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Quantum theory of a swept-gain amplifier. II *

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Small-signal–pulse growth in a homogeneously broadened amplifier with swept gain is discussed from a quantum-mechanical and a semiclassical treatment. The regime in which Beer's law fails is discussed analytically. There we find nonexponential growth and the formation of steady states in the small-signal regime.

I. INTRODUCTION

In the past, the subject of small-signal growth in a laser amplifier has received extensive investigation for both pulsed[1,2] and continuous pumping[3,4] (cw) modes of operation. These calculations are usually carried out in the Fourier domain, and the growth is characterized by Beer's law.[3,4] The condition of validity for Beer's law is that, in the absence of saturation, the combined statistical plus quantum-mechanical averages of the population inversion must vary slowly in time compared to the inverse bandwidth of the amplifier. For lasers as they are currently operated, this condition applies. This is due to the manner in which laser action is usually achieved, namely that once a pump mechanism is found, it is scaled up until the gain is high enough for the medium to lase.

In a recently proposed x-ray laser scheme[5], a different approach was outlined. There the pump was fixed, and the gain was increased by removing all the extraneous effects, such as Doppler broadening, that make the gain small. In the limit of highest possible gain per unit population inversion, i.e., with only natural broadening and instantaneous excitation of all atoms at a time $t = z/c$ ("swept gain"), the conditions of validity for Beer's law no longer apply. As a result, one sees anomalous behavior in the small signal regime. These anomalies were found in previous studies[6,7] that used numerical techniques to investigate this problem.

In this paper we present an analytical discussion of the growth of pulses in the small-signal regime of a collisionless homogeneously broadened amplifier. This discussion applies in principle to general level decays and to the possibility of continuous pumping,[7] and allows a discussion of both the normal and anomalous regimes of pulse growth.[8] The method could be further generalized to include collisions, but the analytical procedure breaks down if one tries to include Doppler broadening. The development also applies to both semiclassical and quantum descriptions of the problem.

In Sec. II we discuss the equations and their solution. We adopt the policy of giving only a limited discussion of special cases in order to get to the answer as expeditiously as possible. The methods of generalization of the equations are discussed extensively in the literature,[7,9] and the generalization of the solution is then straightforward. In Sec. III we choose a particular example to demonstrate the most important anomalies, namely: (a) that the pulse grows as $\exp[(8gz)^{1/2}]$ rather than the usual $\exp(gz)$. Thus the growth rate of the pulse vanishes asymptotically; (b) when one includes a finite loss, the semiclassical description (i.e., a buildup from an initial pulse rather than internal sources) predicts that the pulse vanishes in the limit $z \to \infty$ independently of the value of the gain; (c) a quantum description (i.e., a buildup from internal sources) leads to the prediction of steady states in the small-signal regime of the amplifier.

In the Appendix we show that the normal behavior is recovered from our description in the limits where one expects it to be valid.

II. FORMAL DEVELOPMENT

A. Basic model

In a "swept gain" scheme, atoms are created sequentially in an excited state; that is, at location z on the amplifier (z) axis, an atom is created in an excited state at time $t = z/c$. We describe each atom as a two-level system, and discuss here the special case in which the decay goes from the upper to the lower level. Generalizations to other cases are mentioned at the end of Sec. II B.

Mathematically, the problem of the buildup of an electromagnetic pulse in such an amplifier reduces to an analysis of a linearized version of the following equations[10]:

$$\frac{\partial}{\partial \mu} S^- = -\frac{\gamma}{2} S^- - \frac{2i\wp}{\hbar} S_3(\mathcal{E}_F^+ + \mathcal{E}_R^+),$$
$$\frac{\partial}{\partial \mu} S_3 = -\gamma(\mathfrak{N} + S_3) + \frac{i\wp}{\hbar}[S^+(\mathcal{E}_F^+ + \mathcal{E}_R^+) - (\mathcal{E}_F^- + \mathcal{E}_R^-)S^-], \quad (1)$$
$$\frac{\partial}{\partial z} \mathcal{E}_R^+ = 2\pi i \frac{\omega}{c} \wp S^- - \kappa \mathcal{E}_R^+.$$

Here $\mu = t - z/c$ denotes a retarded time, γ is the Wigner-Weisskopf decay constant, κ represents losses due to scattering and other sources of unsaturable absorption, $\wp$ is the dipole matrix element, $\mathfrak{N}$ is the density of initially excited atoms, and ω may be considered as either the field or the atomic frequency since by hypothesis we only treat homogeneous broadening. $S^{\pm}$ and S_3 are related to the single-atom spin operators (denoted as σ_j) by

$$S^{\pm} = \sum_j \sigma_j^{\pm} e^{\pm i\omega\mu} \delta(z - z_j),$$

with an analogous expression for S_3 which does not contain the term $\exp(i\omega\mu)$. Here z_j denotes the location of the jth atom.

In this paper our primary goal is to compute to a good approximation the field intensity along the amplifier axis as it builds up from the vacuum. For convenience we work with a quantized field, denoting the vacuum state by $|\Phi\rangle$. We regard the operator equations (1) as being approximate Heisenberg equations of motion in the following sense: Namely, we expect to obtain to good approximation the same expectation values in the regime of interest that we would have obtained using the unapproximated Heisenberg equations. In other words, if $E^+ + E^-$ is the full field operator and $\mathcal{E}^{\pm} \equiv \mathcal{E}_F^{\pm} + \mathcal{E}_R^{\pm}$,

$$\langle \Phi | E^- E^+ | \Phi \rangle \simeq \langle \Phi | \mathcal{E}^- \mathcal{E}^+ | \Phi \rangle \quad (2)$$

along the axis of the amplifier.

At this point we make a few remarks concerning our approximations.

(a) We emphasize that the validity of the approximation is guaranteed only in the sense of (2). For example, the plane-wave description, which does violence to the isotropic nature of spontaneous emission, would surely be invalid away from the amplifier axis.

(b) Although we have tailored our approximations directly toward this propagation problem, a heuristic (but systematic) derivation shows that appropriate modifications permit analysis of quite different physical situations.[10]

(c) We have written $\mathcal{E}^{\pm} = \mathcal{E}_F^{\pm} + \mathcal{E}_R^{\pm}$ in order to indicate the manner in which decays have been introduced. $\mathcal{E}_F^{\pm}$ denotes the slowly varying (in time) amplitude of the free field operator. The "radiation field" $\mathcal{E}_R^{\pm}$, which arises from the atomic sources, is responsible for the amplification process. $\mathcal{E}_F^{\pm}$ plays the role of a Langevin noise source and is fundamentally related to the decays by the fluctuation-dissipation theorem. (A similar noise source associated with κ has been left out since it plays no role here.)

(d) The process of handling the decays has the effect of removing the singularities at $z = z_j$, which are implied by the definition of $S^{\pm}$ and S_3. From now on the variables are regarded as continuous functions of z.

(e) Equation (1) is very similar to a semiclassical system; to obtain the latter, take expectation values and assume all products factor; schematically $\langle S\mathcal{E}\rangle = \langle S\rangle \langle\mathcal{E}\rangle$. In the standard usage, semiclassical theory describes the buildup from an incident pulse with no internal noise. One then needs a state $|\psi\rangle$ that is different from the vacuum, and an incident field that is given by $\langle\psi|\mathcal{E}_F|\psi\rangle$.

B. Linearization of the equations

The development of the equations in the small-signal regime is obtained in a straightforward manner by analogy with the standard perturbative discussion as given by Lamb.[9] We remark only about a few points in the logical development where the operator and semiclassical derivations differ. First, the free operators considerably complicate the discussion. In the present case we take the state of the system to be the vacuum in so far as the field is concerned, i.e., $\mathcal{E}_F^+|\phi\rangle = 0$, so that terms containing $\mathcal{E}_F^+$ do not contribute to the answer and we can ignore them for this discussion. To zeroth order, the solutions are found by discarding *all* terms in (1) that contain the field operators and by solving the resulting equations. To find the next order, one substitutes the zeroth-order result into the terms containing $\mathcal{E}_R^+$ and writes the resulting inhomogeneous equation in integral form. This is now sufficient to give the small-signal growth, so the iteration process is terminated at this point. The result for S^- is

$$S^-(\mu, z) \simeq S^-(0, z) e^{-\gamma\mu/2} - \frac{2i\wp}{\hbar} \int_0^{\mu} d\mu' e^{-\gamma(\mu-\mu')/2} \times S_3^0(\mu', z)\mathcal{E}_R^+(\mu', z), \quad (3)$$

where S_3^0 is the zeroth-order approximation for S_3. The final step in linearizing the operator equations involves treating $S_3^0(\mu, z)$ as a c number. This is a standard step which is obvious on physical grounds but which has received little discussion in the literature. At this stage it is convenient to particularize to the case of a spatially homogeneous medium so that S_3^0 is no longer a function of z. One then substitutes Eq. (3) into (1) to obtain

$$\left(\frac{\partial}{\partial z}+\kappa\right)\mathcal{E}_R^+(\mu, z)=\alpha'\int_0^\mu d\mu'\, e^{-(\mu-\mu')/T_2}n(\mu')\times\mathcal{E}_R^+(\mu', z)+\frac{2\pi i\omega}{c}\,\wp e^{-\mu/T_2}S^-(0,z). \quad (4)$$

Equation (4) applies to more general situations than the derivation indicates. No matter what type of decay is under consideration, the function $n(\mu')$ represents a statistical and quantum average small signal population inversion (normalized to unity at its maximum). The decay constant $\gamma/2$ has been replaced by $1/T_2$, where T_2 represents the general decay time of the polarization.[1,2] The explicit source term [the one containing $S^-(0,z)$] is unaffected by considering different decay schemes. Modifications that arise from including an extended (noninstantaneous) excitation are also straightforward and given in Refs. 7 and 9.

C. Connection with semiclassical theory

The procedure for reducing these linearized operator equations to semiclassical equations is straightforward. The details are found in Ref. 7. Here we emphasize the physical interpretation of the reduction.

By definition the operator $\mathcal{E}_R(\mu, z)$ vanishes on the boundary $z=0$. Therefore, the only boundary conditions that contribute to $\mathcal{E}_R$ come from the finite source terms at $\mu=0$. Moreover, since the operator equation (4) is linear, its solution can be written as a linear combination of these source operators $S^-(0,z)$,

$$\mathcal{E}_R^+(\mu, z)=\int_0^z dz'\, G(\mu, z-z')S^-(0,z). \quad (5)$$

Substituting this form into Eq. (4), one sees that G satisfies

$$\frac{\partial G}{\partial z}=\alpha'\int_0^\mu d\mu'\, e^{-(\mu-\mu')/T_2}n(\mu')G(\mu', z-z')-\kappa G(\mu, z-z'), \quad (6)$$

with the initial condition

$$G(\mu, 0)=G_0\, e^{-\mu/T_2}, \quad (7)$$

where G_0 is a coefficient which measures the magnitude of the spontaneous emission. This homogeneous equation is precisely the same as the semiclassical equation, whose derivation was sketched in Sec. II A. The only difference between G and the semiclassical field (other than dimensions) is the restriction to the particular initial data in (7) which is appropriate for spontaneous emission.

In order to establish this relationship more closely, it is useful to see how one constructs answers from G. In the quantized description, the intensity is given as

$$\langle I(\mu, z)\rangle=\langle \mathcal{E}^-(\mu, z)\mathcal{E}^+(\mu, z)\rangle \quad (8)$$

or

$$\langle I(\mu, z)\rangle=\int_0^z\int_0^z dz'dz''\, G(\mu, z-z')G^*(\mu, z-z'')\times\langle S^+(z')S^-(z'')\rangle. \quad (9)$$

If, as is the normal case, the atoms are prepared randomly, then

$$\langle S^+(z)S^-(z')\rangle=\mathfrak{N}|a|^2\delta(z-z'), \quad (10)$$

where $|a|^2$ is the population in the upper state. Then, using (1) and transforming to $z''=z-z'$ gives

$$\langle I(\mu, z)\rangle=\mathfrak{N}|a|^2\int_0^z dz''\, |G(\mu, z'')|^2. \quad (11)$$

Thus, the picture given by the quantum theory is straightforward. Since we have restricted ourselves to the small-signal regime of the amplifier, the equations are linear in the electric field. In this limit the superposition principle is valid, and one can calculate the amplified spontaneous emission by treating each source separately from all the others. Each atom, labeled by its position z', emits a coherent pulse [see Eq. (7)] which is amplified by stimulated emission. This amplification is described semiclassically. Since the sources are statistically independent of each other, the final intensity is an incoherent superposition of the semiclassical solutions, which is described by Eq. (11).

If we call I_{sc} the semiclassical intensity [given, within a constant, by $|G(\mu, z)|^2$], the two theoretical approaches are related by

$$\langle I(\mu, z)\rangle\propto\int_0^z dz''\, I_{sc}(\mu, z''). \quad (12)$$

In the discussion in Sec. III we will use the subscripting in this form to identify and compare the semiclassical and quantum-mechanical results. One final point worth mentioning here is that this quantum-mechanical formulation is equivalent to a semiclassical description of spontaneous emission as a stochastic random function provided the stochastic characteristic of the source is chosen correctly. Precisely, consider Eq. (4) as a stochastic differential equation for a random classical field $\mathcal{E}_R(\mu, z;\Omega)$ (rather than as an operator equation). In this interpretation the randomness is introduced through the source $S^-(0,z;\Omega)$ which is treated as a prescribed random function with zero mean and the first-order correlation function given by

$$\langle S(0,z)S^*(0,z')\rangle \equiv \int P(\Omega)[S(0,z;\Omega)S^*(0,z;\Omega)]\,d\Omega = \mathfrak{N}|a|^2\delta(z-z').$$

Then, the mean intensity of the stochastic field $\mathcal{E}_R$ is given by (11), just as in the quantum-mechanical formulation. The advantage of the quantum-mechanical approach is in setting up the problem, not in the solution of the final equations. Finally, for clarity we remark that the "sc" subscripting, such as in Eq. (12), denotes deterministic semiclassical theory and not the stochastic theory.

D. A representation of the Green's function G

In this section we derive an explicit representation of the Green's function $G(\mu,z)$ which, as we have seen, will be used in both the quantized and the semiclassical approaches. The function G is defined by Eq. (6), together with a prescribed boundary condition at $z=0$ as given by Eq. (7). [Here we seek G only in the first quadrant of the (μ,z) plane; it is chosen to vanish in the other three quadrants.]

The kernel in Eq. (6), $\exp[(\mu'-\mu)/T_2]\,n(\mu')$, is factorizable in the form $f(\mu)h(\mu')$. Although our analysis applies whenever the kernel factorizes in this fashion, we restrict this discussion to the particular case of interest here in order to emphasize the physical quantities that come into the transformations. First, we place Eq. (6) in the form

$$\frac{\partial}{\partial z}\overline{G}(\mu,z)=\alpha'\int_0^\mu d\mu'\,n(\mu')\overline{G}(\mu',z), \qquad (13)$$

where

$$G(\mu,z)\equiv e^{-\mu/T_2}e^{-\kappa z}\,\overline{G}(\mu,z).$$

Differentiating (13) with respect to μ shows $\overline{G}(\mu,z)$ is defined by the differential equation

$$\frac{\partial^2}{\partial\mu\,\partial z}\overline{G}=\alpha' n(\mu)\overline{G},$$

together with the boundary conditions

$$\overline{G}(\mu,z=0)=G(\mu,z=0)e^{\mu/T_2},$$
$$\frac{\partial\overline{G}}{\partial z}(\mu=0,z)=\psi(z). \qquad (14)$$

In the particular case at hand, the boundary data $\overline{G}(\mu,0)$ is constant (in μ) and $\Psi(z)=0$. Next we scale the μ variable to incorporate the factor $n(\mu)$ and obtain

$$\frac{\partial^2}{\partial\tilde{\mu}\,\partial z}\overline{G}=\alpha'\overline{G}, \qquad (15)$$

where $\tilde{\mu}$ is defined by

$$\tilde{\mu}(\mu)\equiv\int_0^\mu n(\mu')\,d\mu'.$$

This quantity $\tilde{\mu}$, called the "reduced time," has been used previously[11] to transform a nonlinear semiclassical problem with decay into the sine-Gordon equation for the special case we discuss in Sec. III. We use it here to reduce a much wider class of (linear) problems to the Klein-Gordon equation (15). Depending upon the specific decay scheme, $n(\mu)$ is either always positive (as in the case of Sec. III) or it is positive for $\mu<\mu_c$ and negative for $\mu>\mu_c$ (as in the example discussed in Sec. II A). In the first case the reduced time $\tilde{\mu}(\mu)$ is a monotonically increasing function of μ, while in the latter case $\tilde{\mu}(\mu)$ reaches a maximum at $\mu=\mu_c$. We assume first that $\tilde{\mu}(\mu)$ is a strictly increasing function of μ.

Equation (15) with boundary conditions (14) is a standard Riemann problem. Its solution is obtained by a straightforward modification of an argument found in Ref. 12. Specializing to the case $\Psi(z)=0$, we find

$$\overline{G}(\tilde{\mu},z)=\overline{G}(0,0)[I_0(2(\alpha' z\tilde{\mu})^{1/2})] + \int_0^{\alpha'\tilde{\mu}} I_0(2[\alpha' z(\tilde{\mu}-\mu')]^{1/2})\frac{\partial}{\partial\mu'}\overline{G}(\tilde{\mu}',0)\,d\tilde{\mu}', \qquad (16)$$

where I_0 denotes the zeroth-order Bessel function of imaginary argument. Returning to physical variables in the case where $\overline{G}(\mu,0)$ is constant (in μ) yields

$$G(\mu,z)=G_0e^{-\kappa z}e^{-\mu/T_2}I_0\Big(2\Big[\alpha' z\int_0^\mu n(\mu')\,d\mu'\Big]^{1/2}\Big). \qquad (17)$$

For that type of decay discussed in Sec. II A, where the reduced time $\tilde{\mu}$ reaches a maximum at μ_c, representation (17) applies for $0\le\mu\le\mu_c$. Using $\overline{G}(\tilde{\mu}(\mu_c),z)$ as boundary data [i.e., giving $\psi(z)$], one then solves

$$\frac{\partial^2}{\partial\tilde{\mu}\,\partial z}\overline{G}=-\alpha'\overline{G}$$

to investigate the domain $\mu\ge\mu_c$ in the manner described in Ref. 12.

III. TYPICAL EXAMPLE

A "normal" laser amplifier[1-4] is characterized by a condition in which, for one of a number of different physical reasons, the ensemble average time variation of $n(\mu)$ is extremely slow compared to either T_2 [see Eq. (4) for definition] or to the Doppler time[2] T_2^* which affects the small-signal regime in a manner analogous to T_2. In this case, the formal development proceeds in the Fourier domain,[3,4] since the kernel in Eq. (4) is a function of the time difference. Our analysis allows us to explore analytically what happens when this condition is not satisfied. Since this problem has already been parametrized using numerical techniques,[7] we deal with a single simple example that shows the anomalies that occur. In the Appendix, we show that the present analysis goes over to the conventional discussion in the appropriate limit.

Although the decay scheme discussed in Sec. II permits a particularly simple derivation of the operator equation, it yields a reduced time $\tilde{\mu}$ which is not a monotonic function of μ. This complication is avoided here by discussing a decay scheme in which both states decay to distant ground states with equal rates denoted by γ. We also take the system to be collisionless so that $1/T_2 = \gamma$. In this case $n(\mu)$ takes on the simple form $\exp(-\gamma\mu)$, and the reduced time $\tilde{\mu}$ is a monotonic function of μ.

Using the solution generated in Sec. IID, we find from Eq. (11) that

$$\langle I(\mu, z)\rangle = \mathfrak{N}|a|^2 G_0^2 e^{-2\gamma\mu} \times \int_0^z dz' I_0^2([2gz'(1-e^{-\gamma\mu})]^{1/2}) e^{-2\kappa z'}, \tag{18}$$

where $g = 2\alpha' T_2$ is the usual gain coefficient for this case as discussed extensively in the literature.[1,2] This solution can be put into a somewhat more transparent form by utilizing the asymptotic expansion $I_0(\rho) \sim \exp(\rho)/(2\pi\rho)^{1/2}$. We can do this since the physically interesting features of the solution occur for $gz \gg 1$ and $\mu \sim \gamma^{-1}$. We have extensively tested the replacement in the region of interest and have found that it introduces errors that are not greater than 10% and are usually much smaller. The singularity in the asymptotic expansion causes no significant difficulties. The I_0 function, because of its exponential character, has the property that when one takes a finite integral a, b where $a \ll b$, the contribution from the lower limit of integration is negligible. We denote such an approximation by not writing the lower limit of integration, i.e., by writing

$$\int_a^b dq\, F'(q) = F(b) - F(a) \sim F(b) \equiv \int^b dq\, F'(q). \tag{19}$$

In other words, we treat the singularity by ignoring it. We consider first the case of lossless pulse growth ($\kappa = 0$). Making the substitutions and integrating as indicated above, we find that

$$\langle I(\mu, z)\rangle \simeq G_0^2 \mathfrak{N}|a|^2 \frac{e^{-2\gamma\mu}}{2\pi g(1-e^{-\gamma\mu})} \times \exp[8gz(1-e^{-\gamma\mu})]^{1/2}. \tag{20}$$

The most interesting feature of this pulse is that its growth does not go as $\exp(gz)$, which is the "normal" growth law discussed in the Appendix. To make this difference more explicit, we consider the pulse energy, where

$$\langle \mathcal{T}(z)\rangle = \int^\infty d\mu\, \langle I(\mu, z)\rangle. \tag{21}$$

This integration gives

$$\langle \mathcal{T}(z)\rangle = \frac{G_0^2 \mathfrak{N}|a|^2}{2\pi g\gamma} \left[\mathrm{Ei}[(8gz)^{1/2}] - \Gamma - e^{(8gz)^{1/2}} \left(\frac{(8gz-1)^{1/2}}{8gz}\right)\right]. \tag{22}$$

Here, Ei is the exponential integral function and Γ is Euler's constant. Ignoring the uninteresting power-law dependences, the pulse energy grows essentially as $\exp(8gz)^{1/2}$. In terms of growth rates, we see that

$$\frac{1}{\langle \mathcal{T}\rangle} \frac{d\langle \mathcal{T}\rangle}{dz} \simeq \left(\frac{8g}{z}\right)^{1/2}, \tag{23}$$

i.e., the pulse growth rate vanishes in the limit $z \to \infty$. This is in contrast to the normal case where the growth rate is essentially constant. To see this more explicitly, let us treat this as a semiclassical problem, in which case instantaneous growth rates have analytical significance. In Fig. 1 we show the semiclassical field amplitude $\mathcal{E}$ (which is equivalent to G in Sec. IIC.) and the instantaneous growth rate denoted $g_{\mathcal{E}}(\mu)$ which is defined by

$$g_{\mathcal{E}}(\mu) = \frac{1}{\mathcal{E}(\mu, z)} \frac{\partial \mathcal{E}(\mu, z)}{\partial z} \tag{24}$$

with the derivative given by Eq. (6). One sees the dropoff in instantaneous growth rate as the pulse reshapes away from the initial exponential with the most rapid drop occurring for small z. When $g_{\mathcal{E}}(\mu)$ is investigated in detail, its maximum turns out to be primarily sensitive to the power law in μ that dominates the rise of the pulse. The growth rate then falls off due to the increasingly gradual rise of the pulse waveform.

In the semiclassical case, the pulse energy is given by

$$\mathcal{T}_{sc}(z) \simeq \frac{G_0^2 \mathfrak{N} |a|^2}{2\pi g\gamma} \frac{e^{(8gz)^{1/2}}}{(8gz)^2} (8gz - 1)^{1/2} . \qquad (25)$$

The growth is dominated by the same $\exp(8gz)^{1/2}$ as the quantum solution so that one can at least qualitatively describe this problem semiclassically. This resemblance is absent in the case of nonzero loss. Then one has an additional multiplicative factor $\exp(-\kappa z)$ in Eq. (25). The energy, and hence the pulse itself, will vanish in the limit $z \to \infty$. The energy in the quantum description obeys an equation analogous to (12). Since the area under curve of $\mathcal{T}_{sc}(z)$ is finite, the value of $\langle \mathcal{T}(z) \rangle$ given by the quantum prediction is a constant in the limit $z \to \infty$. In other words, the mean properties of the pulse reach a "steady state" field, denoted $\langle I_s(\mu) \rangle$ since it is independent of z. This is given by Eq. (18) in the limit $z \to \infty$ and using the asymptotic expansion as

$$\langle I_s(\mu) \rangle = \frac{G_0^2 \mathfrak{N} |a|^2 e^{-2\gamma\mu} \exp[2g(1 - e^{-\gamma\mu})/\kappa]}{2\pi[2g\kappa\pi(1 - e^{-\gamma\mu})]^{1/2}} . \qquad (26)$$

In Fig. 2, we show the mean steady state as a function of g/κ using the unapproximated Bessel function. One sees the change from essentially a simple exponential ($g/\kappa = 1$) characteristic of the spontaneous emission to the more customary shapes associated with amplified light. The curve $g/\kappa = 10$ is near the upper bound of the limits of validity of this analysis. Somewhere in the interval $10 \lesssim g/\kappa \lesssim 20$, depending on the magnitude chosen for the noise source, there is a radical transformation from this type of a steady state to the customary "π pulse" form.[1,11] One final point of importance is the fact that the fluctuations[13] in this system are Gaussian. If one continues the analysis further, one finds for example that $\langle [I(\mu, z) - \langle I_s(\mu) \rangle]^2 \rangle$ is just $\langle I_s(\mu) \rangle^2$. This means that the fluctuations in the steady state are as big as the pulse itself. Hence, we emphasize the mean nature of the steady state.

To summarize, we find in this simple example a number of features that are radically different from the normal case. The growth law is characterized by $\exp(8gz)^{1/2}$. The growth rate is thus a monotonically decreasing function of distance, and when the analysis is carried through in detail, the rate is found to be substantially smaller than the one predicted by the usual formulas. As a consequence of this growth law, one finds that steady states are formed in the small-signal regime of the amplifier in the presence of nonzero losses.

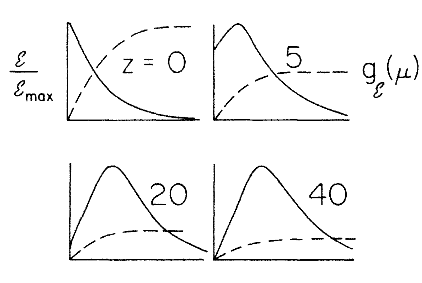

FIG. 1. Electric field amplitude $\mathcal{E}/\mathcal{E}_{max}$ and its instantaneous growth rate $g_{\mathcal{E}}(\mu)$ (on a fixed scale) as a function of z.

APPENDIX

In this appendix we show how the normal growth law is recovered from our solution. This is expected to occur whenever $n(\mu)$ varies slowly compared to T_2. We therefore discuss an example in which the atoms are not all excited at $\mu = 0$, but rather are excited uniformly over an interval $0 < \mu < T_p$, where $T_p \gg \gamma^{-1}$. We take the atoms to have the type of decay as described in Sec. III, in which case it is straightforward to show that $n(\mu)$ is constant over the interval $(0, T_p)$ except for small domains near 0 and T_p which can be ignored. The generalization of the source term in Eq. (4) and the resulting generalization in the discussion of Sec. III C are discussed in Ref. 7. Following that discussion, we find

$$\langle I(\mu, z) \rangle = \mathfrak{N} |a|^2 G_0^2 \times \int_0^z dz' \int_0^\mu d\mu_0 \, e^{-2\gamma(\mu-\mu_0)} \times I_0^2([2gz'\gamma(\mu - \mu_0)]^{1/2}) . \qquad (A1)$$

We note (a) that μ_0 enters as a new parameter in the Green's function that labels the retarded time at which the atoms were excited and (b) the reduced time and the retarded time are proportional here by virtue of $n(\mu)$ being constant. (c) The coefficient G_0 has different units in this case from those in Sec. III. (d) Finally, we emphasize that the integral over μ_0 in the above expression comes

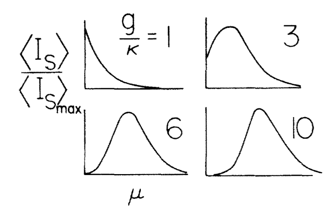

FIG. 2. Shape of the mean steady state $\langle I_s(\mu) \rangle$ as a function of the gain to loss ratio g/κ.

from the generalization of the source term described in Ref. 7. Writing $r^2 = 2g\gamma z'(\mu - \mu_0)$ yields

$$\langle I(\mu, z)\rangle = \frac{\mathfrak{N}|a|^2 G_0^{\,2}}{g\gamma} \times \int_0^z \frac{dz'}{z'} \int_0^{(2g\gamma z'\mu)^{1/2}} dr\, r \exp\left(\frac{-r^2}{gz'}\right) I_0^2(r). \tag{A2}$$

We are interested in the regime $\mu \gg 0$, $z \gg 0$. In the integrand the neighborhood of $z' = 0$ is negligible because of the factor $\exp(-r^2/gz')$; hence, because $\mu \gg 0$, we replace the upper limit of the r integration by $+\infty$ and use Weber's second exponential integral[14] to find

$$\langle I(\mu, z)\rangle \simeq \frac{\mathfrak{N}|a|^2 G_0^2}{2\gamma} \int^z dz' \exp(\tfrac{1}{2}gz') I_0(gz'/2). \tag{A3}$$

For large z, this last integral takes the form

$$\langle I(\mu, z)\rangle \simeq \frac{\mathfrak{N}|a|^2 G_0^2}{\sqrt{\pi}\,\gamma} \int^z \frac{dz'}{\sqrt{gz'}} \exp(gz') \quad (z \gg 0,\ \mu \gg 0), \tag{A4}$$

which is the result one obtains from the conventional discussion of the gain.

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