Transient processes for incidence of a light signal on a vacuum-medium interface

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We extend the classical Brillouin-Sommerfeld treatment of precursors in several ways: We consider oblique incidence on an interface, the vector nature of the field, the effect of different forms of the incident signal, and the different refractive indices of the media. The amplitudes, frequencies, directions of propagation, and duration of the signal at all stages of the transient process are calculated. The steepness of the form of signals is one of the crucial factors for the experimental detection of the phenomena under consideration. The different possibilities of the experimental detection of the refracted and reflected signals are discussed.

I. INTRODUCTION

When a monochromatic electromagnetic plane wave with frequency ω impinges at an angle α on a boundary between two media with relative index of refraction $n(\omega)$, the amplitudes of the refracted (d) and reflected (r) waves are determined by Fresnel's formulas, which can be written in the following form [see Eqs. (4) and (5)]:

$$\begin{aligned} f_\perp^r &= \frac{\cos\alpha - [n^2(\omega)-\sin^2\alpha]^{1/2}}{\cos\alpha + [n^2(\omega)-\sin^2\alpha]^{1/2}}, \\ f_\parallel^r &= \frac{[n^2(\omega)-\sin^2\alpha]^{1/2} - n^2(\omega)\cos\alpha}{[n^2(\omega)-\sin^2\alpha]^{1/2} + n^2(\omega)\cos\alpha}, \\ f_\perp^d &= \frac{2\cos\alpha}{\cos\alpha + [n^2(\omega)-\sin^2\alpha]^{1/2}}, \\ f_\parallel^d &= \frac{2[n^2(\omega)-\sin^2\alpha]^{1/2}}{[n^2(\omega)-\sin^2\alpha]^{1/2} + n^2(\omega)\cos\alpha}. \end{aligned} \tag{1}$$

Here the indices $\parallel$ and $\perp$ correspond to the electric vector of the incident wave being parallel and perpendicular to the plane of incidence, respectively.

Relations (1) are simple consequences of conservation of energy and are associated with processes that proceed for all times from $-\infty$ to $+\infty$.

In order to be able to study transient processes, we must consider incidence on the boundary between two media of a wave train limited in extent ("signal"), for example, a plane wave bounded on one side,

$$\begin{aligned} \vec{E}^e(r,t) &= 0, \quad t < (x\sin\alpha + z\cos\alpha)/c, \\ &= \vec{E}_0 \exp\{-i\omega_0[t-(x\sin\alpha+z\cos\alpha)/c]\}, \\ & \qquad t > (x\sin\alpha + z\cos\alpha)/c. \end{aligned} \tag{2}$$

Here the x-z plane is the plane of incidence, $z=0$ is the boundary plane between vacuum and the medium with a refractive index $n(\omega)$, and the z axis is directed towards the inside of the medium.

In order to find the amplitude and phase relations between the signal (2) and the corresponding refracted and reflected signals, it is convenient to write (2) as a contour integral in the complex ω plane in the form of a superposition of monochromatic plane waves:

$$\vec{E}^e(r,t) = -\frac{\vec{E}_0}{2\pi i} \times \int_{-\infty+ia}^{\infty+ia} \frac{\exp\{-i\omega[t-(x\sin\alpha+z\cos\alpha)/c]\}}{\omega-\omega_0}\,d\omega. \tag{3}$$

Now the above transient nonstationary problem is reduced to a stationary one, because for each of the plane waves in (3) the laws of refraction and reflection are described by the Fresnel formulas (1) with the corresponding refractive index $n(\omega)$.

Then the reflected (r) and the refracted (d) signals are described by summing over stationary harmonics:

$$\vec{E}^{r,d}(r,t) = -\frac{1}{2\pi i}\int_{-\infty+ia}^{\infty+ia} \vec{E}_0^{r,d} f^{r,d}(\omega) \times \frac{\exp[-i(\omega t - \vec{k}_{r,d}\cdot\vec{r})]}{\omega-\omega_0}\,d\omega, \tag{4}$$

where

$$\begin{aligned} \vec{E}_0^{e,r} &= \{-E_0''\cos\alpha, E_0', \pm E_0''\sin\alpha\}, \\ \vec{E}_0^d &= \{-E_0''\cos\alpha, E_0', E_0''\cos\alpha\tan\beta\}, \\ \vec{k}_{e,r} &= \{(\omega/c)\sin\alpha, 0, \pm(\omega/c)\cos\alpha\}, \\ \vec{k}_d &= \{(\omega/c)n(\omega)\sin\beta, 0, (\omega/c)n(\omega)\cos\beta\}. \end{aligned} \tag{5}$$

The angle of refraction β is related to the incident angle α by the relation $\sin\alpha = n(\omega)\sin\beta$.

The magnetic field is obtained via Maxwell's equation from (4):

$$\vec{H}^{r,d}(r,t) = -\frac{1}{2\pi i}\int \frac{c}{\omega}(\vec{k}_{r,d}\times\vec{E}^{r,d})f^{r,d}_{(\omega)} \times \frac{\exp[-i(\omega t - \vec{k}_{r,d}\cdot\vec{r})]}{\omega-\omega_0}\,d\omega\,. \tag{6}$$

It then remains to calculate the integrals (4)–(6).

On the molecular level the formation of reflected and refracted fields is connected with the forced oscillations of the charged particles of the material, both electrons and ions, in the field of the incident signal. However, it takes some time for these particles to execute the forced oscillations. During this time transient processes take place. It is obvious that these processes are determined by the inertial and relaxational properties of the medium [i.e., by the refractive index $n(\omega)$] as well as by the character of the incident signal. It is the object of our work to discuss these transient processes in the optical frequency range of incident signals.

The above-mentioned method of reducing the nonstationary problem to a stationary one was proposed in 1914 in the classical works of Sommerfeld[1,2] and Brillouin[3] and has been used for studying the propagation of scalar signals in dispersive media at normal incidence. The refractive index was assumed to have the following form:

$$n(\omega) = [1 + \Omega^2/(\omega_1^2 - \omega^2 - 2i\gamma_1\omega)]^{1/2}\ ,$$
$$\Omega^2 = 4\pi Ne^2/m. \tag{7}$$

Here N, ω_1, γ_1, e, and m are the number per unit volume, the characteristic frequency, the damping constant, the charge, and the mass of the particles considered as an aggregate of charged harmonic oscillators.

Sommerfeld studied the transient processes occurring immediately after the arrival of the signal. He showed that the corresponding transient response (precursor, or according to Brillouin's terminology,[3] forerunner) is propagated in the direction of the incident signal with the velocity of light c in vacuum and is damped as the time increases.

Sommerfeld also showed that the end of the transient processes and the onset of the stationary signal with frequency ω_0 are mathematically determined by looping the contour of the integrals (4) and (6) around the pole ω_0.

Brillouin considered the intermediate stages of the transient processes using the saddle-point method for an approximate calculation of the integrals. The branch points in the contour integral are essential in this case. He found that the damping of first "Sommerfeld" precursor is followed by a second precursor, which then turns into the stationary refracted signal.

Sommerfeld and Brillouin were interested in the question of signal velocity in dispersive media and in the resolution of the associated paradoxes arising from the theory of relativity. The experimental investigation of precursors was considered by them to be impossible with the experimental techniques available at the time.

However, the development of experimental techniques during the last 60 years, particularly the generation of ultrashort powerful light pulses with steep wave fronts, may allow experimental verification of these theoretical predictions.

Such experiments could be useful for studying the microscopic properties of media on ultrashort signals propagating in the media, for example, by transfer of information along optical channels. They may also have some relevance in applied physics, such as the measurement of ultrashort time intervals.

Following Sommerfeld's article,[2] analytic expressions were found by Skrotskaya *et al.*[4] for the beginning of transient processes (first precursor) when the front of a light pulse impinges on the border of a medium with a simple refractive index and also when the pulse goes through a vacuum-medium-vacuum system. The interest of the last case lies in the fact that if one can satisfy the condition $\sin\alpha = n(\omega_0)$ on the second interface, only the precursor will be refracted, while if the electric vector is parallel to the plane of incidence and if the relation $\tan\alpha = n(\omega_0)$ is satisfied, only the precursor and not the stationary signal will be reflected.

The present work has some relation to Ref. 4. However, we shall use Brillouin's method, which will enable us to consider all stages rather than just the initial stage of the transient process. Moreover, we shall properly take into account properties of the medium and of the incident signal corresponding more closely to the real situation.

The paper is organized as follows: In Sec. II transient processes are considered for incident signals of the type described in Eq. (2) with a medium of refractive index described by Eq. (7). These considerations are similar to those of Brillouin, but in our case the vector character of the field gives us the possibility of studying the time variation of the direction of propagation of the signal as well as the way in which the Fresnel formulas are established. The properties specific to an incident step-signal, in contrast to those for a monochromatic signal (2), are investigated in Sec. III. We find both the reflected and the refracted signal as functions of time in analytic form for a plasma-type refractive index [simpler than in Eq. (7)]. In Sec. IV we present results of the analysis of transient processes in a medium with several characteristic frequencies, i.e., with a refractive index which is more complicated than that in (7). The number of precursors is found to increase accordingly. In Sec. V the influence of the steepness of the pulse on the transient processes are analyzed. It was found that the steepness of the front is a crucial condition for the experimental observation of transient processes. Finally, Sec. VI contains qualitative results, quantitative estimates, and conclusions.

II. FORMATION OF PRECURSORS FOR A MONOCHROMATIC INCIDENT SIGNAL

Let the monochromatic signal (2), bounded on only one side, with frequency ω_0 close to one of the characteristic frequencies ω_1 of the medium, impinge on the dispersive medium extending from $z=0$, where it joins the vacuum, to $z=\infty$. Let us now assume that the medium is described by the refractive index (7), i.e., the transient processes are generated only by particles of one kind. The different types of particle motions (electronic oscillations in atoms and molecules, oscillations and rotations of ions in molecules, displacements of ions in ionic crystals, polar oscillations of particles with dipole moments, elastic displacements of the dipole groups, etc.) are characterized by different inertial and relaxational properties. The intensity of the forced oscillations is a maximum when the frequency of incident light ω_0 is in resonance with the frequency of any of these motions, i.e., $\omega_0 \sim \omega_j$, and the oscillations settle down after a time $t \gtrsim \gamma_j^{-1}$ [see (7)]. It is obvious that the electronic mechanism of polarization is the least inertial, i.e., at each point of the media the electrons will be the first to react to the incident field. Therefore we suppose that only the electrons give a contribution to $n(\omega)$ in (7), and the more difficult case will be deferred to Sec. IV.

For analytical evaluation of the integrals in (4) and (6), it is necessary first of all to investigate the analytic properties of the integrand in (4) in the complex ω plane. (For brevity we omit the calculations for the magnetic field and give only the results.) The argument of the exponential function in (4) can be written in the following form:

$$\phi(\omega) = -i\omega[t - n(\omega)(x\sin\beta + z\cos\beta)/c]$$
$$= -i\omega[t - (x\sin\alpha)/c - (z\cos\alpha/c)n_1(\omega)], \quad (8)$$

where

$$n_1(\omega) = \left(1 - \frac{\Omega_1^2/\cos^2\alpha}{\omega^2 - \omega_1^2 + 2i\gamma_1\omega}\right)^{1/2}.$$

The last expression in (8) is found by using the formulas (5) and (7) and the law of refraction, $\sin\alpha = n\sin\beta$.

From (8) and (4) one can see that the integrand in (4) has a number of singularities[5]:

$$\text{A pole at the point } \omega_0:\ \omega = \omega_0\,, \qquad \text{Branch points of the function } \phi(\omega)\,, \quad (9)$$

$$\omega_{1,2} = -i\gamma_1 \pm (\omega_1^2 - \gamma_1^2)^{1/2}, \quad (10)$$

$$\omega_{3,4} = -i\gamma_1 \pm (\omega_1^2 - \gamma_1^2 + \Omega_1^2/\cos^2\alpha)^{1/2}. \quad (11)$$

If we join the points ω_1 to ω_3 and ω_2 to ω_4 by two branch lines, the integrand will be single valued. The position of the branch points (10) and (11) and the path of integration Γ are shown in Fig. 1 for the case $\omega_1 > \omega_0$.

For an approximate calculation of the integral (4) we shall use, following Brillouin,[3] the saddle-point method. In order to do this it is necessary to find the allowed region of integration, i.e., where $\mathrm{Re}[\phi(\omega)] < 0$, the location of the saddle points $d\phi(\omega)/d\omega = 0$, and the line of the steepest descent through the saddle points.

From (8) we can see that for a given point x, z of the medium the location of the saddle point $\omega = \omega_j$ changes with time t. The connection between ω_j (the solution of the equation $d\phi/d\omega = 0$) and the corresponding t_j is determined by the group velocity v_j:

$$t_j = \frac{x\sin\alpha}{c} + \frac{z\cos\alpha}{v_j}, \quad v_j = \mathrm{Re}\left.\frac{d\omega}{dk_1}\right|_{\omega=\omega_j}, \qquad k_1 = \frac{\omega}{c}n_1 = \frac{\omega}{c}\left(1 - \frac{\Omega_1^2/\cos^2\alpha}{\omega^2 + 2i\gamma_1\omega - \omega_1^2}\right)^{1/2}. \quad (12)$$

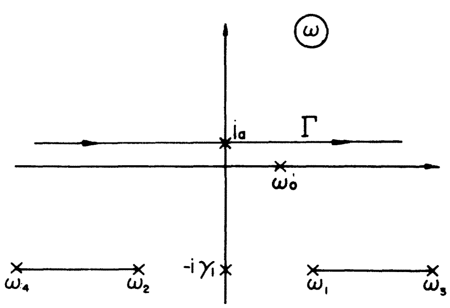

FIG. 1. Singularities of the integrand of $E^d(\omega)$ in the complex ω plane. ω_0 is the pole, $\omega_{1,2,3,4}$ are branch points. Γ is the path of integration.

Since $\phi(\omega)$ [Eq. (8)] depends both on ω and on the function $k_1(\omega)$, it is convenient to consider the location of the saddle points on the dispersion curve $\omega_1(k)$ (Fig. 2). It is clear from (12) that at some time t_j a signal with the group velocity v_j will arrive at the point x, z. In other words, at the given point x, z the main contribution to the integral (4) will be given by those points on the dispersion curve $\omega_1(k)$ at which the tangent $d\omega/dk_1$ equals

$$v_j = \frac{z \cos\alpha}{t_j - (x \sin\alpha)/c} .$$

The maximum value of the tangent, equal to the light velocity c in vacuum, is reached at $\omega \to \infty$. Hence the signal does not propagate with a velocity greater than c ($v_j \leq c$), and a time $t_0 = (x \sin\alpha + z\cos\alpha)/c$ corresponds to the beginning of the transient process at the point x, z and to the formation of the first precursor.[6]

For $t > t_0$ the saddle point moves down along the upper branch of the dispersion curve (Fig. 2), as far as some time $t_1 = (x \sin\alpha)/c + (z \cos\alpha)/v_1$ at which the first contribution from the lower branch of the dispersion curve appears corresponding to the second precursor. Then for $t > t_1$ the saddle point moves up along this lower branch. Finally at some time t_2 the path of integration will go around the pole ω_0 [see Eq. (9)] which correspond to the formation of the stationary refracted signal and the end of the transient processes.

The locations of the saddle points in the complex frequency plane depend, then, upon the time t relative to the times t_0, t_1, and t_2. At the beginning of the transient period, i.e., for $t \simeq t_0$, only the high frequencies contribute. For subsequent times

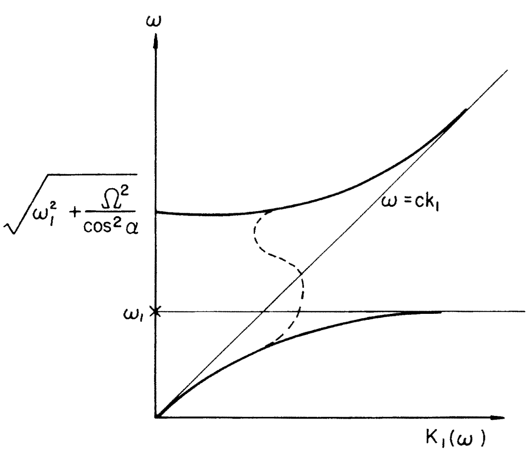

FIG. 2. Dispersion curve $\omega(k_1)$. Solid line—without damping ($\gamma = 0$); dotted line—schematic behavior when $\gamma \neq 0$.

lower frequencies contribute. With this in mind, we introduce the following approximations to the modified complex index of refraction, $n_1(\omega)$ appropriate to the high- and low-frequency regions:

$$n_1(\omega) \simeq 1 - \frac{\Omega_1^2/\cos^2\alpha}{2\omega(\omega + 2i\gamma_1)}, \quad \omega > \frac{\Omega_1}{\cos\alpha}, \omega_1 \tag{13}$$

and

$$n_1(\omega) \simeq A + B\omega(\omega + 2i\gamma_1), \quad \omega < (\Omega_1/\cos\alpha), \omega_1,$$

where

$$A = (1 + \Omega_1^2/\omega_1^2 \cos^2\alpha)^{1/2}, \quad B = \Omega_1^2/2A\,\omega_1^4 \cos^2\alpha . \tag{14}$$

By using Eqs. (8) and (13) we find that the saddle points at the beginning of the transient process are given by

$$\omega_{5,6} = -2i\gamma_1 \pm [\Omega_1^2 z/2c(t - t_0)\cos\alpha]^{1/2} ; \quad \text{for } t_0 \lesssim t. \tag{15}$$

Somewhat later in the transient process we find from Eqs. (14) and (8) that the saddle points are now located at

$$\omega_{7,8} = \{ -\tfrac{2}{3}\gamma_1 \pm [c(t_1 - t)/3Bz\cos\alpha]^{1/2}\}\, i , \quad \text{for } t_0 < t < t_1 \tag{16}$$

$$\omega_9 = -\tfrac{2}{3}\gamma_1 i \quad \text{for} \quad t = t_1 = x\sin\alpha/c + z\cos\alpha/v_1 ,$$

where $v_1 = c(A + \frac{4}{3}\gamma_1^4 B)^{-1}$. For still later times the saddle points are located at

$$\omega_{10,11} = -\tfrac{2}{3}\gamma_1 i \pm [c(t - t_1)/3Bz\cos\alpha]^{1/2}, \quad t_1 < t < t_2.$$

Here

$$t_2 = x(\sin\alpha)/c + (A + 3B\omega_0^2)z(\cos\alpha)/c$$

is the time at which the stationary signal is formed at the point x, z with frequency ω_0, group velocity $c/\mathrm{Re}[n(\omega_0)]$, and angle of propagation β. In other words, the duration of the transient process at points z away from the interface is[7]

$$\Delta t = t_2 - t_0 \simeq z\cos\alpha(A + 3B\omega_0^2 - 1) . \tag{17}$$

According to the saddle-point method the essential contributions to the integral will be

$$\int \psi(\omega) e^{\varphi(\omega)} d\omega \approx \sum_j \psi(\omega_j) e^{\varphi(\omega_j)} \frac{1}{2}\left[\frac{1}{\pi}\left(\frac{d^2 \mathrm{Re}[\varphi(\omega)]}{d\omega^2}\right)_{\omega=\omega_j}\right]^{-1/2} \tag{18}$$

This part of our calculation is similar to that of Brillouin.[8] Hence we show here only the path of integration and the results of the calculation (for details see Ref. 3).

At small $t-t_0$ the path begins at the point where $\mathrm{Re}[\phi(\omega)]$ is negatively infinite and passes through ω_6 along the line of steepest descent to ω_2 where $n=\infty$, i.e., $v=0$. Then it goes through ω_7 to ω_1 and through ω_5 to the point where $\mathrm{Re}[\phi(\omega)]$ is negatively infinite. This contour corresponds to a time $t_0<t<t_1$, i.e., to the velocities $c>v>v_1=c(A+\frac{4}{3}\gamma_1^2 B)^{-1}$. The velocity v_1 is a solution of the equation $d^2k_1/d\omega^2=0$ and for a time $t_1=(x\sin\alpha)/c+(z\cos\alpha)/v_1$ the path of integration has a break at ω_9 at which the two saddle points coalesce.

The second precursor exists for $t_1<t<t_2$ and the path of integration has the following form:

$$\omega_6\to\omega_2\to\omega_{11}\to -i\infty\to\omega_{10}\to\omega_1\to\omega_5 .$$

If $\omega_0<\omega_1$ this contour can approach the pole ω_0 and it must then be taken around the pole.

As a result of the calculation we find the refracted field in the following forms:

$$\begin{pmatrix}E_y\\ H_x\\ H_z\end{pmatrix}_\perp = E_0\begin{pmatrix}1\\ -n_1\cos\alpha\\ \sin\alpha\end{pmatrix} f_\perp^d F(t) , \qquad (19)$$

$$\begin{pmatrix}E_x\\ E_z\\ H_y\end{pmatrix}_\parallel = E_0\begin{pmatrix}-\cos\alpha\\ (\sin\alpha)/n_1\\ -n^2/n_1\end{pmatrix} f_\parallel^d F(t) , \qquad (20)$$

corresponding to the electric vector of the incident signal being parallel (‖) or perpendicular (⊥) to the plane of incidence. Here, by virtue of (7) and (8), $n(\omega)=n_1(\omega,\alpha=0)$.

Let us give now the form of f and F in (19) and (20) for different times.

A. $t_0<t<t_1$

In this case the main contribution in the integral (4) is given—for $t\gtrsim t_0$—by the saddle points $\omega_{5,6}$ in Eq. (15) and for t close to t_1, by the saddle point ω_7 in Eq. (16).

Neglecting the small imaginary parts of $\omega_{5,6}$ everywhere except in the argument of the exponential function, we receive the contribution of the saddle points $\omega_{5,6}$,

$$F(t)=-[a\pi^2(t-t_0)]^{-1/4}\exp[-2\gamma_1(t-t_0)] \times\{\sin([4a(t-t_0)]^{1/2}+\tfrac14\pi) + i\omega_0[(t-t_0)/a]^{1/2}\cos([4a(t-t_0)]^{1/2}+\tfrac14\pi)\},$$

$$a=z\Omega_1^2/2c\cos\alpha,\quad f_\perp^d=1+c(t-t_0)/2z\cos\alpha,$$

$$f_\parallel^d=1+c(t-t_0)\cos2\alpha/2z\cos\alpha . \qquad (21)$$

For saddle point ω_7 we have

$$F(t)=-\left(\frac{c}{2\pi Bz\cos\alpha(2\gamma_1+3|\omega_7|)}\right)^{1/2}\frac{|\omega_7|-i\omega_0}{|\omega_7|^2+\omega_0^2} \times\exp\left(-\frac{2Bz\cos\alpha}{c}|\omega_7|^2(\gamma_1+|\omega_7|)\right), \qquad (22)$$

$$f_\perp^d=\frac{2}{1+n_1},\quad f_\parallel^d=\frac{2n_1}{n_1+n},\quad n_1\approx A-\frac{c(t_1-t)}{3z\cos\alpha},$$

where A and B are determined in (14).

Let us note that the expression (22) is a small correction to (21) everywhere in the region $t_0<t<t_1$ except in the immediate vicinity of t_1.

From (21) it follows that for $t\gtrsim t_0$ the field oscillates with a frequency which decreases continuously as $(t-t_0)^{-1/2}$. The amplitude of $\mathrm{Re}[F(t)]$ (response to an incident cosinusoidal signal) decreases, and the amplitude of $\mathrm{Im}[F(t)]$ (response to a sine signal) originally increases and then decreases.

Formula (21) is the first term of an asymptotic expansion of the exact solution (for the beginning of the transient process), which is a series of Bessel functions (see Sec. III and Refs. 1 and 4). This means that (21) loses its validity for very small $t-t_0$ which is mathematically connected with the fact that the integrand along the line of steepest descent does not decrease fast enough for very distant saddle points.

Knowing the amplitudes of the refracted signal, we can calculate the energy-flux vector $\vec{S}_\perp^d$:[9]

$$\vec{S}_\perp^d=\frac{c}{4\pi}E_0^2\{(\sin\alpha)\hat{x}+[1-c(t-t_0)/z\cos\alpha](\cos\alpha)\hat{z}\} \times(f_\perp^d)|F(t)|^2 . \qquad (23)$$

From (23) it follows that in the beginning of the transient process (small $t-t_0$) the direction α_e of the energy propagation in the medium coincides with the direction of incident signal, $\alpha_e\approx\alpha$. The angle α_e increases with increasing $t-t_0$,

$$\sin\alpha_e=\frac{\sin\alpha}{n(\omega_5)}\approx\frac{\sin\alpha}{1-\Omega^2/|\omega_5|^2\cos^2\alpha}>\sin\alpha,\qquad \alpha_e>\alpha. \qquad (24)$$

As time increases the energy of the precursor decreases and it is damped out. But as is seen from (24) the direction in which the energy of the first precursor is propagated changes with time away from the direction of refracted signal. This result which is at first sight surprising is connected with the properties of the refractive index of the medium at high frequencies ($|\omega_5|>\omega_1$) of the incident signal.

These components of the signal swing the charged particles of the medium and they are responsible for the first precursor. For such high frequencies the refractive index for the medium is smaller than the refractive index for vacuum, $n(\omega_5)<1$ [cf. Eq. (30) below]. This means that for these frequencies the medium is optically less dense than vacuum, and it is natural that $\alpha_e > \alpha$.

B. $t \sim t_1$

Now the path of integration goes across the saddle point ω_9 in Eq. (15). As mentioned above this point corresponds to the extremum of the group velocity $d^2k/d\omega^2=0$, i.e., the expansion $k(\omega)$ cannot be stopped at the second order and the corresponding formula (18) is invalid. Also taking into account terms of the third order in ω, we find

$$F(t)=\frac{i}{2\pi(\omega_0+\frac{2}{3}i\gamma_1)}\left(\frac{c}{Bz\cos\alpha}\right)^{1/3} \times \mathrm{Ai}\left[\left(\frac{c}{Bz\cos\alpha}\right)^{1/3}(t-t_1)\right] \times\exp\left[-\frac{2\gamma_1}{3}\left(t-t_1+\frac{4Bz\gamma_1^2\cos\alpha}{9c}\right)\right], \quad (25)$$

$$f_\perp^d=\frac{2}{1+n_1}\,,\quad f_\parallel^d=\frac{2n_1}{n_1+n^2}\,,\quad n_1\approx A\,,$$

where

$$\mathrm{Ai}(x)\equiv\int_{-\infty}^{\infty}\exp[-i(xu-u^3)]\,du$$

is the Airy integral.

The function (25) describes the transition from an aperiodic signal (22), the end of the first precursor, to the beginning of the second precursor (for $t>t_1$), which is determined by the low-frequency branch of the dispersion curve in Fig. 2.

The second precursor propagates in the direction α_e',

$$\sin\alpha_e'=(\sin\alpha)/n(\omega_9)<\sin\alpha,\quad \alpha_e'<\alpha_e\,, \quad (26)$$

i.e., unlike the first precursor, the direction of the propagation of the second precursor changes towards the direction of the refractive signal as time increases.

C. $t>t_1$

Now the contribution from the saddle points $\omega_{5,6}$ is small and the main contribution is provided by the saddle points $\omega_{10,11}$.

As a result we find for the functions f and F in (19) and (20)

$$F(t)=-\left(\frac{c}{3\pi^2 zB\cos\alpha\,(t-t_1)}\right)^{1/4}\frac{\exp[-\frac{2}{3}\gamma_1(t-t_1+4\gamma_1^2Bz\cos\alpha/9c)]}{(|\omega_{10}|^2+\omega_0^2)^2-4\omega_0^2c(t-t_1)/3Bz\cos\alpha}$$
$$\times\left[(|\omega_{10}|^2-\omega_0^2)\mathrm{Re}(\omega_{10})\sin\psi-\tfrac{2}{3}\gamma_1(|\omega_{10}|^2+\omega_0^2)\cos\psi-i\omega_0\left(\omega_0^2-\frac{c(t-t_1)}{3Bz\cos\alpha}+\tfrac{4}{9}\gamma_1^2\right)\cos\psi\right.$$
$$\left.+\tfrac{4}{3}\gamma_1 i\left[\frac{c(t-t_1)}{3Bz\cos\alpha}\right]^{1/2}\omega_0\sin\psi\right], \quad (27)$$

where

$$\psi=\left(\frac{4c(t-t_1)^3}{27Bz\cos\alpha}\right)^{1/2}-\tfrac{1}{4}\pi,\quad f_\perp^d=\frac{2}{1+A+c(t-t_1)/3z\cos\alpha}\,,\quad f_\parallel^d=2\bigg/\left\{1+\frac{f_\perp^d}{2-f_\perp^d}\left[\left(1+\frac{\Omega_1^2}{\omega^2}\right)^{1/2}+\frac{c(t-t_1)}{3z}\right]^2\right\}.$$

From (27) it follows that E^d oscillates with increasing frequency. The amplitude increases very rapidly when the path of integration reaches the pole $c(t-t_1)/3Bz\cos\alpha\approx\omega_0^2$ and then damps exponentially. In this case the path of integration is very near the pole and now, in addition to the saddle points, it is necessary to consider also the path encircling the pole. This gives

$$E^d=\frac{i\varphi}{2\pi i}E_0 f_{\perp,\parallel}^d(\omega_0)\exp\{-i[\omega_0 t-k_d(\omega_0)r]\}. \quad (28)$$

The time at which the curve is completed ($\varphi=2\pi$) corresponds to the formation of the stationary refracted signal and the end of the transient process. It is easy to see that as in the previous case the energy-flux vector continues to turn towards the direction of the refracted signal as time increases.

The factor $f_{\perp,\parallel}^d$ in (19) and (20) also changes with time from $f^d\approx1$ at small $t-t_0$ [see (21)] going (through the maximum $f_\perp^d$) to $f^d=f^d(\omega_0)$ when the line of steepest descent approaches the point ω_0. It is natural that the Fresnel coefficients are not the stationary values, but that they are formed during the transient process.

III. STEP-LIKE SIGNAL

Let us consider the case of incidence of a step-signal instead of the monochromatic signal in (2), i.e.,

$$x(t)=\begin{cases}0, & t<0\ ,\\ 1, & t>0\ .\end{cases} \tag{29}$$

Using the simplicity of this case we shall receive clear results both for refracted and reflected signals at the beginning of the transient process. For this purpose we can employ the following form for the refractive index of the medium:

$$n=(1-\Omega^2/\omega^2)^{1/2},\quad \Omega^2=4\pi Ne^2/m\ . \tag{30}$$

In the spirit of our model of the medium as an aggregate of charged oscillators the expression (30) means that immediately after the arrival of the signal a charge does not have time to acquire a velocity and a displacement from the equilibrium position. In other words, at the beginning of the transient process it is the high-frequency components of the signal which arrive first at the given point of the medium. But for them, $\omega > \omega_1,\ \gamma_1$ and formula (7) goes over into the "plasma formula" (30).

In this case it is possible by analogy with Refs. 1 and 4 to reduce the integrand for the first precursor in (4) to a generating function for Bessel functions. If the electric vector of the incident signal is perpendicular to the plane of incidence we find for the reflected (r) and refracted (d) signals

$$\begin{pmatrix}E_y\\ H_x\\ H_z\end{pmatrix}_r = E_0\begin{pmatrix}1\\ \cos\alpha\\ \sin\alpha\end{pmatrix}[J_0(k_r)-1+J_2(k_r)], \tag{31}$$

$$\frac{E_y^d}{E_0}=\frac{H_z^d}{E_0\sin\alpha}=J_0(k_d)+J_2(k_d)\delta^2\ ,$$

$$-\frac{H_x^d}{E_0\cos\alpha}=J_0(k_d)+3J_2(k_d)\delta^2$$

$$+4[J_4(k_d)\delta^4+J_6(k_d)\delta^6]+\cdots\ ,$$

where

$$k_r=\frac{\Omega}{\cos\alpha}(t-t_0)\ ,$$

$$k_d=\frac{\Omega}{\cos\alpha}\left[(t-t_0)\left(t-t_0+\frac{2z\cos\alpha}{c}\right)\right]^{1/2},$$

$$\delta=\left(\frac{t-t_0}{t-t_0+(2z\cos\alpha)/c}\right)^{1/2},$$

and J_n is the Bessel function of order n.

From (31) we can see that the reflected signal, unlike the refracted one, immediately begins to propagate in the direction given by the law of reflection and depends on t and z only through $t-t_0$, i.e., the reflected field has a similar behavior at any distance from the interface.

Using the power-series expansions of Bessel functions for small arguments [$k_r<1$, i.e., $t-t_0 < (\cos\alpha)/\Omega$] we find that the reflected field E^r is proportional to k_r^2, i.e., the amplitude of the reflected signal is zero on the front of the signal, $k_r=0$.

On the interface the amplitude of the reflected signal has a spatial maximum and increases as a function of time. The variation of the reflected field at the beginning of the transient process is shown in Fig. 3(a).

The refractive signal has a completely different behavior. On the interface ($z=0$, $\delta=1$) and near it E^d oscillates and decays to zero. On the front of the signal the amplitude has a maximum, which

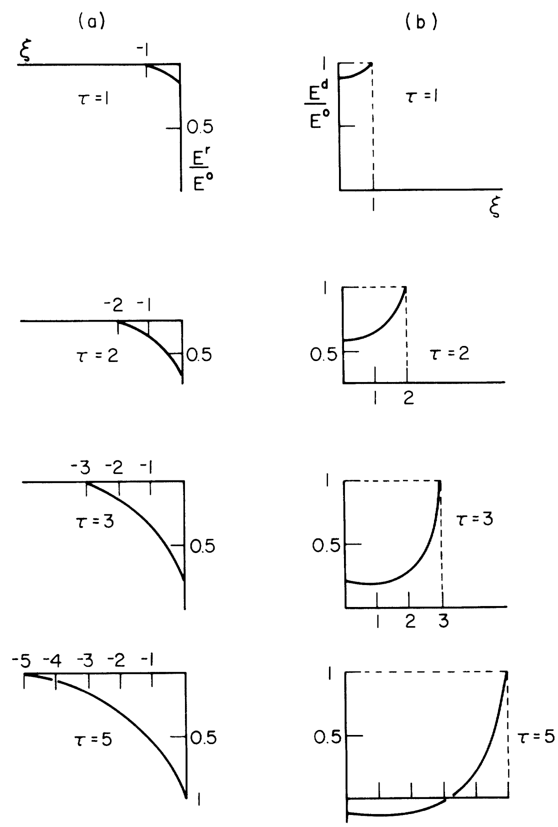

FIG. 3. Ratios E^r/E^0 (a) and E^d/E^0 (b) of the amplitudes of the refracted signals to the amplitude of the incident step signal as a function of dimensionless distance from the interface $\xi=\Omega_1 z/c$ for different times $\tau=(\Omega_1/\cos\alpha)[t-(x\sin\alpha)/c]$.

propagates into the medium as time increases [see Fig. 3(b)]. Let us note that formula (21) represents the asymptotic expansion of Eq. (31) at large arguments of the Bessel functions.

We now turn to the second precursor. In this case we cannot find the exact solution and it is necessary once more to use the saddle point method.

As shown in Sec. II the beginning of the second precursor at $t \sim t_1$ is determined by the saddle point ω_9 in Eq. (16) which corresponds to the condition $d^2k/d\omega^2 = 0$. The important difference from the case $\omega_0 \neq 0$ is that here the saddle point $\omega_9 = -\frac{2}{3}i\gamma_1$ is located very close to the pole $\omega = 0$ of the integrand in (4).

Near the saddle point ω_9 we have [cf. (4) and (25)] for $t \sim t_1$

$$F(t) \sim e^{\Phi(\omega_9)} \int \frac{f_\perp(\omega)}{\omega} \exp[i(\theta u - u^3)]\, d\omega , \qquad (32)$$

where

$$\theta = \left(\frac{c}{Bz\cos\alpha}\right)^{1/3}(t - t_1) ,$$

$$u = \left(\frac{Bz\cos\alpha}{c}\right)^{1/3}(\omega - \omega_9) ,$$

$$f_\perp(\omega) = \frac{2}{1+A}\left(1 - \frac{B}{1+A}\,\omega(\omega + 2i\gamma_1\omega)\right) ,$$

and A and B are determined in (14).

Using the definition of the Airy integral $\mathrm{Ai}(\theta)$, differentiating and integrating it with respect to the parameter θ, we find

$$F(t) = \frac{2}{1+A}\, e^{\Phi(\omega_9)} \left\{ \frac{1}{2\pi} \int_0^\theta \mathrm{Ai}(p) e^{-ipu_0}\, dp + \frac{1}{2\pi}\frac{B}{1+A}\left(\frac{c}{Bz\cos\alpha}\right)^{1/3}\left[\tfrac{4}{3}\gamma_1 \mathrm{Ai}(\theta) + \left(\frac{c}{Bz\cos\alpha}\right)^{1/3}\frac{d\mathrm{Ai}}{d\theta}\right]\right\} . \qquad (33)$$

For small θ,

$$\int_0^\theta \mathrm{Ai}(p)\, e^{-ipu}\, dp \sim \Gamma(\tfrac{1}{3})\theta/\sqrt{3} ,$$

the two last terms in (33) are small; thus

$$F(t) \approx \frac{2}{1+A} \exp[-\tfrac{2}{3}\gamma_1(t - t_1)] \times\left[1 + \frac{\Gamma(\frac{1}{3})}{2\pi\sqrt{3}}\left(\frac{c}{Bz\cos\alpha}\right)^{1/3}(t - t_1)\right] . \qquad (34)$$

As $t - t_1$ increases the contributions of the saddle points $\omega_{10,11}$ in Eq. (16) become important. These points are located far from the pole $\omega = 0$ and, according to (27) with $\omega_0 = 0$, we find

$$F(t) \approx -\left(\frac{3Bz\cos\alpha}{c\pi^2(t - t_1)^3}\right)^{1/4} \exp[-\tfrac{2}{3}\gamma_1(t - t_1)] \times \sin\left[\left(\frac{4c(t - t_1)^3}{27Bz\cos\alpha}\right)^{1/2} - \tfrac{1}{4}\pi\right]. \qquad (35)$$

Hence the second precursor arrives at the given point x, z at a time

$$t_1 = (x\sin\alpha)/c + (z\cos\alpha)/v_1$$

with velocity $v_1 = c(A + \frac{4}{3}B\gamma_1^2)^{-1}$ and with relative amplitude of order 1. At $t > t_1$ the amplitude at first slightly increases [see (34)] and then has damped oscillations [see (35)]. At the front of the precursor the relative amplitude equals 1, and behind the front it damps at large t.

The stationary field is determined by the pole $\omega = 0$. For large enough t, when the transient processes determined by saddle points will terminate, only the contribution from the pole $\omega = 0$ will remain. Hence in agreement with the Fresnel formulas for a constant field, the following field will be established in the medium:

$$\begin{pmatrix} E_y \\ H_x \\ H_z \end{pmatrix}_{\text{stat}} = \frac{2}{1+A}\begin{pmatrix} 1 \\ -A\cos\alpha \\ \sin\alpha \end{pmatrix} E_0 . \qquad (36)$$

The second precursor is observed against the background of this stationary signal.

IV. TRANSIENT PROCESSES IN A MEDIUM WITH A NUMBER OF CHARACTERISTIC FREQUENCIES

Let us now consider the refraction of a monochromatic signal (2) incident on the interface of a medium with, say, two different characteristic frequencies ω' and ω'' corresponding to oscillations of the electrons and ions which are located in the ultraviolet and infrared regions of the spectrum, respectively.

The field of the refractive signal is determined as before by formulas (4) and (6) but the refractive index is now the following generalization of (7):

$$k_1 = (\omega/c)n_1 ,$$

$$n_1 = \left(1 - \frac{\Omega_1^2/\cos^2\alpha}{\omega^2 + 2i\omega'\gamma' - \omega'^2} - \frac{\Omega_2^2/\cos^2\alpha}{\omega^2 + 2i\omega''\gamma'' - \omega''^2}\right)^{1/2} . \qquad (37)$$

Now the argument of the exponential function [see Eq. (8)] in the integrand (4) and (6) has four pairs of branch points rather than two as in (10) and (11).

After joining the corresponding points by four branch lines the integrand will be single valued and one can perform the integrations in the complex ω plane. As before we have no singularities in the upper half of the ω plane and $E^d = 0$ at $t < t_0 = (x \sin\alpha + z\cos\alpha)/c$. However, the dispersion curve $\omega_1(k)$ consists now of three branches rather than the two in Fig. 2, i.e., three and not two precursors will now propagate in the medium.

The saddle points corresponding to extrema of the group velocity are determined from the condition $d^2k/d\omega^2 = 0$ and are given by

$$\omega_9' = -i\tfrac{1}{6}\gamma' \pm \omega'(\Omega_2/\sqrt{3}\,\Omega_1)^{1/2}, \qquad \omega_9'' = -\tfrac{2}{3}\gamma' i. \tag{38}$$

For the corresponding velocities we receive

$$v' \approx c[A_1 + 2(3B_1B_2)^{1/2}]^{-1}, \qquad v'' \approx c(A_2 + \tfrac{4}{3}B_3\gamma''^2)^{-1}, \tag{39}$$

where

$$A_1 = \left(1 + \frac{\Omega_1^2}{\omega'^2}\right)^{1/2}, \quad A_2 = \left(1 + \frac{\Omega_1^2}{\omega'^2} + \frac{\Omega_2^2}{\omega''^2}\right)^{1/2},$$

$$B_1 = \frac{\Omega_2^2}{2A_1}, \quad B_2 = \frac{\Omega_1^2}{2\omega'^2 A_1}, \quad B_3 = \frac{\Omega_2^2}{2\omega''^2 A_2}.$$

From (39) we can see that $v' > v''$ and the following general picture of the propagation of precursors emerges: At a given point x, z the transient processes (the first precursor) begin at a time $t_0 = (x\sin\alpha + z\cos\alpha)/c$. The second precursor will arrive at this point at $t' = (x\sin\alpha)/c + (z\cos\alpha)/v'$ with a velocity v' and the third precursor at $t'' = (x\sin\alpha)/c + (z\cos\alpha)/v''$ with a velocity v''.

For a calculation of the refractive signal in the medium we shall again use the saddle-point method. It is clear that the results will be similar to those in Sec. II. For different times we obtain the following:

(i) $t_0 < t < t'$. The signal is determined by the same formula (21) as in case of one characteristic frequency.

(ii) $t \sim t'$. The value of the integral (4) is essentially determined by the saddle point ω_9' in Eq. (38) and as a result we find

$$F(t) = \frac{1}{2\pi}\left(\frac{c}{4B_2 z\cos\alpha}\right)^{1/3} \exp\left[-\frac{\gamma'}{6}\left(t - t' + \frac{8z\cos\alpha}{3c}(3B_1B_2)^{1/2}\right)\right] \times \mathrm{Ai}\left[\left(\frac{c}{4B_2 z\cos\alpha}\right)^{1/3}(t-t')\right]\frac{2}{[\omega_0 - \omega'(\Omega_2/\sqrt{3}\,\Omega_1)^{1/2}]^2} \times \left\{\omega_0 \sin\left[\omega'\left(\frac{\Omega_2}{\sqrt{3}\,\Omega_1}\right)^{1/2}(t-t')\right] + i\omega'\left(\frac{\Omega_2}{\sqrt{3}\,\Omega_1}\right)^{1/2}\cos\left[\omega'\left(\frac{\Omega_2}{\sqrt{3}\,\Omega_1}\right)^{1/2}(t-t')\right]\right\}. \tag{40}$$

If $\omega_0 \approx |\omega_9'|$, the path of integration is very near the pole and it is necessary to go around the pole. The influence of the pole will then become apparent before the beginning of the third precursor. At normal incidence the second and third precursors are observed against the background of a growing signal with frequency ω_0, but at oblique incidence they are separated in space.

(iii) $t \sim t''$. The signal is determined by (25) by changing $A, B, \gamma_1 \to A_2, B_3, \gamma''$ correspondingly.

The directions of propagation and the amplitudes (Sec. VI) of the precursors depend on the relations between the frequency of the incident signal and the characteristic frequencies of the medium: If $\omega_0 \approx |\omega_9'| \sim \omega'(\Omega_2/\Omega_1)^{1/2}$, the second precursor begins to propagate at the angle α_2, where $\sin\alpha_2 = (\sin\alpha)/n(\omega_9')$. For the angle of refraction β we have $\sin\beta = (\sin\alpha)/n(\omega_0)$, i.e., $\alpha_2 \approx \beta$, and the second precursor propagates almost along the direction of propagation of the refracted signal. For the third precursor we have $\sin\alpha_3 = (\sin\alpha)/n(\omega_9'')$, i.e., $\alpha_3 < \beta$; thus in principle it could be separated from the refracted signal.

If $\omega_0 < \omega''$, then $\alpha_3 \gtrsim \beta$, but now the difference $\alpha_2 - \beta$ is not small; therefore it is now the second precursor which can be separated from the refracted signal.

It will be shown in Sec. VI that depending on the relation between ω_0 and ω', ω'', the precursors differ not only in their respective directions, but also in their amplitudes. This latter fact would facilitate the observation of precursors.

V. INFLUENCE OF THE STEEPNESS OF THE FRONT OF THE SIGNAL ON THE TRANSIENT PROCESSES

In all previous discussions we assumed that the front of the incident signal was very sharp, i.e., the signal reaches its maximum amplitude instantly, while the real signal will of necessity smoothly increase at the front and smoothly decrease at the trailing end of the signal.

This circumstance is not important for a qualitative description of the transient processes, i.e., the appearance of few precursors before the stationary signal is established. However, it is essential for quantitative estimates, because the smoothness of the signal "smears" all transient process.

Let us discuss first the case of a medium with the plasma refractive index (30). Let the signal impinge on such a medium, but, in contrast to (29), with its front smeared,

$$\vec{E}^e = \begin{cases} 0, & t<0 \\ \vec{E}_0(1-e^{-bt}), & t>0 \end{cases}$$

$$= \frac{\vec{E}_0}{2\pi i} \int_{-i\infty}^{i\infty} \frac{b\exp(st)}{s(s+b)}\, ds\ . \tag{41}$$

Let us note that in this case it is more convenient to use Laplace-type rather than Fourier-type integral representations.

The steepness of the front is determined by the parameter b. From the theoretical point of view the limit of sharp front corresponds to the case when the field reaches its maximum over a distance of the order of a wavelength of the incident signal, $\Delta z \gtrsim h/\Delta t = \lambda$. In the microwave range ($\omega_0 \sim 10^9 - 10^{12}$ Hz) sharp fronts can be achieved experimentally ($b \sim \omega_0$), while in the optical range ($\omega_0 \sim 10^{14} - 10^{15}$ Hz) the steepness of the front is usually not more than 10^{12} Hz, i.e., $b/\omega_0 \ll 1$, and the front accommodates 10^3–10^4 wavelengths. Of course, under such circumstances formula (29) for the incident signal is insufficient and it is necessary to use the expression (41).

Transient processes in a medium with a plasma refractive index were studied in Sec. III. But now $b \neq \infty$ and, restricting ourselves to the case of a normally incident signal, we have using (4), (1), (30), and (41) for the refracted (d) and reflected (r) signals,

$$E_y^d = E_0 \frac{2b}{2\pi i} \int_{-i\infty}^{i\infty} \frac{\exp[st-(z/c)(s^2+\Omega^2)^{1/2}]}{s(s+b)[1+(1+\Omega^2/s^2)^{1/2}]} ds, \tag{42}$$

$$E_y^r = E_0 \frac{b}{2\pi i} \times \int_{-i\infty}^{i\infty} \frac{[s-(s^2+\Omega^2)^{1/2}]\exp(st-|z|/c)}{s(s+b)[s+(s^2+\Omega^2)^{1/2}]}\, ds\ . \tag{43}$$

By a change of variables in the integrals (42) and (43) it is possible to reduce the exponential factors to the generating function for Bessel functions of real arguments.[1,4] To a good approximation we can set $b/\Omega \ll 1$ and find the following results:

$$E_y^d \approx \frac{2b}{\Omega}\left(J_1(k_d)\delta + 2\sum_{l=1}^{\infty} J_{2l+1}(k_d)\delta^{2l+1}\right),$$
$$E_y^r \approx -(2b/\Omega)[J_3(k_r)+3J_5(k_r)+\cdots]\ . \tag{44}$$

The definitions of all quantities were given above [see (31)].

Comparing (44) and (31) we see that the change in the integrand in (4), $1/s \to b/s(s+b)$, associated with the finite steepness of the front of a signal (41), induces important departures from the previous results (compare with Fig. 3):

(i) On the interface ($z=0$) E^d increases first, then at $t\sim 3/\Omega$ reaches a maximum, at $t\sim 10/\Omega$ begins to decrease, and at $t\sim 100/\Omega$ practically vanishes. Let us note that for $t\sim 100/\Omega$ the incident signal E^e has only 1% of its maximum amplitude on the interface. The reflected signal E^r increases rapidly on the interface and at a time $t\sim 10/\Omega$ we have $E^r = E^e$, i.e., from this time on the incident signal is completely reflected (see Fig. 4).

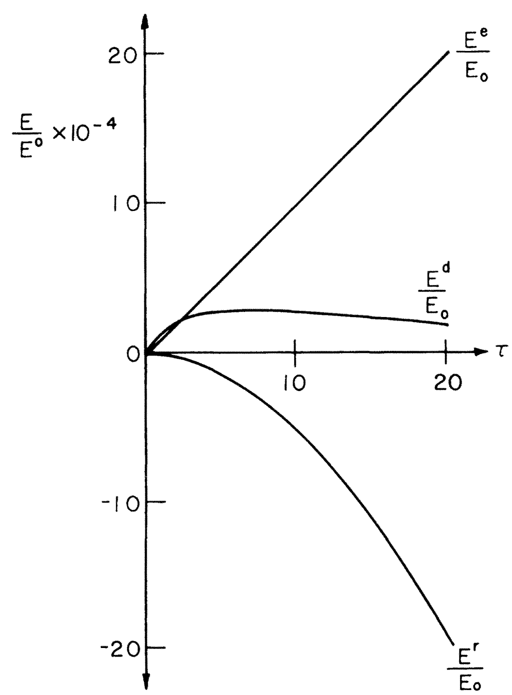

FIG. 4. Incident (E^e), refracted (E^d), and reflected (E^r) signals on the interface as a function of time $\tau = (\Omega_1/\cos\alpha)[t-(x\sin\alpha)/c]$ for the case of smooth increase of the amplitude E_0 of the incident signal.

(ii) On the front of the signal

$$E^r = E^d = 0. \quad (45)$$

Recall that for the case of the step signal (29) $E^d = E^e$ on the front of the signal [see Fig. 3(b)]. Behind the front E^d increases somewhat and then decreases. This increase is smaller for points in the medium more distant from the interface. The reflected signal E^r continuously increases behind the front and approaches E^e.

Let us consider now the incidence of the monochromatic signal with a smeared front:

$$E_y^e = \begin{cases} 0, & t<0 \\ E_0 e^{-i\omega_0 t}(1-e^{-bt}), & t>0 \end{cases}$$

$$= \frac{E_0}{2\pi i}\int_{-i\infty}^{i\infty} \frac{b\exp(st)}{(s+i\omega_0)(s+b+i\omega_0)}\, ds\ . \quad (46)$$

The refracted (d) and reflected (r) fields are similar to (31) and (44) and are given by

$$E_y^d \approx E_0(2b/\Omega)[J_1(k_d)\delta + 2J_3(k_d)\delta^3 + 2J_5(k_d)\delta^5 + \cdots]$$
$$- iE_0(2b/\Omega)(4\omega_0/\Omega)[J_2(k_d)\delta^2 + 3J_4(k_d)\delta^4 + 5J_6(k_d)\delta^6 + \cdots],$$
$$E_y^r \approx -E_0(2b/\Omega)[J_3(k_r) + 3J_5(k_r) + \cdots] + iE_0(2b/\Omega)(4\omega_0/\Omega)[J_4(k_r) + 4J_6(k_r) + 9J_8(k_r) + \cdots]. \quad (47)$$

Comparing (47) and (44) we conclude that if $b/\Omega \ll 1$ and $\omega_0 < \Omega$, then, the response $\mathrm{Re}[E^{r,d}]$ to the cosinusoidal incident signal in (46) and $E^{r,d}$ for the signal (41) have the same properties. The response $\mathrm{Im}[E^{r,d}]$ to an incident sine signal in (46) has the same properties, but its amplitude is smaller by a factor ω_0/Ω. This is because the field on the front of the incident sine signal increases more slowly.

Hence for the incident signal (46) also the amplitude of the reflected signal at a time $t \sim 10/\Omega$ is of the same order of magnitude as the amplitude of the incident signal. After this time the reflected signal increases up to the amplitude E_0 and then oscillates with the same frequency ω_0 as the incident signal.

Let us consider the influence of the steepness of the front of an incident signal on the transient processes in the more general case of a medium with the refractive index (7). We consider first the step signal (41). The Fourier components of this signal are proportional to $1/\omega(1+\omega^2/b^2)^{1/2}$, i.e., the smoothness of the front causes a decrease in the amplitude of the spectral components of the incident signal corresponding to the high frequencies $\omega \gtrsim b$. However, the high frequencies correspond to the first precursor (see Fig. 2) and for the first precursor it is enough to take into account the plasma reflective index (30). Therefore the influence of the steepness of the front is similar to that mentioned above. Namely, the intensity of the refracted signal is small, the reflected signal rapidly increases, and its intensity becomes equal to the intensity of the incident signal before the latter reaches its maximum value. Thus the smoothness of the front of the incident signal is not important for the subsequent stages of the transient process.[10]

If the incident signal is the monochromatic wave (46) with a smeared front, its Fourier-components are proportional to

$$\frac{1}{\omega_0}\left|\frac{1}{1-(\omega/\omega_0)^2} - \frac{1}{1-[(\omega+b)/\omega_0]^2}\right| .$$

This means that as result of the smoothness of the front both high-frequency ($\omega > \omega_0$) and low-frequency ($\omega < \omega_0$) components of the incident signal will be weakened. The weakening of the first precursor will be as calculated above.

The second precursor is determined by the value of the integral

$$E_y^d = E_0\left(-\frac{bi}{2\pi i}\right) \times \int \frac{f^d(\omega)\exp\{-i\omega[t-(z/c)n(\omega)]\}}{(\omega-\omega_0)(\omega+ib-\omega_0)}\, d\omega \quad (48)$$

near the saddle point $\omega_9 = -\frac{2}{3}\gamma_1 i$.

The influence of the steepness of the front is to change the amplitude factor $1/(\omega_0 + \frac{2}{3}i\gamma_1) \sim 1/\omega_0$ in the expression (25) to

$$\frac{b}{(-\frac{2}{3}i\gamma_1 - \omega_0)(-\frac{2}{3}i\gamma_1 + ib - \omega_0)} \sim \frac{b}{\omega_0^2}$$

in (48), i.e., the amplitude of the second precursor decreases by a factor $b/\omega_0 \sim 10^{-2}$–10^{-3}. Hence both precursors have small intensities and as a result of the smoothness of the front of the incident signal the transient processes are smeared.

VI. CONCLUSIONS

We shall now give a complete qualitative description of the transient processes when a signal impinges on the vacuum-medium interface. At normal incidence a number of oscillations with different amplitudes, frequencies, and durations may be observed in the medium. These oscillations are connected with the establishment of forced vibrations of different kinds of charged particles. Because the different particles have different inertial properties (the characteristic frequencies of the electrons $\omega \sim 10^{16}$ sec^{-1} are in the ultraviolet range of a spectrum and for ions $\omega \sim 10^{13}$–10^{14} sec^{-1}, i.e., in the infrared part) and different relaxation times, the transient processes, caused by the electrons and the ions, do not occur simultaneously. It is obvious that the intensity of any transient process has a maximum when the frequency of the incident signal is in resonance with the given polarization mechanism. Therefore by changing ω_0 it is possible to analyze any transient process, once faster transient processes have been damped out.

If the incidence is oblique the different relaxation processes turn out to be separated both temporally and spatially.

Electronic polarization is the least inertial mechanism for all substances. Therefore it is the electrons which first react to the incident signal and which give rise to the first precursor.

At the beginning of the transient process the electrons are quasifree, i.e., they are described by the plasma refractive index (30). Thus the properties of the first precursor do not in general depend on the nature of the substance.

The next precursors are associated with the properties of the substance and depend on the characteristic frequencies and relaxation times of the corresponding oscillations. All these precursors are gradually damped out and are replaced by a stationary signal which is determined by the forced oscillations of particles.

Let us now give some numerical estimates for the case of two precursors at normal incidence of a signal with frequency $\omega_0 = 4\times 10^{15}$ sec^{-1} with infinite steepness of its front, the frequency of the medium being $\omega_1 = \Omega_1 = 4\times 10^{16}$ sec^{-1}.

The first precursor arrives at any point z, say, 1 cm from the interface at a time $t_0 = z/c \sim 3\times 10^{-11}$ sec with an amplitude equal to the amplitude of the incident signal. Then the amplitude changes, and after $\Delta t \sim 10^{-12}$ sec becomes about 10^{-3} of its initial value for an incident cosinusoidal signal and about 10^{-5} for a sinusoidal form [see (21)]. The ratio of the average energy flux, transferred by the first precursor across unit area located at distance d from the interface, to the energy of the incident signal is equal to[4] $c/d\Omega_1 \sim 10^{-6}$.

At a time $t_1 = z/v_1 \sim 5\times 10^{-11}$ sec the second precursor arrives at the point z with a relative amplitude of about 10^{-5} for an incident cos-signal and about 3×10^{-2} for a sin-signal [see (25)]. To start with, the amplitude of the second precursor decreases as the Airy integral, after $\Delta t \sim 10^{-13}$ sec reaches relative values of 10^{-6} and 10^{-3}, respectively, and then increases quickly as $\omega \to \omega_0$.

The behavior of the reflected signal is quite different (see Sec. III and Ref. 4). The reflected field is equal to zero on the front of the incident signal. The reflected signal has a delay of time Δt relative to the refracted signal because its formation requires the excitation of several layers of the medium. For normal incidence at $\Delta t \sim 3/\Omega$ the depth of this "active region" is of the order $c\Delta t \sim 3c/\Omega \sim 10^{-6}$ cm, i.e., about 100 atomic layers. If $\omega_0 < \Omega$, the reflected signal near the interface reaches the value of amplitude of the incident signal quite quickly (after $\Delta t \sim 5/\Omega \sim 10^{-16}$ sec), then oscillates with the frequency of the incident signal. However, if $\omega_0 > \Omega$ there is no "total reflection" from the medium with plasma refractive index and the stationary state is reached considerably slower. The whole transient process will come to an end after a time $\Delta t = t_s - t_0 \sim 1.7 \times 10^{-11}$ sec [see (17)].

From these estimates it is clear that the order of magnitude of the energy transferred by precursors is at the limits of sensitivity of modern detectors of radiation. The duration of the transient process and the amplitudes of the precursors depend on the values of the parameters ω_0, ω_1, and Ω_1. The duration of the transient process turns out to be quite small with the above choice of parameters. However, apart from decreasing the frequency of the incident signal, we have a few other possibilities:

(i) Since the characteristic frequencies of the precursors are different, it is possible, in principle, to separate them by means of frequency filters.

(ii) For oblique incidence the directions of propagation of the precursors are different. Roughly speaking, the first precursor propagates along the direction of the incident signal and the second along the refracted signal. This fact may also be utilized to separate the precursors.

(iii) The dependence of the amplitudes of the two precursors on the distance z from the interface is also not the same, namely $E_{\mathrm{I}} \sim 1/\sqrt{z}\, \exp(-az/\omega^2)$ and $E_{\mathrm{II}} \sim 1/\sqrt{z}\, \exp(-bz\omega^2)$. That is, high-frequency components in the spectrum of the first precursor and low-frequency components for the second one are damped less than other frequencies. Therefore the further one is from the interface the greater the difference between the characteristic frequencies of the precursors and the longer the duration of the transient process. At the same time, unfortunately, the intensities of the precursors are sharply decreased.

(iv) If the medium is characterized by the more complicated refractive index (36), the number of precursors increases to three, the additional one occuring at an intermediate frequency which is more convenient for detection. Let us make some numerical estimates. For $\omega_0 = 10^{13}\,\text{sec}^{-1}$, $\omega' = \Omega_1 = 4\times10^{16}\,\text{sec}^{-1}$ and $\omega'' = \Omega_2 = 10^{14}\,\text{sec}^{-1}$ and at normal incidence of the signal the first precursor will arrive at the point $z = 1$ cm at a time $t_0 \sim 3\times10^{-11}$ sec, the second one at a time $t' \sim 5\times10^{-11}$ sec, and the third one at a time $t'' \sim 5.7\times10^{-11}$ sec. The whole transient process will terminate after $t_s \sim 5.8\times10^{-11}$ sec.

It was shown in Sec. IV that it is not only these three times which differ, but also the intensities and directions of propagation of the precursors. It was noted that results are essentially dependent on relations between the frequency of the incident signal ω_0 and the characteristic frequencies ω', ω'', and $\omega'(\Omega_2/\sqrt{3}\,\Omega_1)^{1/2}$.

Thus if $\omega_0 \sim \omega'(\Omega_2/\Omega_1)^{1/2} \sim 3\times10^{15}\,\text{sec}^{-1}$ (and $\omega' = \Omega_1 = 4\times10^{16}\,\text{sec}^{-1}$, $\omega'' = \Omega_2 = 10^{14}\,\text{sec}^{-1}$, $z = 1$ cm) the first precursor, which does not depend on the nature of substance, propagates at an angle which is slightly larger than the angle of incidence. The second and the third precursors propagate at angles α_{II} and α_{III} given by $\sin\alpha_{II} \sim 0.667\sin\alpha$, $\sin\alpha_{III} \sim 0.567\sin\alpha$. The stationary signal propagates at the angle of refraction β, where $\sin\beta \approx 0.665\sin\alpha$, i.e., the directions of propagation of the second and third precursors are close to that of the refracted signal. Under these conditions, the amplitudes relative to incident cosine and sine signals are for the first precursor equal to 10^{-3} and 10^{-5}, for the second equal to 10^{-1}, and for the third equal to 10^{-7} and 10^{-3}. Hence for $\omega_0 \sim \omega'(\Omega_2/\Omega_1)^{1/2}$ the amplitude of the second precursor is much larger than that for the first and the third.

If $\omega_0 < \omega''$, the intensity of the third precursor increases significantly. For example, for $\omega_0 = 10^{13}\,\text{sec}^{-1}$ the ratio of intensity of the third precursor to the intensity of the incident signal is 10^{-3} and 10^{-1} for cosinusoidal and sinusoidal forms of the incident signal, respectively. In this case the direction of propagation of the third precursor almost coincides with the angle of refraction ($\alpha_{III} \gtrsim \beta$), whereas for the second precursor $\alpha_{II} - \beta$ it is of the order of a few degrees.

Hence for $\omega_0 > \omega'$ the intensity of the first precursor is larger than the intensities of the other two, for $\omega_0 < \omega''$ the intensity of the third is the largest, while for $\omega_0 \sim \omega'(\Omega_2/\Omega_1)^{1/2}$ the second possesses the largest amplitude. Let us note that the frequency of the second precursor is in the optical range of the spectrum, which facilitate observation.

Unfortunately, the difficulties of observation of precursors are magnified by the influence of the steepness of the front of the signal on the transient processes (Sec. V). The spectral components of the signals (41) and (46) with smoothly increasing fronts are smaller than those for the signals (29) and (2) with infinite steepness, and as a result the intensities of the precursors decrease. Furthermore, the sharp impulse, imparted to the particles of the medium by the signals (29) and (2) with infinite steepness is absent for the smooth signals (41) and (46). The smooth increase of the incident signal results in a smooth increase of the refracted and reflected signals.

Both of the above-mentioned circumstances result in a smearing of the transient processes, especially for the refracted signal. For example (see Sec. V), in the case of a monochromatic incident signal the amplitude of the second precursor decreases by more than a factor of 100. Therefore for experimental investigation of transient processes it is necessary to use signals with sharpest possible fronts ($b \to \infty$) or, failing that ($b \ll \omega_0$), to observe the reflected and not the refracted signal.

The passage of finite short pulses across the interface could be of some interest. If the duration of the pulse is more than the duration of the transient processes in the medium, all transient phenomena will come to an end and the stationary signal will settle in the moment the smooth end of the pulse arrives at the interface. Therefore the smoothness of the beginning and of the end of the pulse will stimulate equal and nonoverlapping phenomena in antiphase. A series of such pulses will not change the picture because the medium will have come to rest by the time subsequent pulses arrive, and each new pulse will be perceived as the first one. If the duration of the pulse is less than the duration of the transient processes, the reaction of the medium will be much more complicated. This question demands special consideration.

The Sommerfeld-Brillouin theory has been widely used in investigations of the propagation of radio waves, particularly in the ionosphere. However, in the microwave region we know of only one experimental work[11] where both precursors were observed in good agreement with the theory. We hope that experimental investigation with light signals of different frequencies and durations will be carried out in the near future. For comparison with a specific experiment it will be necessary to do a computer simulation of the expected signal response using a specific form of the incident signal with a specific index of refraction of the medium. Such computer simulation has been done recently by Birman and Frankel[12] for a certain medium containing polaritons and surface plasmons.

One of the defects of our consideration is the restriction to a linear theory. This means that the durations of the incident signals must not be too small and their power should not be too large. Accounting for the nonlinear dependence of the polarization on the amplitude of the electric field requires—for the oscillator under consideration—the inclusion of anharmonic terms in the oscillator equation. However, the anharmonic terms will only become apparent after the harmonic ones, i.e., our considerations apply for the description of the beginning of the transient processes also in the nonlinear case.

Furthermore, we are interested in frequencies in the normal dispersion regions. In the anomalous dispersions region it is necessary to take into account the energy exchange between the field and the dispersive medium, that the group velocity is not simply related to the signal velocity, etc. All of these problems are more complicated and demand special consideration.

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[5]By choosing the corresponding Riemann sheet it is possible to eliminate the poles of the complex function $\phi^{r,d}(\omega)$, corresponding to the physical requirement of damping and propagation of the refracted field deep inside the medium. Furthermore, it is assumed that the angle α is not too close to $\frac{1}{2}\pi$. When $\alpha \approx \frac{1}{2}\pi$ the branch points $\omega_{3,4} \to \infty$ and this case needs the special consideration.

[6]This result also follows immediately from Fig. 1, because at $t < t_0$ $\phi(\omega)$ has a negative real part in the upper half of the complex ω plane, and the integrand has no singularities in this half-plane. Hence deforming the contour Γ into the upper half-plane we obtain $E^d = 0$ at $t < t_0$.

[7]It is possible to show that the group velocity corresponding to the saddle point when the path of integration passes through ω_0 practically coincides with the velocity of the signal.

[8]The distinction lies in the vector character of the signal, the existence of Fresnel's coefficients, and the obliqueness of the incidence of the signal.

[9]Equation (23) represents the energy-flux vector averaged over a period. For a discussion of the instantaneous energy flux, see J. A. Stratton, *Electromagnetic Theory* (McGraw-Hill, New York, 1941).

[10]Of course, it is assumed that $b >> \gamma$. Otherwise all transient processes will terminate (for a time $\sim 1/\gamma$) before the incident signal reaches its maximum value on the interface.

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