one sees that the population difference has a dc term and population pulsations oscillating at integral multiples of the intermode beat frequency:

$$D(z,\omega,t) = N(z,\omega,t)\sum_{k=-\infty}^{\infty} d_k(\omega)\exp[ik(\Delta t - \pi z/L)]. \quad (41)$$

The equation of motion for D is real, indicating that D itself is real. Hence in (41) we have the reality condition

$$d_{-k}(\omega) = d_k^*(\omega). \quad (42)$$

The polarization (7) of the medium is given by adding the contributions of all atoms at z at time t regardless of their times or frequencies of excitation. This is given by

$$P(z,t) = \int_{-\infty}^{\infty} d\omega\, \text{tr}(\rho e r) = \int_{-\infty}^{\infty} d\omega\, \rho_{ab}(z,\omega,t) + \text{c.c.} \quad (43)$$

Substituting (40) for ρ_{ab}, we find

$$P(z,t) = \sum_m \exp[-i(\nu_m t - K_m z)] \times \int_{-\infty}^{\infty} d\omega\, N(z,t)W(\omega)p_m(\omega) + \text{c.c.} \quad (44)$$

Identifying coefficients of $\exp[-i(\nu_n t + \phi_n - K_n z)]$ in (44) and (7), we find

$$\mathcal{P}_n = 2\wp\bar{N}\int_{-\infty}^{\infty} d\omega\, W(\omega)p_n(\omega)e^{i\phi_n}, \quad (45)$$

where the average excitation is

$$\bar{N} = L^{-1}\int_0^L dz\, N(z,t). \quad (46)$$

Our problem thus reduces to finding the $p_n(\omega)$. This is accomplished in Sec. IV. Here we have neglected terms in $\mathcal{P}_n$ which vary in time at intermode beat frequencies, for the nth mode is constrained to oscillate essentially within the frequency range $\Omega_n + \nu/Q_n \rightarrow \Omega_n - \nu/Q_n$ and hence cannot respond to intermode frequency variations.

IV. SOLUTION OF THE EQUATIONS OF MOTION

In this section, we substitute (40) for $\rho_{ab}(z,\omega,t)$ and (41) for $D(z,\omega,t)$ into their equations of motion (9) and (20) and combine the results to obtain Eq. (48), the polarization Fourier coefficients $p_n(\omega)$ in terms of the population-pulsation coefficients $d_k(\omega)$. We then obtain a difference equation (53) for the d_k's alone, which is conveniently written in terms of a continued fraction (66).

We have from (27), (40), and (41)

$$N(z,\omega,t)\sum_m p_m(-i\nu_m)\exp[-i(\nu_m t - K_m z)] = -(i\omega + \gamma)N\sum_m p_m \exp[-i(\nu_m t - K_m z)]$$
$$- \tfrac{1}{2}i(2\gamma_a\gamma_b)^{1/2}N\sum_n\sum_k \mathcal{E}_n d_k \exp[-i(\nu_n - k\Delta)t]\exp[i(K_n - k\pi/L)z],$$

where, for convenience, we have introduced the dimensionless, complex amplitude

$$\mathcal{E}_n = \frac{\wp E_n}{\hbar} \frac{e^{-i\phi_n}}{(2\gamma_a\gamma_b)^{1/2}} . \tag{47}$$

Equating the coefficients of $\exp[-i(\nu_m t - K_m z)]$, we find ($n-k=m$ in last sum)

$$(-i\nu_m)p_m = -(i\omega+\gamma)p_m - \tfrac{1}{2}i(2\gamma_a\gamma_b)^{1/2}\sum_k \mathcal{E}_{k+m}d_k ,$$

which yields

$$p_m = -\tfrac{1}{2}i(2\gamma_a\gamma_b)^{1/2}\mathfrak{D}(\omega-\nu_m)\sum_k \mathcal{E}_{k+m}d_k , \tag{48}$$

where

$$\mathfrak{D}_x(\Delta\omega) = (\gamma_x + i\Delta\omega)^{-1} \tag{49}$$

is a convenient abbreviation for a frequently occurring complex denominator. Combining (48) and (45), we have the polarization coefficients (we have set $k=m-n$):

$$\mathcal{P}_n = -i\frac{\wp^2\overline{N}}{\hbar}\sum_m E_m \exp[i(\phi_n-\phi_m)] \times \int_{-\infty}^{\infty} d\omega\, W(\omega)\mathfrak{D}(\omega-\nu_n)d_{m-n}(\omega) . \tag{50}$$

Similarly substituting (40) and (41) into (39), we find

$$N\sum_k d_k ik\Delta \exp\left[ik\left(\Delta t - \frac{\pi z}{L}\right)\right] = -\gamma_{ab}N\sum_k d_k \exp\left[ik\left(\Delta t-\frac{\pi z}{L}\right)\right] + \frac{\gamma_a\gamma_b}{\gamma_{ab}}N + \tfrac{1}{4}(\gamma_a-\gamma_b)^2 N\sum_k d_k\exp\left[ik\left(\Delta t - \frac{\pi z}{L}\right)\right]\mathfrak{D}_{ab}(k\Delta) + \left(i(2\gamma_a\gamma_b)^{1/2}N\sum_n\sum_m \mathcal{E}_n p_m^* \exp[i(\nu_m-\nu_n)t - i(K_m-K_n)z] + \text{c.c.}\right) .$$

Equating coefficients of $\exp[ik(\Delta t - \pi z/L)]$ (in the last two sums $k=m-n$ and $n-m$, respectively), we find

$$d_k = \delta_{k,0} + i\sqrt{2}\,\gamma_{ab}(\gamma_a\gamma_b)^{-1/2}\mathfrak{F}(k\Delta)\sum_n (\mathcal{E}_n p_{n+k}^* - \mathcal{E}_n^* p_{n-k}) , \tag{51}$$

where the complex factor

$$\mathfrak{F}(k\Delta) = \frac{(\gamma_a\gamma_b/\gamma_{ab})(\gamma_{ab}+ik\Delta)}{(\gamma_{ab}+ik\Delta)^2 - \frac{1}{4}(\gamma_a-\gamma_b)^2} = \tfrac{1}{2}[\mathfrak{D}_a(k\Delta)+\mathfrak{D}_b(k\Delta)]\frac{\gamma_a\gamma_b}{\gamma_{ab}} . \tag{52}$$

Further substituting (48) into (51), we find the relation for the d_k alone,

$$d_k = \delta_{k,0} - \gamma_{ab}\mathfrak{F}(k\Delta)\sum_n\sum_i [\mathcal{E}_n\mathfrak{D}(\nu_{n+k}-\omega)\mathcal{E}_{n+k+i}^* d_{-i} + \mathfrak{D}(\omega-\nu_{n-k})\mathcal{E}_n^*\mathcal{E}_{n-k+i}d_i] , \tag{53}$$

in which we have used the reality condition (42).

It is convenient to label the N modes of operation as $\mathcal{E}_1, \ldots, \mathcal{E}_N$. We can then write the d_k in simpler form:

$$\begin{aligned} d_k &= \delta_{k,0} - \gamma_{ab}\mathfrak{F}(k\Delta)\sum_{n=1}^{N}\left(\mathcal{E}_n\mathfrak{D}(\nu_{n+k}-\omega)\sum_{i=1-n-k}^{N-n-k} d_{-i}\mathcal{E}_{n+k+i}^* + \mathcal{E}_n^*\mathfrak{D}(\omega-\nu_{n-k})\sum_{i=1-n+k}^{N-n+k} d_i\mathcal{E}_{n-k+i}\right) \\ &= \delta_{k,0} - \gamma_{ab}\mathfrak{F}(k\Delta)\sum_{n=1}^{N}\sum_{i=1}^{N}[\mathcal{E}_n\mathcal{E}_i^*\mathfrak{D}(\nu_{n+k}-\omega)d_{n+k-i} + \mathcal{E}_n^*\mathcal{E}_i\mathfrak{D}(\omega-\nu_{n-k})d_{i-n+k}] \\ &= \delta_{k,0} - \gamma_{ab}\mathfrak{F}(k\Delta)\sum_{n=1}^{N}\sum_{i=1}^{N}\mathcal{E}_n\mathcal{E}_i^* d_{n+k-i}[\mathfrak{D}(\nu_{n+k}-\omega)+\mathfrak{D}(\omega-\nu_{i-k})] . \end{aligned} \tag{54}$$

In this last line, we have interchanged the dummy indices n and i in the second summation. We now define the subscript $j=n-i$ which runs from $-(N-1)$ to $N-1$. In terms of j, Eq. (54) now reads

$$d_k = \delta_{k,0} - \gamma_{ab}\mathfrak{F}(k\Delta)\sum_{j=-(N-1)}^{N-1} d_{j+k}\sum_{n=1+j\geq N}^{N+j\leq N}\mathcal{E}_n\mathcal{E}_{n-j}^*\,[\mathfrak{D}(\nu_{n+k}-\omega)+\mathfrak{D}(\omega-\nu_{n-j-k})] . \tag{55}$$

This can be written in the succinct form

$$\sum_{j=-(N-1)}^{N-1} c_{jk}d_{j+k} = 0 , \tag{56}$$

where the dimensionless, complex coefficients c_{jk} are given by

$$c_{jk}=\begin{cases}\gamma_{ab}\sum_{n=1+j\geq 1}^{N+j\leq N}\mathcal{E}_n\mathcal{E}^*_{n-j}[\mathfrak{D}(\nu_{n+k}-\omega)+\mathfrak{D}(\omega-\nu_{n-j-k})]\,, & j\neq 0\,,\\ \left(1-\frac{\delta_{k,0}}{d_0}\right)[\mathfrak{F}(k\Delta)]^{-1}+\gamma_{ab}\sum_{n=1}^{N}I_n[\mathfrak{D}(\nu_{n+k}-\omega)+\mathfrak{D}(\omega-\nu_{n-k})]\,, & j=0\,,\end{cases}\tag{57}$$

where the dimensionless intensity

$$I_n=\tfrac{1}{2}(\wp E_n)^2/(\hbar^2\gamma_a\gamma_b)\,.\tag{58}$$

Here we have divided (55) by $\mathfrak{F}(k\Delta)$ to avoid many multiplications in the evaluation of (57).

The dc population-difference coefficient d_0 is given by (55) as

$$d_0=1-2\left(\frac{\gamma_{ab}}{\gamma}\right)d_0\sum_{n=1}^{N}I_n\mathfrak{L}(\omega-\nu_n)-d_0\sum_{j=1}^{N-1}\left(\frac{c_{j0}d_j}{d_0}+\text{c.c.}\right)\,,\tag{59}$$

where the dimensionless Lorentzian

$$\mathfrak{L}(\omega-\nu_n)=\gamma^2/[\gamma^2+(\omega-\nu_n)]\,.\tag{60}$$

We can solve this for d_0 and find a formula (63) which shows effects of saturation explicitly. For present and later convenience we introduce the ratio r_j of population difference coefficients,

$$r_j=d_j/d_{j-1}\,.\tag{61}$$

In terms of this, the ratio d_j/d_0 appearing in (59) is given by the product

$$\frac{d_j}{d_0}=\prod_{m=1}^{j}r_m\,.\tag{62}$$

Combining this with (59), we have

$$d_0=1/(1+S_h+S_p)\,,\tag{63}$$

where the saturation due to hole burning, S_h, is given by

$$S_h=2\frac{\gamma_{ab}}{\gamma}\sum_{n=1}^{N}I_n\mathfrak{L}(\omega-\nu_n)\,,\tag{64}$$

and that due to population pulsations, S_p, is given by

$$S_p=\sum_{j=1}^{N-1}\left(c_{j0}\prod_{m=1}^{j}r_m+\text{c.c.}\right)\,.\tag{65}$$

We now show that the ratios r_j can be calculated in terms of a continued fraction and its remainders. Then d_0 can be calculated from (63)–(65) and $d_{k\neq 0}$ is given by (62) together with the reality condition (42), $d_{-k}=d_k^*$.

We divide Eq. (56) by $d_{-(N-2)+k}$, write the resulting ratio $d_{j+k}/d_{-(N-2)+k}$ in terms of the r_i, and find

$$\sum_{j=-(N-3)}^{N-1}c_{jk}\prod_{m=-(N-3)}^{j}r_{m+k}+c_{-(N-2),k}+\frac{c_{-(N-1),k}}{r_{-(N-2)+k}}=0\,.$$

Solving this for the ratio r_l with the smallest subscript,

$$l=-(N-2)+k\tag{66}$$

we have

$$r_l=-\left(\frac{c_{-(N-1),k}}{c_{-(N-2),k}+\sum_{j=-(N-3)}^{N-1}c_{j,k}\prod_{m=l+1}^{j+k}r_m}\right)\,.\tag{67}$$

This yields the ratio r_l in terms of higher r_m and thus generates an infinite continued fraction.

For example, consider two-mode operation ($N=2$, $k=l$); the ratio (67) becomes a simple continued fraction,

$$r_l=-c_{-1,l}/(c_{0,l}+c_{1,l}r_{l+1})\,.\tag{68}$$

For example,

$$r_1=\frac{-c_{-1,1}}{c_{0,1}+c_{1,1}r_2}=\frac{-c_{1,1}}{c_{-0,1}-c_{1,1}c_{1,2}/(c_{0,2}+c_{1,2}r_3)},\quad\text{etc.}$$

For $N=3$ ($k=+1$), the ratio is more complicated:

$$r_l=\frac{-c_{-2,l+1}}{c_{-1,l+1}+c_{0,l+1}r_{l+1}+c_{1,l+1}r_{l+1}r_{l+2}+c_{2,l+1}r_{l+1}r_{l+2}r_{l+3}}\,.\tag{69}$$

Summarizing our results, we have the amplitude (E_n) and relative phase (ψ_n) determining Eqs. (6) and (11), the zeros of which are mode-locked solutions. These equations contain the complex polarization components $\mathcal{P}_n$, given by Eq. (50). This equation, in turn, contains the population difference coefficients d_k given by (62) with the ratios (67). $\mathcal{P}_n$ has been programmed for digital computer. Discussion is given in Sec. VII on numerical analysis.

V. RATE-EQUATION APPROXIMATION (REA) SOLUTION

In the numerical analysis and in the non-mode-locked case for which the intermode spacing $\Delta \gg \gamma_a$ and γ_b, the REA solution for the complex polarization $\mathcal{P}_n$ is valuable. In this approximation, population pulsations are neglected, that is, $d_{k\neq 0} = 0$, and the phase equations (11) decouple from the amplitude equations (5). Mode-locking cannot be predicted in this approximation. We find the only nonvanishing coefficients

$$\mathcal{P}_n = -i\wp^2 \overline{N}\hbar^{-1}E_n \int_{-\infty}^{\infty} d\omega \frac{W(\omega)\mathfrak{D}(\omega - \nu_n)}{1+S_h}, \tag{70}$$

where the hole-burning saturation is given by (64) as

$$S_h = 2\frac{\gamma_{ab}}{\gamma}\sum_{l=1}^{N} I_l \mathfrak{L}(\omega - \nu_l). \tag{71}$$

Here the dimensionless Lorentzian $\mathfrak{L}(\omega - \nu_l)$ and intensity I_l are given by Eqs. (60) and (58), respectively. The steady-state solutions of (5) in the REA are given by ($\dot{E}_n = 0$)

$$Q_n^{-1} = \wp^2\overline{N}(\hbar\epsilon_0\gamma)^{-1}\int_{-\infty}^{\infty} d\omega \frac{W(\omega)\mathfrak{L}(\omega - \nu_n)}{1+S_h}, \tag{72}$$

which comprises a coupled set of transcendental equations in the dimensionless intensities I_n. Numerical analysis is required for (72) as well as for (68) with (11), but the calculation is considerably quicker and provides starting values for the more elaborate version. Similarly the starting values for the beat frequency Δ can be obtained by calculating the average value of the frequency differences $(\nu_n + \phi_n) - (\nu_{n-1} + \phi_{n-1})$ in the REA, namely,

$$\Delta = \frac{1}{2}\frac{c}{L} + \frac{1}{N-1}\sum_{n=1}^{N-1}\left(-\frac{1}{2}\frac{\nu}{\epsilon_0}\right)\mathrm{Re}\left(\frac{\mathcal{P}_{n+1}}{E_{n+1}} - \frac{\mathcal{P}_n}{E_n}\right)$$
$$\cong \frac{1}{2}\frac{c}{L} + \frac{1}{2}\frac{\nu\wp^2\overline{N}}{\hbar\epsilon_0\gamma}(N-1)^{-1}$$
$$\times \sum_{n=1}^{N-1}\int_{-\infty}^{\infty} d\omega\, W(\omega)\frac{\omega - \nu_n}{\gamma}\frac{\mathfrak{L}(\omega - \nu_n)}{1+S_h}. \tag{73}$$

VI. ONE- AND TWO-MODE OPERATION

For single-mode operation, there are no population pulsations ($d_{k\neq 0} = 0$), and the appropriate complex polarization $\mathcal{P}_n$ is a special case of the REA term (70) in which the saturation (64) derives from mode n alone:

$$\mathcal{P}_n = -i\wp^2\overline{N}\hbar^{-1}E_n\int_{-\infty}^{\infty} d\omega \frac{W(\omega)\mathfrak{D}(\omega - \nu_n)}{1+2(\gamma_{ab}/\gamma)I_n\mathfrak{L}(\omega - \nu_n)}. \tag{74}$$

Combining this with the amplitude equation (5), we find

$$\dot{E}_n = g_n E_n - \tfrac{1}{2}(\nu/Q_n)E_n, \tag{75}$$

where the (saturated) gain of the medium is given by

$$g_n = \frac{1}{2}\frac{\nu\wp^2\overline{N}}{\hbar\epsilon_0\gamma}\int_{-\infty}^{\infty} d\omega \frac{W(\omega)\mathfrak{L}(\omega - \nu_n)}{1+2(\gamma_{ab}/\gamma)I_n\mathfrak{L}(\omega - \nu_n)}. \tag{76}$$

In particular for homogeneous broadening, the integral over $W(\omega)$ is trivial and the gain (76) is given by

$$g_n = \frac{1}{2}\frac{(\nu\wp^2\overline{N}/\hbar\epsilon_0\gamma)\mathfrak{L}(\omega - \nu_n)}{1+2(\gamma_{ab}/\gamma)I_n\mathfrak{L}(\omega - \nu_n)}. \tag{77}$$

These results are well known. See, for example, the book by Yariv[14] in which a slightly different notation is used.

We obtain a threshold condition by setting the unsaturated ($I_n = 0$), central tuned [$\nu = \omega_0$, the line center of $W(\omega)$] gain equal to the loss:

$$g_n(I_n = 0, \nu = \omega_0) = \tfrac{1}{2}\nu/Q_n. \tag{78}$$

With (76), this becomes

$$\frac{1}{Q_n} = \frac{\wp^2\overline{N}_T}{\hbar\epsilon_0\gamma}\int_{-\infty}^{\infty} d\omega\, W(\omega)\mathfrak{L}(\omega - \omega_0). \tag{79}$$

We choose the Gaussian inhomogeneity distribution

$$W(\omega) = (\sqrt{\pi}\,\Delta\omega)^{-1}\exp[-(\omega - \omega_0)^2/(\Delta\omega)^2] \tag{80}$$

for the remainder of this work. The threshold condition (78) with (80) can be written in terms of the plasma dispersion function,

$$Z(\gamma + i(\omega_0 - \nu)) = iK\pi^{-1/2}\int_{-\infty}^{\infty} dv\left(\frac{\exp[-(v/u)^2]}{\gamma + i(\omega_0 - \nu) + iKv}\right), \tag{81}$$

through the coordinate transformation $\omega - \omega_0 = Kv$ and the relation $\Delta\omega = Ku$. The function (81) is then written

$$Z(\gamma + i(\omega_0 - \nu)) = i\pi^{-1/2}\int_{-\infty}^{\infty} d\omega\left(\frac{\exp[-(\omega - \omega_0)^2/(\Delta\omega)^2]}{\gamma + i(\omega - \nu)}\right). \tag{82}$$

Combining (79), (80), and (82), we have the threshold condition

$$1/Q = \wp^2\overline{N}_T Z_i(\gamma)/\hbar\epsilon_0\Delta\omega. \tag{83}$$

Furthermore, the polarization (74) can be integrated by partial fractions to yield

$$\mathcal{P}_n(t) = -(\wp^2/\hbar\Delta\omega)\bar{N}E_n[Z_r(\gamma'+i\omega-i\nu_n) + i(\gamma/\gamma')Z_i(\gamma'+i\omega-i\nu_n)]\,, \tag{84}$$

where the *power*-broadened decay constant is

$$\gamma' = \gamma[1+2(\gamma_{ab}/\gamma)I_n]^{1/2}\,, \tag{85}$$

and where Z_r and Z_i are the real and imaginary parts of Z.

It is inconvenient to express the gain (76) and especially the general coefficient $\mathcal{P}_n$ in terms of this threshold condition. Specifically, we define the relative excitation

$$\mathfrak{N} = \bar{N}/\bar{N}_T = \bar{N}\wp^2 QZ_i(\gamma)/\hbar\epsilon_0\Delta\omega\,. \tag{86}$$

The gain (76) with distribution (80) is then

$$g_n = \tfrac{1}{2}\nu\mathfrak{N}[QZ_i(\gamma)]^{-1}(\gamma/\gamma')Z_i(\gamma'+i\omega-i\nu_n)\,. \tag{87}$$

Similarly, the general coefficient (50) is given by

$$\mathcal{P}_n = -i\epsilon_0\mathfrak{N}[Q\sqrt{\pi}Z_i(\gamma)]^{-1}\sum_m E_m\exp[i(\phi_n-\phi_m)]\int_{-\infty}^{\infty}d\omega\exp\left(\frac{-(\omega-\omega_0)^2}{(\Delta\omega)^2}\right)\mathfrak{D}(\omega-\nu_n)d_{m-n}\,. \tag{88}$$

The frequency-determining equation (6) yields, with (84) and (86),

$$\nu_n+\dot{\phi}_n = \Omega_n + \tfrac{1}{2}\nu\mathfrak{N}[QZ_i(\gamma)]^{-1}Z_r(\gamma'+i\omega-i\nu_n)\,. \tag{89}$$

The two-mode case has considerable formal similarity with the Doppler-broadened standing-wave and ring lasers. The theories for these are given in the Appendix. Neither problem there contains the phase ϕ_n. Our two-mode case does not either, for the phases invariably cancel out. In fact, the coefficients (57) $c_{1,k}\propto\mathcal{E}_1^*\mathcal{E}_2$, $c_{-1,k}\propto\mathcal{E}_1\mathcal{E}_2^*$, and $c_{0,k}$ have no phase dependence. The two-mode ratio r_k [Eq. (68)] is, in turn, proportional to $c_{-1,k}$ and within (68) is always multiplied by $c_{1,k-1}$ so that the phase difference factor $\exp[i(\phi_2-\phi_1)]$ cancels out. Furthermore, the only other place r_k appears is as r_1 in the d_1 and d_{-1} of (86), where the phase factors again cancel. Hence we set the products $\mathcal{E}_1^*\mathcal{E}_2$ and $\mathcal{E}_1\mathcal{E}_2^*$ both equal to $(I_1I_2)^{1/2}$ with no loss of generality. This simplification does not occur, of course, in higher-mode operation, for nonzero relative phase angles (13) exist.

The two-mode polarizations $\mathcal{P}_1$ and $\mathcal{P}_2$ from (88) have the explicit values

$$\mathcal{P}_1 = -i\epsilon_0\mathfrak{N}[Q\sqrt{\pi}Z_i(\gamma)]^{-1}\times\int_{-\infty}^{\infty}d\omega\exp\left(\frac{-(\omega-\omega_0)^2}{(\Delta\omega)^2}\right)\mathfrak{D}(\omega-\nu_1)(E_1d_0+E_2d_1)\,, \tag{90}$$

$$\mathcal{P}_2 = -i\epsilon_0\mathfrak{N}[Q\sqrt{\pi}Z_i(\gamma)]^{-1}\times\int_{-\infty}^{\infty}d\omega\exp\left(\frac{-(\omega-\omega_0)^2}{(\Delta\omega)^2}\right)\mathfrak{D}(\omega-\nu_2)(E_1d_{-1}+E_2d_0)\,,$$

where the dc population coefficient is given by (63),

$$d_0 = 1/(1+S_h+S_p)\,, \tag{91}$$

with the hole-burning contribution

$$S_h = 2(\gamma_{ab}/\gamma)[I_1\mathfrak{L}(\omega-\nu_1)+I_2\mathfrak{L}(\omega-\nu_2)] \tag{92}$$

and the population-pulsation contribution

$$S_p = c_{1,0}r_1 + \text{c.c.} \tag{93}$$

Here the ratio r_1 is given by (68) as

$$r_1 = -\frac{c_{-1,1}}{c_{01}+c_{11}r_2} = -\frac{c_{-1,1}}{c_{01}-c_{11}c_{-1,2}/(c_{02}+c_{12}r_3)}\,,\ \text{etc.}\,, \tag{94}$$

where the coefficients (57) have the simple form

$$c_{-1,k} = \gamma_{ab}(I_1I_2)^{1/2}[\mathfrak{D}(\nu_{1+k}-\omega)+\mathfrak{D}(\omega-\nu_{2-k})]\,, \tag{95a}$$

$$c_{1,k} = \gamma_{ab}(I_1I_2)^{1/2}[\mathfrak{D}(\nu_{2+k}-\omega)+\mathfrak{D}(\omega-\nu_{1-k})]\,, \tag{95b}$$

$$c_{0,k\neq0} = [\mathfrak{F}(k\Delta)]^{-1}+\gamma_{ab}\sum_{n=1}^{2}I_n[\mathfrak{D}(\nu_{n+k}-\omega)+\mathfrak{D}(\omega-\nu_{n-k})]\,. \tag{95c}$$

The population coefficients $d_1 = d_{-1}^* = d_0r_1$.

Appropriate transformations of the fundamental population-pulsation frequency (Δ) and the Doppler-shifted mode frequencies allow these two-mode equations to be used for the two-mirror single-mode standing-wave laser and the bidirectional ring laser. These transformations are summarized in Table I. For the standing-wave laser we have the additional requirements

$$E = E_1+E_2,\ (I_1 = I_2 = \tfrac{1}{4}I)\,,$$

$$\mathcal{P} = \mathcal{P}_1+\mathcal{P}_2\,,$$

$$\theta_1\to\theta_1+\tfrac{1}{2}\pi\,,\quad \theta_2\to\theta_2-\tfrac{1}{2}\pi\,.$$

TABLE I. Summary of the parameters Δ and ν for the two-mode cases.

	Unidirectional	Standing wave	Bidirectional
Δ	$\nu_2-\nu_1$	$2Kv$	$\nu_--\nu_++2Kv$
ν_1	ν_1-Kv	$\nu-Kv$	ν_+-Kv
ν_2	ν_2-Kv	$\nu+Kv$	ν_-+Kv

For the bidirectional ring it has been shown, for example, by Menegozzi and Lamb[6] that the effect of small rotation rates on the self-consistency equations is to introduce a shift in the empty-cavity resonance frequencies $\Omega_n = |Kn|c$ by an amount $2A\dot{\theta}\Omega_n/Lc$, where A is the area enclosed by the laser path and $\dot{\theta}$ is the laser rotation rate. Traveling-wave modes are up (ν_-) or down (ν_+) shifted depending on whether they are going in the opposite or same direction as the cavity rotation. Hence in the self-consistency equations,

$$\Omega_n \to \Omega_n - (2TA\dot{\theta}\,\Omega_n/Lc)\mathrm{sgn}K_n = \Omega_n - \tfrac{1}{2}\Delta_r \mathrm{sgn}K_n ,$$

where T is the number of turns if a multiturn loop is used.

VII. NUMERICAL ANALYSIS

Inasmuch as the medium has finite bandwidth, the population-pulsation coefficients d_k must converge to zero as k approaches infinity, allowing the recursion relation (56) to be truncated for sufficiently large k. Alternatively, especially for two-mode operation, the ratios $r_l = d_l/d_{l-1}$ must converge to a value less than unity in this limit. We can truncate the continued fraction for some appropriately large index which we call n^*. For analytic approximations and phenomena such as various types of saturation spectroscopy[15] n^* may be as small as 1, e.g., the case of a "nonsaturating" probe in the presence of a saturating beam.

While we use the continued fraction for the two-mode cases, we have found it inconvenient for the multimode case. Instead the recursion relation (56) may be solved by an iterative method using the d's rather than the r's. We have in addition the exact expression for d_0 in terms of S_h and S_p given by Eq. (63), written here in terms of the d's,

$$d_0 = \left[\, 1 + S_h + \sum_{j=1}^{N-1}\left(\frac{c_{j,0}d_j}{d_0} + \mathrm{c.c.}\right)\right]^{-1} . \qquad (96)$$

Initially a guess value of 1.0 is assumed for d_0 and zero for the remaining d_k. d_0 is calculated from the exact equation (96) and used in the evaluation of the recursion relation with $k=1$. This yields, in turn, a value for d_1 which is used with the previous value of d_0 to obtain the remaining d's appearing in Eq. (96). These new values for the d's and the old value of d_0 are used to compute a new value for d_0 [Eq. (96)]. Then the recursion relation is evaluated for d_1, etc. Using new values of the d's, the process is iterated, each time allowing the value of k to increase by one, thus evaluating one higher-order d_k. The iteration is terminated when a check of the recursion relation (56), normalized to d_0 for the d's used in the polarization calculations (50), is satisfied to the desired accuracy, typically one part in 10^5 for the three-mode case.

We desire the solution to the coupled set of equations given by steady-state solutions of (5) and (11) ($\dot{E}_n = \dot{\psi}_n = 0$). We use a multimode Newton-Raphson procedure which is based on a first-order Taylor-series approximation. Specifically, we seek to establish values a_i for each of the variables x_i (these are the E_n and ψ_n) which yield a desired set of values for M functions f_i (these are the derivatives $\dot{E}_n$ and $\dot{\psi}_n$). The desired values for the functions are zero (steady state). Assuming the f_i can be expanded in a Taylor series about the a_i one has

$$f_i(x_1, \ldots, x_M) = f_i(a_1, \ldots, a_M) + \sum_{j=1}^{M} F_{ij}(x_j - a_j) + \cdots , \qquad (97)$$

where

$$F_{ij} = \frac{\partial f_i(x_1 \cdots x_M)}{\partial x_j}.$$

Truncating the series with the first-order terms, one inverts (96) to find approximate values for the a_j,

$$a_j = x_j - \sum_{j=1}^{M} (F^{-1})_{ji}[f_i(a_1 \cdots a_M) - f_i(x_1 \cdots x_M)] . \qquad (98)$$

When possible, it is desirable to calculate the derivatives exactly. In our case, we must find them numerically. A simple set of guess values for the a_i consists of the E's determined from a REA and 0 for the relative phase angles. Taking these values for the x_i, we use (97) to find better values for the a_i. We iterate this procedure until the f's are sufficiently close to the desired values (for us, 0).

In addition to the Newton-Raphson zeroing procedure for the intensity and relative phase equations, the condition of mode locking must be met, that is, the modes oscillating in the laser must be equally spaced in frequency. To this end we calculate the pulled oscillation frequency from the self-consistency equations for the frequency (6) for each mode. In the unlocked case these would be used as the next trial values for the oscillating frequencies; however, in our case we must calculate a new mode separation and average pulling. The mode separation is computed as the average separation between pairs of pulled mode frequencies. The average of the frequency pulling for the individual modes is used for the frequency shift for the new set of equally spaced frequencies. In effect, a straight line is fitted to the pulled frequencies, with the condition that the resulting points have equal mode spacing. These new mode-locked frequencies are used in the next iteration during which the intensity and relative phase equations are zeroed. This process is repeated until the errors in periodic mode spacing from Eq. (6) are 1 Hz or less, or until convergence appears unlikely, that is, mode locking does not take place.

Failure of the zeroing procedure or the mode-locking iteration to converge means that either no mode-locked solution exists for the parameters chosen or that the numerical procedure cannot handle the configuration. Convergence does not imply stability, but the latter is determined immediately from the matrix of partial derivatives. In fact, using the first-order Taylor series about the points a_i,

$$f_i(a_1+\epsilon_1, \ldots, a_M+\epsilon_M)=\sum_{j=1}^{M} F_{ij}\epsilon_j . \tag{99}$$

Since the f's are the derivatives in question for a small-vibrations analysis, we see that the small deviations ϵ_j return to 0 (and hence that the solution is stable) provided the eigenvalues of the matrix F are all negative.

We generally calculate the matrix F with previous intensity and relative phase values that were used to predict the final values. Thus for a stability analysis either a new matrix F must be computed or the difference in the old and final intensities and phases must be small. Typically, in the preceding step, the equations were zeroed to within 1 part in 10^8, and the equations were mode locked to within 5–10 Hz in that step. The final values ordinarily were 10^{-10} for zeroing and 1 Hz for mode-locking. Thus, usually, the last matrix of partial derivatives obtained may be legitimately used for the stability analysis. It might be noted that the stability analysis applied with less stringent error requirements may predict unstable results when the final values give a stable solution, or vice versa. We refer the interested reader to Hambenne[16] for a more detailed discussion of the material in this section.

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APPENDIX

The single-mode standing-wave electromagnetic field has the form

$$E(z,t)=\tfrac{1}{2}E(t)e^{-i\nu t}\sin Kz+\text{c.c.} \tag{A1}$$

The self-consistency equations for the slowly varying amplitude $E(t)$ and for the optical frequency ν are

$$\dot{E}=-\tfrac{1}{2}(\nu/Q)E-\tfrac{1}{2}(\nu/\epsilon_0)\,\text{Im}(\mathcal{P}) , \tag{A2}$$

$$\nu=\Omega-\tfrac{1}{2}(\nu/\epsilon_0)\,\text{Re}(\mathcal{P})/E , \tag{A3}$$

where $\mathcal{P}$ is a complex, slowly varying polarization component (in Lamb's theory[1] $\mathcal{P}=C+iS$, where C and S are the in-phase and in-quadrature components of the atomic polarization) defined in terms of the polarization of the medium by

$$P(z,t)=\tfrac{1}{2}\mathcal{P}\exp(-i\nu t)\sin Kz+\text{c.c.} \tag{A4}$$

Our problem reduces to the calculation of the polarization component $\mathcal{P}$.

We consider a medium of two-level atoms with line center ω as depicted in Fig. 1. We describe the medium by a population matrix

$$\rho(z,v,t)=\sum_{\alpha=a,b}\int_{-\infty}^{t}dt\int_0^L dz_0\,\lambda_\alpha(z_0,v,t_0)\rho(\alpha,z,v,t,t_0)\times\delta(z-z_0-v(t-t_0)) , \tag{A5}$$

where $\rho(\alpha,z,v,t,t_0)$ is the single-atom density matrix for an atom excited at time t_0 and place z_0 to the state α with z component of velocity v. $\lambda_\alpha(z_0,v,t_0)$ is the pumping rate to the αth state and varies slowly enough to be factored outside the t_0 integration. Using the Schrödinger equation for the single-atom density matrix, one can show[12] that the corresponding equations for the components of the population matrix are just those for $\rho(z,\omega,t)$, Eqs. (27)–(30), where the time derivative is given by

$$\frac{d}{dt}=\frac{\partial}{\partial t}+v\frac{\partial}{\partial z} . \tag{A6}$$

In these equations, $\mathcal{V}_{ab}$ is given by the electric-dipole perturbation energy for the single-mode field (A1) (in the rotating-wave approximation),

$$\mathcal{V}_{ab}=-\tfrac{1}{2}\wp Ee^{-i\nu t}\sin Kz . \tag{A7}$$

The equation of motion for the population difference D of (32) is given by (38), in which we set $D(z,\omega,t')\rightarrow D(z',v,t')$, where z' is given by

$$z'=z-v(t-t') , \tag{A8}$$

and the frequency ω is replaced by the z component of velocity, v.

The solutions of Eqs. (27) and (38) written for $\rho(z,v,t)$ are given by Fourier series in terms of the position coordinate z. To understand this approach, recall in Lamb's perturbational approach that to zeroth order in the electric dipole interaction energy (A7), the population difference is given simply by $N(z,v,t)$, which contains no optical frequency time or space variations. The first-order contribution to ρ_{ab} contains the factor $\sin Kz$ corresponding to one interaction with the electric field. The second-order term for the population difference contains two interactions and hence has $\sin^2 Kz$ dependence, and the third-order term for ρ_{ab} has $\sin^3 Kz$ dependence. In general, we see that the population difference D contains even powers of e^{iKz} and ρ_{ab} contains odd powers. Thus we expand ρ_{ab} in an odd-term Fourier series

$$\rho_{ab}(z,v,t)=N(z,v,t)\sum_{m=-\infty}^{\infty} p_m(v)e^{(2m+1)iKz}e^{-i\nu t} \tag{A9}$$

and the population difference in the even-term Fourier series

$$D(z,v,t)=N(z,v,t)\sum_{k=-\infty}^{\infty} d_k(v)e^{2ikKz}. \tag{A10}$$

Here time dependence not due to excitation variation could be included in the expansion coefficients p_m and d_k. We restrict our analysis to steady state.

In terms of (A9) the polarization of the medium is given by

$$P(z,t)=\int_{-\infty}^{\infty} dv\,\rho_{ab}(z,v,t)+\text{c.c.} \tag{A11}$$

Projecting this and Eq. (A4) onto $\sin Kz$ and identifying negative frequency components, we find the complex polarization component

$$\mathcal{P}(t)=2\wp\frac{2}{L}(2i)^{-1}\int_0^L dz\,\sin Kz\int_{-\infty}^{\infty} dv\,\rho_{ab}(z,v,t). \tag{A12}$$

Substituting (A9), one has

$$\mathcal{P}(t)=4\wp(2iL)^{-1}\int_0^L dz\,(e^{iKz}-e^{-iKz})\times\sum_m e^{(2m+1)iKz}\int_{-\infty}^{\infty} dv\,p_m$$
$$=2i\wp\int_{-\infty}^{\infty} dv\,(p_0-p_{-1}). \tag{A13}$$

Hence our problem reduces to finding p_0 and p_{-1}.

Substituting (A9) and (A10) into (27) for ρ_{ab} and using the time derivative (A6), we find

$$N\sum_m p_m[(2m+1)iKv-i\nu]e^{(2m+1)iKz}e^{-i\nu t}=-(i\omega+\gamma)N\sum_m p_m e^{(2m+1)iKz}e^{-i\nu t}$$
$$+\tfrac{1}{2}iN(\wp E/\hbar)e^{-i\nu t}(2i)^{-1}(e^{-iKz}-e^{iKz})\sum_k d_k e^{2kiKz}.$$

Equating coefficients of $e^{(2m+1)iKz}$, we have

$$p_m=\tfrac{1}{4}(\wp E/\hbar)\mathfrak{D}[(2m+1)Kv+\omega-\nu](d_{m+1}-d_m). \tag{A14}$$

Similarly substituting (A9) and (A10) into (38) for D, we have

$$N\sum_k d_k(2kiKv)e^{2kiKz}=-\gamma_{ab}N\sum_k d_k e^{2kiKz}+\frac{\gamma_a\gamma_b}{\gamma_{ab}}N+\tfrac{1}{4}(\gamma_a-\gamma_b)^2N\sum_k\mathfrak{D}_{ab}(2kKv)e^{2kiKz}$$
$$+2iN\frac{\frac{1}{2}\wp E}{\hbar\,2i}(e^{iKz}-e^{-iKz})\sum_m(p_m^*e^{-(2m+1)iKz}-p_m e^{(2m+1)iKz}).$$

Equating coefficients of e^{2kiKz}, we have

$$d_k[2kiKv+\gamma_{ab}-\tfrac{1}{4}(\gamma_a-\gamma_b)^2\mathfrak{D}_{ab}(2kKv)]$$
$$=(\gamma_a\gamma_b/\gamma_{ab})\delta_{k,0}+\tfrac{1}{2}(\wp E/\hbar)(p_{-k}^*-p_{-k-1}^*+p_k-p_{k-1})$$

or

$$d_k=\delta_{k,0}+\wp E(\hbar\gamma_a\gamma_b)^{-1}\gamma_{ab}\mathfrak{F}(2kKv)(p_{-k}^*-p_{-k-1}^*+p_k-p_{k-1}), \tag{A15}$$

where the complex factor

$$\mathfrak{F}(2kKv)=\tfrac{1}{2}[\mathfrak{D}_a(2kKv)+\mathfrak{D}_b(2kKv)](\gamma_a\gamma_b/\gamma_{ab}). \tag{A16}$$

Equation (A15) leads to a two-mode interaction in which we identify

$$\nu_1=\nu-Kv,\quad \nu_2=\nu+Kv,\quad \Delta=2Kv. \tag{A17}$$

We further have that $\nu_{1-k}=\nu_1-k\Delta$, etc. Then Eq. (A14) is written

$$p_m=\tfrac{1}{4}(\wp E/\hbar)\mathfrak{D}(\omega-\nu_{1-m})(d_{m+1}-d_m). \tag{A18}$$

In terms of this, the complex polarization component (A13) is then

$$\mathcal{P}(t) = -\tfrac{1}{2}i\wp^2 E\bar{N}(\hbar\sqrt{\pi}u)^{-1}\int_{-\infty}^{\infty} dv\, e^{-(v/u)^2}[\mathfrak{D}(\omega-\nu_1)(d_0-d_1)+\mathfrak{D}(\omega-\nu_2)(d_0-d_{-1})]\,, \tag{A19}$$

or in terms of the relative excitation $\mathfrak{N}$ of (61),

$$\mathcal{P}(t) = -\tfrac{1}{2}i\epsilon_0\mathfrak{N}[QZ_i(\gamma)\sqrt{\pi}\,]^{-1}I^{1/2}\int_{-\infty}^{\infty} dv\, e^{-(v/u)^2}[\mathfrak{D}(\omega-\nu_1)(d_0-d_1)-\mathfrak{D}(\omega-\nu_2)(d_{-1}-d_0)]\,. \tag{A20}$$

Substituting (A18) into (A15), we eliminate p_m:

$$d_k = \delta_{k,0}+\tfrac{1}{4}\gamma_{ab}I\mathfrak{F}(k\Delta)[\mathfrak{D}(\nu_{1+k}-\omega)(d_{k-1}-d_k)-\mathfrak{D}(\nu_{2+k}-\omega)(d_k-d_{k+1})+\mathfrak{D}(\omega-\nu_{1-k})(d_{k+1}-d_k)-\mathfrak{D}(\omega-\nu_{2-k})(d_k-d_{k-1})]\,,$$

where the dimensionless intensity is

$$I=\tfrac{1}{2}(\wp E)^2/\hbar^2\gamma_a\gamma_b\,. \tag{A21}$$

This becomes

$$d_k = \delta_{k,0}+\tfrac{1}{4}I\gamma_{ab}\mathfrak{F}(k\Delta)\Big(d_{k-1}[\mathfrak{D}(\nu_{1+k}-\omega)+\mathfrak{D}(\omega-\nu_{2-k})] - d_k\sum_{n=1}^{2}[\mathfrak{D}(\nu_{n+k}-\omega)+\mathfrak{D}(\omega-\nu_{n-k})]+d_{k+1}[\mathfrak{D}(\nu_{2+k}-\omega)+\mathfrak{D}(\omega-\nu_{1-k})]\Big)\,. \tag{A22}$$

This yields the recurrence relation for $d_{k>1}$,

$$c_{1,k}d_{k+1}+c_{0,k}d_k+c_{-1,k}d_{k-1}=0\,,$$

where the coefficients are defined as

$$c_{-1,k}=\gamma_{ab}I[\mathfrak{D}(\nu_{1+k}-\omega)+\mathfrak{D}(\omega-\nu_{2-k})]\,, \tag{A23a}$$

$$c_{0,k}=-4[\mathfrak{F}(k\Delta)]^{-1}-\gamma_{ab}I\sum_{n=1}^{2}[\mathfrak{D}(\nu_{n+k}-\omega)+\mathfrak{D}(\omega-\nu_{n-k})]\,, \tag{A23b}$$

$$c_{1,k}=\gamma_{ab}I[\mathfrak{D}(\nu_{2+k}-\omega)+\mathfrak{D}(\omega-\nu_{1-k})]\,. \tag{A23c}$$

These are the same as the recursion relation (36) and coefficients (71) for the two-mode unidirectional ring laser with $I_1=I_2$, except that $c_{0,k}$ contains minus signs and a factor of 4. Note that the minus signs can be uniformly canceled throughout the continued fraction, yielding an overall minus sign. This sign, in turn, cancels that for $-d_{\pm 1}$ in (A19), so that Eq. (A19) has the same form as the unidirectional, two-mode versions of Eq. (90).

Now consider a ring laser with the two oppositely directed running waves given by the field

$$E(z,t)=\tfrac{1}{2}E_+\exp[-i(\nu_+t+\phi_+-Kz)]+E_-\exp[-i(\nu_-t+\phi_-+Kz)]+\text{c.c.} \tag{A24}$$

We describe the atoms by the Doppler-broadened population matrix (A5) as for the standing-wave case. Our Fourier expansion for the population difference is almost the same as (A10) for the standing-wave case, but must include the possible frequency difference between the two running waves, that is,

$$D(z,v,t)=N(z,v,t)\sum_k d_k\exp[ik(2Kz-\psi)]\,, \tag{A25}$$

where the relative phase angle is

$$\psi=(\nu_+-\nu_-)t+\phi_+-\phi_-\,. \tag{A26}$$

The off-diagonal element ρ_{ab} is similarly expanded as

$$\rho_{ab}=N(z,v,t)\exp[-i(\nu_+t+\phi_+-Kz)]\times\sum_m p_m\exp[im(2Kz-\psi)]\,. \tag{A27}$$

Substituting (A27) and (A25) into the equations of motion and performing the appropriate projections, we find

$$p_m=-\tfrac{1}{2}i(\wp/\hbar)\mathfrak{D}(m\Delta+\omega-\nu_1)(E_+d_m+E_-d_{m+1})\,, \tag{A28}$$

where the frequencies

$$\nu_1=\nu_+-Kv\,, \tag{A29}$$

$$\nu_2=\nu_-+Kv\,, \tag{A30}$$

$$\Delta=\nu_2-\nu_1=\nu_--\nu_++2Kv\,. \tag{A31}$$

Similarly we find

$$d_k=\delta_{k,0}+i(\wp/\hbar)\gamma_{ab}(\gamma_a\gamma_b)^{-1}\mathfrak{F}(k\Delta)\times[E_+p_{-k}^*+E_-p_{-k-1}^*-E_+p_k-E_-p_{-k-1}]\,,$$

where $\mathfrak{F}(k\Delta)$ is given by (52). Combining this with (A28), we obtain the recurrence relation

$$d_k = \delta_{k,0} - \gamma_{ab}\mathfrak{F}(k\Delta)\Big(d_{k-1}(I_+I_-)^{1/2}[\mathfrak{D}(\nu_{1+k}-\omega)+\mathfrak{D}(\omega-\nu_{2-k})] + d_k\sum_{n=1}^{2} I_n[\mathfrak{D}(\nu_{n+k}-\omega)+\mathfrak{D}(\omega-\nu_{n-k})]$$
$$+ d_{k+1}(I_+I_-)^{1/2}[\mathfrak{D}(\nu_{2+k}-\omega)+\mathfrak{D}(\omega-\nu_{1-k})]\Big)\,. \tag{A32}$$

This is the same as the two-mode case of our multimode result (55). The physics, of course, is different, and this fact is reflected in the way in which the frequencies (A29)–(A31) are evaluated in the integral over the inhomogeneous medium. The length of the polarization calculation on the computer is the same in either case.

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