Analytically solvable problems in radiative transfer. IV

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(Received 30 June 1975)

The Biberman-Holstein integral equation is solved for an infinite cylinder in the limit of large optical thickness and for a wide class of absorption profiles. Special attention is paid to the Voigt profile.

I. INTRODUCTION

In solving time-dependent and time-independent problems in the theory of radiative transfer in a spectral line, an important role is played by the solutions of the following eigenvalue problem:

$$A(2,1)\Psi(\vec{r}) - A(2,1)\int_V K(|\vec{r}-\vec{r}'|)\Psi(\vec{r}')\,d\vec{r}' = \tilde{A}(2,1)\Psi(\vec{r}),$$

$$K(|\vec{r}-\vec{r}'|) = \int_0^\infty \mathcal{L}(\nu)k(\nu)\frac{\exp(-k(\nu)|\vec{r}-\vec{r}'|)}{4\pi|\vec{r}-\vec{r}'|^2}\,d\nu. \quad (1)$$

In this equation, $A(2,1)$ is the Einstein probability per second for the spontaneous transition $2\rightarrow 1$, $\mathcal{L}(\nu)$ is the emission, and $k(\nu)$ is the absorption profile. The density in state 1 is assumed to be independent of position. The integration in Eq. (1) extends over a volume V. The kernel $K(|\vec{r}-\vec{r}'|)$ has been introduced by Biberman[1] and Holstein;[2] it is required in order to solve Eq. (1) for the eigenvalues $\tilde{A}(2,1)$ and the corresponding eigenfunctions $\Psi(\vec{r})$. For example, let some spatial distribution of excited atoms be present in the volume V at time $t=0$. If no excitation or deexcitation by other mechanisms (e.g., collisions with electrons) takes place, the solution of the continuity equation for $n_2(\vec{r},t)$ for $t\geq 0$ is given by

$$\sum a_j\Psi_j(\vec{r})\exp[-\tilde{A}_j(2,1)t],$$

where $a_j=\int n_2(\vec{r})\Psi_j(\vec{r})\,d\vec{r}$ and the integration extends over V. The smallest eigenvalue $\tilde{A}_0(2,1)$ has been determined from the analysis of such a decay experiment.[3]

In stationary—and in particular nonequilibrium—problems, when there is also excitation and deexcitation of atoms by electrons, for instance, the use of the eigenvalues $\tilde{A}_j(2,1)$ and the corresponding eigenfunctions makes it possible to reduce the continuity equation for state 2 to a set of coupled linear equations.[4] Each of these equations is formally similar to the continuity equation for the excited state in the optically *thin* case. The solution of this set is either elementary or can be performed by elementary numerical techniques. We have reported on such nonequilibrium calculations for a slab elsewhere,[5] using results obtained previously (paper I of this series[6]). However, experiments on the nonequilibrium behavior of plasmas are performed almost exclusively in cylindrical geometries. Therefore, there is interest in the solutions of Eq. (1) when V is an infinite cylinder. We shall give these in Sec. II for the situation where the optical thickness k_0R is large for a wide class of line shapes. Though the calculations are more complicated than in the slab case, the principle is the same. For the Doppler and Lorentz profiles, the results will be compared to those of Payne and Cook,[7] which have been obtained by a direct numerical solution of Eq. (1) using essentially the Fredholm method. In many situations also, the spectral line shape is a Voigt profile. Section III contains a discussion of the values of the a parameter and k_0R for which the Voigt profile may be replaced by either a Doppler or a Lorentz profile. Furthermore, approximations to the solutions of Eq. (1) are derived for those values of a and k_0R where this cannot be done. The approximations are estimated to be accurate within 10%. We shall repeat a few calculations already performed in previous papers in order to make this one self-contained.

II. SOLUTION OF THE EIGENVALUE PROBLEM

In the following, it will be assumed that the symmetric line shape $\mathcal{L}(u)$, normalized to unity, exhibits a power-law dependence in the wings, i.e.,

$$\begin{aligned} &\mathcal{L}(\nu)\,d\nu=\mathcal{L}(u)\,du \sim D|u|^{1/(\alpha-1)}\,du\\ &|u|=2|\nu-\nu_0|/\Delta\nu \gg 1,\\ &k(\nu)=k(u)=k_0\mathcal{L}(u),\\ &k_0=(2\pi e^2/mc)n_1f/\Delta\nu. \end{aligned} \quad (2)$$

Here D and α are parameters, ν_0 is the central frequency, $\Delta\nu$ is a half-width, the density in the state 1 is denoted by n_1 (independent of position), and f is the oscillator strength. We have necessarily defined $0<\alpha<1$; otherwise $\mathcal{L}(u)$ could not be normalized. A particular case is a Lorentz profile which has $D=\pi^{-1}$ and $\alpha=\frac{1}{2}$. Asymmetric line shapes too can be dealt with if some simple modifications are introduced. (See Sec. III of Ref. 6). It then follows that the (asymmetric) statistical line shape has $D=1$ and $\alpha=\frac{1}{3}$. The Doppler profile requires separate treatment. The appropriate expressions for this case will be given without proof, being a straightforward modification of the argument given in the case of Eq. (2). It will turn out that the formulas for a Doppler profile have many features of the limit $\alpha\to 1$ from below in Eq. (2). In Eq. (2) neither D nor α is affected by possible hyperfine structure (hfs) of the line. Therefore our formulas invariably apply in this case [but not[6] if $\mathcal{L}(u)$ is a Doppler profile with hfs]. In solving equations like Eq. (1), an important role is played by the Fourier transform of the kernel, and in particular its behavior near $k=0$. It has been shown elsewhere[6] that

$$\hat{K}(\vec{k})\equiv\int e^{i\vec{k}\cdot\vec{r}}K(r)\,d\vec{r}\sim 1-C\frac{k_0}{k}\int_{u_0(k)}^{\infty}\mathcal{L}^2(u)\,du,\quad k\to 0, \tag{3}$$

where $u_0(k)$ is defined as the positive solution for $k\to 0$ of the equation

$$k_0\mathcal{L}(u_0)=k\,,$$

and where C is a constant and equal to $\pi/\sin(\frac{1}{2}\alpha\pi)$.

In Eq. (3) the integration extends over the entire three-dimensional space

$$\vec{r}=(x,y,z),\quad \vec{k}=(k_x,k_y,k_z),\quad k=(k_x^2+k_y^2+k_z^2)^{1/2}\,.$$

No confusion will arise between k, the absolute value of the wave vector $\vec{k}$, and the absorption profile $k(\nu)=k(u)$ in Eq. (2), since the latter will always be accompanied by its variable. Equation (3) has the following physical interpretation: It is easy to see that the mean free path of a photon at a frequency ν is equal to $k(\nu)^{-1}$. It will be shown below that in solving the eigenvalue problem, that part of the Fourier spectrum in Eq. (3) is important where k is of the order of R^{-1}, R being the radius of the cylinder. Hence $u_0(k)$ is the (dimensionless) frequency at which a photon has a mean free path of the order R. The integration on the right-hand side of Eq. (3), therefore, is over all frequencies at which a photon has a mean free path longer than R. Hence the behavior of $\hat{K}(k)$ near $k=0$, and (as will be shown) of the solutions of Eq. (1) for $k_0R\gg 1$, is determined by the behavior of $\mathcal{L}(u)$ in its far wings.

It is a simple matter to calculate the integral in Eq. (3) if $\mathcal{L}(u)$ is given by Eq. (2). We have

$$\hat{K}(k)\sim 1-\frac{1-\alpha}{1+\alpha}\frac{\pi D^{1-\alpha}}{\sin(\frac{1}{2}\alpha\pi)}\left(\frac{k}{k_0}\right)^{\alpha},\quad \frac{k}{k_0}\to 0\,. \tag{4a}$$

If $\mathcal{L}(u)$ is a Doppler profile,

$$\hat{K}(k)\sim 1-\frac{\pi}{4}\frac{k}{k_0}\left(\ln\frac{k_0}{k\sqrt{\pi}}\right)^{-1/2},\quad \frac{k}{k_0}\to 0\,. \tag{4b}$$

For a Doppler profile with hfs, see Ref. 6. If V is the infinite cylinder $(x^2+y^2)^{1/2}\leq R$ and $\Psi(\vec{r})$ in Eq. (1) is independent of z, the integration over z can be carried out. The two-dimensional Fourier transform of the new kernel is most easily obtained from Eqs. (3) or (4) by putting $k_z=0$. Let us now reconsider Eq. (1) where the integration has been carried out over z. In operator notation it can be written

$$(I-K)\Psi=\mu\Psi\,,$$

where I is the identity operator and $\mu=\tilde{A}(2,1)/A(2,1)$. Widom[8] has shown how the solutions of equations of this type can be obtained from the behavior of the Fourier transform for $k/k_0\to 0$ given in Eq. (4). The procedure is to show that the difference $I-K$ for $k_0R\to\infty$ becomes proportional to some other integral operator which is determined by the exponent α in the second term in Eq. (4) [$\alpha=1$ for Eq. (4b)]. In terms of this new integral operator, Eq. (1) can be reformulated and we have the following result: For j, l fixed and $k_0R\gg 1$, the eigenvalues of Eq. (1) are given by

$$\frac{\tilde{A}_{j,l}(2,1)}{A(2,1)}\sim\frac{1-\alpha}{1+\alpha}\frac{\pi D^{1-\alpha}}{\sin(\frac{1}{2}\alpha\pi)}\frac{\lambda_{j,l}^{-1}(\alpha)}{(k_0R)^{\alpha}} \tag{5a}$$

if Eq. (4a) applies, and if Eq. (4b) holds true, by

$$\frac{\tilde{A}_{j,l}(2,1)}{A(2,1)}\sim\frac{\pi}{4}\frac{\lambda_{j,l}^{-1}(\alpha=1)}{k_0R}\left(\ln\frac{k_0R}{\sqrt{\pi}}\right)^{-1/2}. \tag{5b}$$

Here $\lambda_{j,l}(\alpha)$ are the eigenvalues of the following integral equation[9]:

$$\lambda_{j,l}f_{j,l}^{(\alpha)}(\vec{\rho})=\int_{\rho\leq 1}K^{(\alpha)}(\vec{\rho},\vec{\rho}\,')f_{j,l}^{(\alpha)}(\vec{\rho}\,')\,d\vec{\rho}\,', \tag{6}$$

where $\vec{\rho}=(x/R,y/R)$; the integers j, l take the values $0,1,\ldots$. The eigenvalues with $l=0$ correspond to solutions of Eq. (6) where $f_{j,l}^{(\alpha)}(\vec{\rho})$ depends only on $\rho=|\vec{\rho}|$, and those with $l=1,\ldots$ to solutions $f_{j,l}^{(\alpha)}(\vec{\rho})$ dependent on both ρ and the angle φ. Furthermore, the eigenfunctions of Eq. (1) for $k_0R\to\infty$ tend (in mean square) to those of Eq. (6),

$$\Psi_{j,l}(\vec{\rho})\sim f_{j,l}^{(\alpha)}(\vec{\rho}),\quad \vec{\rho}=(x/R,y/R)\,. \tag{7}$$

The problem, therefore, is to solve Eq. (6). This is done in the Appendix, where some useful properties of the eigenfunctions are also discussed. Here only the procedure is briefly described and the results are given. The eigenfunctions $f_{j,l}^{(\alpha)}(\vec{\rho})$ in Eq. (6) are expanded in some suitably chosen complete set of functions which are mutually orthogonal with respect to some weight function. When this expansion (with expansion coefficients still un-

TABLE I. Lorentz profile values of $(2\pi)^{1/2}/3\lambda_{j,0}(\frac{1}{2})$.

j	Ref. 2	Ref. 7	This paper
0	1.115	1.123	1.122 74
1	···	1.856	1.854 72
2	···	2.37	2.372 51
3	···	2.80	2.796 41
4	···	3.17	3.164 15
5	···	3.50	3.493 46
6	···	···	3.794 33
7	···	···	4.073 04
8	···	···	4.333 89
≥9	···	···	$\sim\pi(j+\frac{9}{16})^{1/2}(\frac{1}{3}\sqrt{2})$

TABLE II. Doppler profile values of $\pi/4\lambda_{j,0}(1)$.

j	Ref. 2	Ref. 7	This paper
0	1.60	1.575	1.575 60
1	···	3.98	4.025 40
2	···	6.37	6.487 36
3	···	8.73	8.952 10
4	···	11.1	11.4179
5	···	13.3	13.8843
6	···	···	16.3510
7	···	···	18.8178
8	···	···	21.2848
≥9	···	···	$\sim\frac{1}{4}\pi^2(j+\frac{5}{8})$

known) is substituted in Eq. (6) and the orthogonality relation applied, it is then found that all the integrals can be calculated in closed form. We are then left with a well-conditioned eigenvalue problem in matrix form [given in Eq. (A8)], from which one can calculate as many eigenvalues and eigenfunctions as one pleases. In Tables I and II we have displayed our results for the eigenvalues $\tilde{A}_{j,0}(2,1)/A(2,1)$ in the cases where $\mathfrak{L}(u)$ is a Lorentz profile ($D=\pi^{-1}$, $\alpha=\frac{1}{2}$) and a Doppler profile, respectively. They are compared with the results of Holstein[2] and of Payne and Cook.[7] It is seen that the agreement for a Lorentz profile is excellent, and is satisfactory for a Doppler profile. Note that for a Doppler profile our k_0R is equal to $k_0R/\sqrt{\pi}$ as used by Holstein[2] and Payne and Cook,[7] owing to the fact that we have used a normalized Doppler profile $\int\mathfrak{L}(u)\,du=1$. The eigenfunctions can be written in cylindrical coordinates for $0\leqslant\rho\leqslant1$ as

$$f_{j,l}^{(\alpha)}(\vec{\rho})=2^{-\alpha/2}(1-\rho^2)^{\alpha/2}e^{\pm il\varphi}\rho^l \times\sum_{m=0}^{\infty}\frac{\Gamma(m+1)}{\Gamma(m+1+\alpha/2)}c_m(j,l)P_m^{(l,\alpha/2)}(1-2\rho^2). \tag{8}$$

They vanish for $\rho>1$. In Eq. (8), j and l take the values $0,1,\ldots,$ and $P_m^{(l,\alpha/2)}(1-2\rho^2)$ is a Jacobi polynomial.[10] The coefficients $c_m(j,l)$ are the numerical solutions of the eigenvalue problem, given in the Appendix [see Eq. (A8)]. They are normalized such that the eigenfunctions are orthonormal, i.e.,

$$\int_0^{2\pi}\int_0^1 f_{j,l}^{(\alpha)*}(\rho,\varphi)f_{j',l'}^{(\alpha)}(\rho,\varphi)\rho\,d\rho\,d\varphi=\delta_{j,j'}\delta_{l,l'}. \tag{9}$$

Tables III and IV give the coefficients $c_m(j,l=0)$ for the first three eigenfunctions for $\alpha=\frac{1}{2}$ (Lorentz

TABLE III. Coefficients of the eigenfunctions for a Lorentz profile ($\alpha=\frac{1}{2}$).

m	$c_m(j=0,l=0)$	$c_m(j=1,l=0)$	$c_m(j=2,l=0)$
0	0.702 27	0.211 281	−0.120 49
1	0.328 22	−1.1791	0.688 36
2	−0.003 36	−1.1106	−0.963 41
3	0.011 41	−0.239 97	−1.8818
4	−0.006 59	−0.045 51	−0.843 69
5	0.004 38	0.007 39	−0.217 70
6	−0.003 11	−0.006 96	−0.019 78
7	0.002 32	0.005 12	−0.010 53
8	0.001 80	−0.003 96	0.005 62
9	0.001 43	0.003 15	−0.004 67

TABLE IV. Coefficients of the eigenfunctions for a Doppler profile ($\alpha=1$).

m	$c_m(j=0,l=0)$	$c_m(j=1,l=0)$	$c_m(j=2,l=0)$
0	0.947 24	0.331 32	−0.198 13
1	0.375 89	−1.9183	1.2177
2	0.009 72	−1.6385	−1.8299
3	0.006 69	−0.371 31	−3.1787
4	−0.003 08	−0.058 44	−1.4206
5	0.001 79	0.003 15	−0.351 97
6	−0.001 12	−0.004 43	−0.041 96
7	0.000 75	0.002 88	−0.010 73
8	−0.000 52	−0.002 04	0.003 62
9	0.000 38	0.001 49	−0.002 98

profile) and $\alpha = 1$ (Doppler profile), and the same eigenfunctions are displayed in Figs. 1 and 2. The rapid convergence of the $c_m(j, l=0)$ as a function of m should be noted.

If the radiative transfer problem in this geometry has cylindrical symmetry, as is usually the case, then only the eigenfunctions with $l=0$ are needed. Therefore we have not displayed the expansion coefficients of the eigenfunctions with $l \neq 0$. A comparison of our eigenfunctions with those obtained by Payne and Cook[7] reveals a larger discrepancy than in the case of the eigenvalues, being of the order of a few percent, and increasing for larger values of j. These are to be attributed to the approximations made by these authors.

For later reference, it should be mentioned that certain asymptotic relations apply to the eigenvalues and eigenfunctions. It can be shown, for example,[11] that in Eqs. (5a) and (5b), for $l=0$,

$$\lambda_j^{-1}(\alpha) = \gamma_j^\alpha, \quad \gamma_j \sim [j + (\tfrac{1}{2} + \tfrac{1}{8}\alpha)]\pi, \quad j \to \infty .$$

This relation is already accurate within 2 and 0.25% for $j = 0, 1$, and within four decimals for $j = 10$. Furthermore, we have for the eigenfunctions $(l=0)$, $j \gg 1$,

$$f_j^{(\alpha)}(\rho) \approx J_0(\gamma_j \rho)\left(2\pi \int_0^1 J_0^2(\gamma_j\rho)\rho\, d\rho\right)^{-1/2} . \qquad (10)$$

This relation is valid within a few percent from $j=5$ on, everywhere except very near $\rho = 1$. In Eq. (10), $J_0(\gamma_j \rho)$ is a Bessel function of order zero. It is not difficult, with the aid of the present results, to solve the various problems that have been solved in previous papers for a slab.

III. VOIGT PROFILE

In many cases the line shape is neither purely Lorentz nor purely Doppler but a "mixture" of both, that is, a Voigt profile[12]:

$$\mathfrak{L}(u) = \frac{a}{\pi\sqrt{\pi}} \int_{-\infty}^{+\infty} \frac{e^{-t^2}\, dt}{(t-u)^2 + a^2}, \quad a = (\Delta\nu_L / \Delta\nu_D)(\ln 2)^{1/2},$$

$$k(u) = k_0 \mathfrak{L}(u), \quad k_0 = (2\pi e^2/mc)(n_1 f/\Delta\nu_D)(\ln 2)^{1/2} . \qquad (11)$$

The limits $a \to 0$ and $a \to \infty$ correspond to a pure Doppler and a pure Lorentz profile, respectively. It is well known that regardless of how small a is, the wings of the Voigt function exhibit a Lorentz character. Since the photons escape through the wings, we have (see[6] Appendix A of Paper I)

$$\hat{K}(k) \sim 1 - (\tfrac{2}{3}\pi)^{1/2}(ak/k_0)^{1/2}, \quad k/k_0 \to 0 . \qquad (12)$$

This equation is the same as Eq. (4a) for $D = \pi^{-1}$ and $\alpha = \frac{1}{2}$, i.e., the same as for a Lorentz profile. The results of the foregoing section, derived in this case, therefore invariably hold for the Voigt function. However, two important restrictions have to be kept in mind. Equation (12) is not valid for natural broadening because Eq. (1) does not then apply, the assumption $\mathfrak{L}(\nu) \propto k(\nu)$ being invalid. Furthermore, the higher-order terms in the asymptotic expansion of $\hat{K}(k)$ in Eq. (12) depend on both the parameter a and k/k_0. Hence Eq. (12) is accurate from a certain value of $k/k_0 \approx (k_0 R)^{-1}$ on, which is dependent on a. In order to see when Eq. (12) is accurate, we have calculated numerically, as a function of k_0/k, the quantity [see Paper I for the expression[6] of $\hat{K}(k)$]

$$1 - \hat{K}(k) = 1 - \int_{-\infty}^{+\infty} \mathfrak{L}(u)\frac{k(u)}{k} \arctan\left(\frac{k}{k(u)}\right) du . \qquad (13)$$

The result is shown in Fig. 3, where the Lorentz broadening is due to foreign gas atoms ($\Delta\nu_L$ inde-

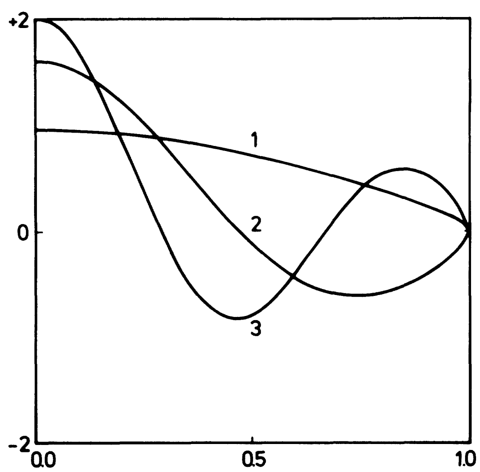

FIG. 1. Eigenfunctions for $\alpha = 1$ (Doppler profile).

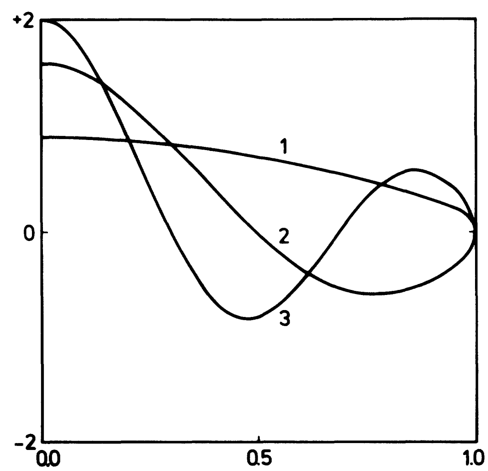

FIG. 2. Eigenfunctions for $\alpha = \frac{1}{2}$ (Lorentz profile).

pendent of n_1) together with the asymptotic expression for $1-\hat{K}(k)$ given in Eqs. (4b) and (12). It should be recalled from Eqs. (5a) and (5b) that $1-\hat{K}(k)$ is essentially equal to the eigenvalues $\tilde{A}_j(2,1)/A(2,1)$ if we put $k/k_0=\lambda_j^{-1/\alpha}(k_0R)^{-1}$. From Fig. 3 we can readily see the values of a and $k_0/k\approx k_0R$ for which Eqs. (12) and (4b) and therefore Eqs. (5a) (for $\alpha=\frac{1}{2}$, $D=\pi^{-1}$) and (5b) apply. Note in particular that noticeable deviations from Eq. (12) are already apparent for very small values of a and moderately large values of k_0/k.

As Fig. 3 shows, there is an important range of values of $k/k_0\approx(k_0R)^{-1}$ and of the parameter a to which neither Eq. (12) nor Eq. (4b) applies and for which, therefore, the results of Sec. II are invalid. We shall try to derive expressions for the eigenvalues $\tilde{A}_j(2,1)/A(2,1)$ of Eq. (1) in this intermediate region.[13] The procedure essentially consists in smoothing together in some reasonable way the asymptotic expressions given in Eqs. (5a) ($\alpha=\frac{1}{2}$, $D=\pi^{-1}$) and (5b). Let us first reconsider Eq. (5a) for $\alpha=\frac{1}{2}$ and $D=\pi^{-1}$ and Eq. (5b) for $l=0$ (the cylindrically symmetric solutions). The equations have been derived on condition that j remains fixed and k_0R is large. These conditions are essential. If we keep k_0R fixed and let $j\to\infty$, for example, then because of the asymptotic relation

$$\lambda_j^{-1}=\gamma_j^\alpha\sim[(j+\tfrac{1}{2}+\tfrac{1}{8}\alpha)\pi]^\alpha ,$$

we see that $\tilde{A}_j(2,1)/A(2,1)\to\infty$. This is clearly incorrect since the decay rates $\tilde{A}_j(2,1)$ can at most

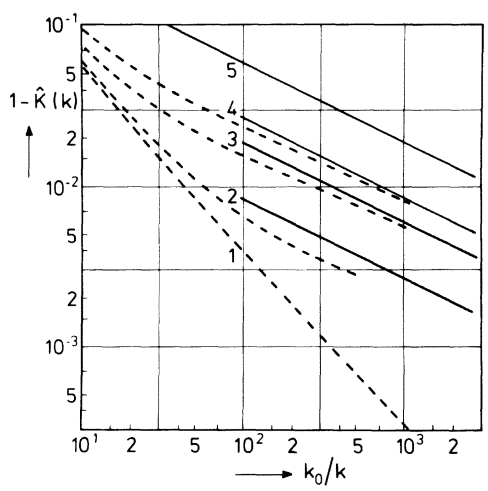

FIG. 3. $1-\hat{K}(k)$ as a function of $k_0/k\approx k_0R$, compared with the asymptotic approximations Eqs. (4b) and (12). Line shape is a Voigt profile with values of the parameter a as follows: (1) $a=0$; (2) $a=0.01$; (3) $a=0.05$; (4) $a=0.01$; (5) $a=0.5$. Equation (4b) coincides everywhere with curve (1); solid lines display Eq. (4a), which coincides for $a=0.5$ with curve (5).

become equal to $A(2,1)$ for $j\to\infty$. In order to correct this situation, let us write tentatively for $k_0R\gg1$, j variable,

$$\tilde{A}_j(2,1)/A(2,1)=1-\hat{K}(k=\gamma_jR^{-1}),\quad \gamma_j=\lambda_j^{-1/\alpha} . \tag{14}$$

Here $\hat{K}(k)$ is given by Eq. (13) and $\mathcal{L}(u)$ is either a pure Lorentz or a pure Doppler profile (and $\alpha=\frac{1}{2}$ or $=1$, respectively). Now, if γ_j/k_0R is small (i.e., j fixed, $k_0R\gg1$), then by Eqs. (4a) (with $\alpha=\frac{1}{2}$, $D=\pi^{-1}$) and (4b), we immediately recover[14] Eqs. (5a) and (5b). On the other hand, if γ_j/k_0R is large (i.e., k_0R is large but fixed, $j\to\infty$), then it is not difficult to show from Eq. (13) that $\hat{K}(k=\gamma_jR^{-1})$ tends to zero so that $\tilde{A}_j(2,1)/A(2,1)$ tends to 1 for $j\to\infty$, as it should. Equation (14) being correct in two opposite limiting cases, it cannot be much in error for all values of γ_j/k_0R on the sole condition that $k_0R\gg1$ (the condition that j is fixed therefore having been withdrawn).

There are strong arguments in favor of Eq. (14), suggesting that it may indeed be possible to prove it to be exact for all j if $k_0R\gg1$. These originate from a closer study of the integral equation in Eq. (1) but cannot be discussed here.[15] We shall accept in the following pages that, in any case, Eq. (14) provides a very good approximation to the eigenvalues $\tilde{A}_j(2,1)/A(2,1)$ of Eq. (1) for j variable, $k_0R\gg1$ if $\mathcal{L}(u)$ is either a Lorentz or a Doppler profile.

Suppose for a moment that the numbers γ_j are independent of α, and consider Eqs. (13) and (14) in which $\mathcal{L}(u)$ is now a Voigt profile. It is then clear that Eq. (14) with $\mathcal{L}(u)$ a Lorentz or a Doppler profile can be obtained from the same equation with $\mathcal{L}(u)$ a Voigt profile, letting $a\to\infty$ or $a\to0$. Actually the $\gamma_j[\approx(j+\frac{1}{2}+\frac{1}{8}\alpha)\pi]$ are not independent of α, but fortunately this dependence is very weak. We lose at most a few percent accuracy if we replace the γ_j by numbers γ_j' which are independent of α. With the choice

$$\gamma_j'=(j+\tfrac{5}{8})\pi,\quad(\lambda_j^{-1}=\gamma_j'^{\alpha}),$$

it appears that the numbers in Tables I and II for $j=0$ are reproduced within 5% (too large) and 2% (too small) and with much greater accuracy for $j\geq1$. With a somewhat different choice of γ_j' even a slightly better fit can be made, but this seems hardly worthwhile. The reasoning above is now repeated: Eq. (14) with $\mathcal{L}(u)$ a Voigt profile and $\gamma_j'=(j+\frac{5}{8})\pi$ replacing γ_j, being valid within a few percent for $k_0R\gg1$ and j variable in the two limiting cases $a\to\infty$ and $a=0$, we expect it to not be greatly in error for all values of a. The estimated error is roughly double the maximum value mentioned above; therefore it is of the order of 10%, let us say, for $j=0$, and much less for $j \geq 1$. In view of the uncertainties in current experiments, theories and parameters (cross sections, etc.) used in building models of plasmas, this value is quite acceptable. The result also holds true for resonance broadening, but, as has been mentioned, not for natural boradening. In the latter case, the diffusion approximation is appropriate for sufficiently large k_0R leading to $\bar{A}_j(2,1)/A(2,1) \propto (k_0R)^{-2}$ and therefore considerably more trapping.

The question about the corresponding eigenfunctions is more difficult to answer. Fortunately, as Figs. 1 and 2 show, the difference between the cases $\alpha = \frac{1}{2}$ (Lorentz) and $\alpha = 1$ (Doppler) is small. It is of the order of 5% for $j=0$, except very close to the wall. It becomes rapidly smaller for larger values of j, because, as Eq. (10) shows, the eigenfunctions for j large tend to the same limit everywhere except very close to the wall. For most purposes, therefore, it is rather immaterial whether in the general Voigt case one uses the eigenfunctions for $\alpha = 1$ (Doppler) or for $\alpha = \frac{1}{2}$ (Lorentz), provided, of course, one uses the same set consistently.

IV. CONCLUSION AND REMARKS

It has been shown how the eigenvalues and eigenfunctions of the Biberman-Holstein integral equation are calculated for a variety of line shapes when the volume is an infinite cylinder of radius R. These include the Doppler and Lorentz profiles. Approximate formulas have been given for the Voigt profile. The results can be applied to nonequilibrium calculations in an infinite cylinder[16] in the same manner as in the case of a slab.[5]

APPENDIX

In this appendix we solve for $\rho = |\vec{\rho}| \leq 1$ the eigenvalue problem

$$\lambda(\alpha) f^{(\alpha)}(\vec{\rho}) = \int_{\rho \leq 1} K^{(\alpha)}(\vec{\rho}, \vec{\rho}\,') f^{(\alpha)}(\vec{\rho}\,') d\vec{\rho}\,' . \quad \text{(A1)}$$

The kernel $K^{(\alpha)}(\vec{\rho}, \vec{\rho}\,')$ vanishes for $\rho, \rho' > 1$. For $\rho, \rho' \leq 1$ it is given by[9]

$$K^{(\alpha)}(\vec{\rho}, \vec{\rho}\,') = \frac{2^{1-\alpha}}{\pi \Gamma^2(\frac{1}{2}\alpha)} \int_{\max(\rho,\rho')}^{1} \frac{s^{1-\alpha}(s^2-\rho^2)^{(\alpha/2)-1}(s^2-\rho'^2)^{(\alpha/2)-1}(s^4-\rho^2\rho'^2)\, ds}{s^4 - 2s^2 \vec{\rho}\cdot\vec{\rho}\,' + \rho^2\rho'^2} .$$

It appears convenient to transform Eq. (A1) to Fourier space. In the following, $\vec{k} = (k_1, k_2)$ and $\vec{k}\cdot\vec{\rho} = k_1\xi_1 + k_2\xi_2$. We multiply both sides of Eq. (A1) by $(2\pi)^{-1} e^{i\vec{k}\cdot\vec{\rho}}$ and integrate over all $\vec{\rho}$ with $\rho \leq 1$.

To the integral with respect to $\vec{\rho}\,'$, we apply Parseval's identity.[17] In Fourier space Eq. (A1) then becomes

$$\lambda(\alpha) \hat{f}^{(\alpha)}(k) = \int \hat{K}^{(\alpha)}(\vec{k}, \vec{k}') \hat{f}^{(\alpha)}(\vec{k}')\, d\vec{k}' , \quad \text{(A2)}$$

with

$$\hat{f}^{(\alpha)}(\vec{k}) = (2\pi)^{-1} \int_{\rho \leq 1} e^{i\vec{k}\cdot\vec{\rho}} f^{(\alpha)}(\vec{\rho})\, d\vec{\rho} ,$$

$$\hat{K}^{(\alpha)}(\vec{k}, \vec{k}') = (2\pi)^{-2} \int\int \exp(i\vec{k}\cdot\vec{\rho} - i\vec{k}'\cdot\vec{\rho}\,') \times K^{(\alpha)}(\vec{\rho}, \vec{\rho}\,')\, d\vec{\rho}\, d\vec{\rho}\,' .$$

For $\hat{K}^{(\alpha)}(\vec{k}, \vec{k}')$, we have the following representation in cylindrical coordinates[9]:

$$K^{(\alpha)}(\vec{k}, \vec{k}') = (kk')^{-\alpha/2} \sum_{m=0}^{\infty} \frac{\epsilon_m}{2\pi} \cos[m(\Psi - \Psi')] \times \int_0^1 s J_{m+\alpha/2}(sk) J_{m+\alpha/2}(sk')\, ds . \quad \text{(A3)}$$

In Eq. (A3), $\epsilon_0 = 1$, $\epsilon_m = 2$, $m \geq 1$, and $\Psi - \Psi'$ is the angle between the two-dimensional vector $\vec{k}$ and $\vec{k}'$. In the eigenfunctions $\hat{f}^{(\alpha)}(\vec{k})$, the radial and angular parts are separated by putting

$$\hat{f}^{(\alpha)}(\vec{k}) = i^l e^{\pm i l \Psi} \hat{f}_l^{(\alpha)}(k) . \quad \text{(A4)}$$

Equation (A4) is substituted in Eq. (A2). The integration over Ψ is trivial, and we are left with the eigenvalue problem

$$\lambda(\alpha) \hat{f}_l^{(\alpha)}(k) = k^{-\alpha/2} \int_0^1 s J_{l+\alpha/2}(sk) \int_0^{\infty} J_{l+\alpha/2}(sk') \hat{f}_l^{(\alpha)}(k') k'^{1-\alpha/2}\, dk' .$$

It has been shown in Ref. 9 that with $p(m) = 2m + l + 1 + \frac{1}{2}\alpha$,

$$(kk')^{-\alpha/2} \int_0^1 s J_{l+\alpha/2}(sk) J_{l+\alpha/2}(sk')\, ds = 2(kk')^{-1-\alpha/2} \sum_{m=0}^{\infty} p(m) J_{p(m)}(k') J_{p(m)}(k') .$$

Hence the eigenvalue problem becomes

$$\lambda(\alpha) \hat{f}_l^{(\alpha)}(k) = 2k^{-1-\alpha/2} \sum_{m=0}^{\infty} p(m) J_{p(m)}(k) \int_0^{\infty} J_{p(m)}(k') \hat{f}_l^{(\alpha)}(k') k'^{-\alpha/2}\, dk' . \quad \text{(A5)}$$

We now expand the radial function as follows:

$$\hat{f}^{(\alpha)}_{j,l}(k) = k^{-1-\alpha/2} \sum_{m=0}^{\infty} c_m(j,l) J_{p(m)}(k), \tag{A6}$$

with coefficients $c_m(j,l)$, dependent on α, to be determined below. We substitute Eq. (A6) in Eq. (A5), multiply both sides of the resulting equation by $k^{\alpha/2} J_{p(n)}(k)$, integrate them over k from zero to infinity, and apply the orthogonality relation[18]

$$\int_0^\infty k^{-1} J_{p(n)}(k) J_{p(m)}(k) = 2p(n)\delta_{n,m}. \tag{A7}$$

Equation (A5) is then reduced to the matrix problem

$$\lambda_j(\alpha) c_n(j,l) = \sum_{m=0}^{\infty} \kappa_{n,m}(l) c_m(j,l), \quad n = 0, 1, \ldots, \tag{A8}$$

with[19]

$$\kappa_{n,m}(l) = 2p(n) \int_0^\infty J_{p(n)}(k') J_{p(m)}(k') k'^{-1-\alpha/2}\, dk' = \frac{\Gamma(1+\alpha)(2n+l+1+\frac{1}{2}\alpha)\Gamma(n+m+l+1)}{2^\alpha \Gamma(1+\frac{1}{2}\alpha+m-n)\Gamma(1+\frac{1}{2}\alpha+n-m)\Gamma(m+n+l+2+\alpha)}.$$

The matrix becomes symmetric upon introducing

$$c_n(j,l) = (2n+l+1+\tfrac{1}{2}\alpha)^{1/2} c_n'(j,l).$$

Equation (A8) can now be solved by truncating and applying standard numerical techniques. The matrix truns out to be very well conditioned. For instance, if truncated at order $M=8$, the first eigenvalue appears to be accurate to seven decimal places, and the first eigenfunction is correct to at least four decimal places. From Eqs. (A6) and (A4) we have the following representation of the eigenfunction:

$$\hat{f}^{(\alpha)}_{j,l}(\vec{k}) = i^l e^{\pm i l \psi} k^{-1-\alpha/2} \sum_{m=0}^{\infty} c_m(j,l) J_{2m+l+1+\frac{1}{2}\alpha}(k). \tag{A9}$$

Our primary interest is not in $\hat{f}^{(\alpha)}(k)$ but in the function $f^{(\alpha)}(\vec{\rho})$ in ordinary space. They are related by the inverse Fourier transformation

$$f^{(\alpha)}(\vec{\rho}) = (2\pi)^{-1} \int e^{-i\vec{k}\cdot\vec{\rho}} \hat{f}^{(\alpha)}(\vec{k})\, d\vec{k}.$$

Substitution of Eq. (A9) gives integrals that are all standard and can be obtained from existing tables.[20] The result is that $f^{(\alpha)}(\vec{\rho})$ vanishes for $\rho > 1$ and that for $\rho \le 1$ in cylindrical coordinates

$$f^{(\alpha)}_{j,l}(\vec{\rho}) = 2^{-\alpha/2}(1-\rho^2)^{\alpha/2} e^{\pm i l \varphi} \rho^l \times \sum_{m=0}^{\infty} \frac{\Gamma(m+1)}{\Gamma(m+1+\frac{1}{2}\alpha)} c_m(j,l) P_m^{(l,\alpha/2)}(1-2\rho^2). \tag{A10}$$

In Eq. (A9), $P_m^{(l,\alpha/2)}(1-2\rho^2)$ is a Jacobi polynomial.[21] The solutions $c_m(l,j)$ of Eq. (A8) are unique, apart from a multiplication constant. The constant is fixed by requiring that the eigenfunctions, in addition to being orthogonal [which they are, since the kernel in Eq. (A1) is symmetric], also be orthonormal, i.e.,

$$\int_{\rho \le 1} f^{(\alpha)*}_{j,l}(\vec{\rho}) f^{(\alpha)}_{j',l'}(\vec{\rho})\, d\vec{\rho} = \delta_{j,j'}\delta_{l,l'}. \tag{A11}$$

By Parseval's relation,[17] Eq. (A11) is equivalent to

$$\int \hat{f}^{(\alpha)*}_{j,l}(\vec{k}) \hat{f}^{(\alpha)}_{j',l'}(\vec{k})\, d\vec{k} = \delta_{j,j'}\delta_{l,l'}. \tag{A12}$$

In order to determine the orthonormality relation for the coefficients $c_m(j,l)$ in explicit form, Eq. (A9) is substituted in the left-hand side of Eq. (A12). The integration over the angle yields $2\pi\delta_{l,l'}$. The left-hand side of Eq. (A12) becomes with $p(m) = 2m+l+1+\frac{1}{2}\alpha$,

$$2\pi\delta_{l,l'} \sum_{m,m'=0}^{\infty} c_m(j,l) c_{m'}(j',l) \int_0^\infty J_{p(m)}(k) J_{p(m')}(k) k^{-1-\alpha/2}\, dk = 2\pi\delta_{l,l'} \sum_{m,m'=0}^{\infty} c_m(j,l)(4m+2l+2+\alpha)^{-1} \kappa_{m,m'}(l) c_{m'}(j',l).$$

The summation over m' is carried out by Eq. (A8). Comparing the result with the right-hand side of Eq. (A12), it is seen that the $c_m(j,l)$ must obey

$$2\pi\lambda_j(\alpha) \sum_{m=0}^{\infty} (4m+2l+2+\alpha)^{-1} c_m(j,l) c_m(j',l) = \delta_{j,j'}. \tag{A13}$$

That the coefficients $(4m+2l+2+\alpha)^{-1/2} c_m(j,l)$ are orthogonal also follows, of course, from the fact that the matrix problem of Eq. (A8) is symmetric for these quantities. Another useful orthogonality relation follows from the fact that the kernel in Eq. (A5) can be expanded in terms of its eigenfunctions normalized to unity on the interval $(0,\infty)$ as follows $[p(m) = 2m+l+1+\frac{1}{2}\alpha]$:

$$2(kk')^{-1-\alpha/2}\sum_{m=0}^{\infty} p(m)J_{p(m)}(k)J_{p(m)}(k')$$

$$=(2\pi)^{-1}\sum_{j=0}^{\infty}\lambda_j \hat{f}_{j,l}(k)\hat{f}_{j,l}(k').$$

We substitute Eq. (A6), multiply both sides by $(kk')^{\alpha/2}J_{p(n)}(k)J_{p(n')}(k')$, integrate over k and k' from 0 to ∞, and apply Eq. (A7). We obtain

$$2\pi\sum_{j=0}^{\infty}\lambda_j c_n(j,l)c_{n'}(j,l)=(4n+2l+2+\alpha)\delta_{n,n'}. \quad \text{(A14)}$$

Finally, it should be noted that a number of integrals can be expressed directly in terms of the expansion coefficients. For example,

$$\int_0^1 \rho^{2p} f_{j,0}(\rho)\rho\, d\rho = 2^{-\alpha/2}\sum_{m=0}^{\infty}\frac{\Gamma(m+1)}{\Gamma(m+1+\frac{1}{2}\alpha)}c_m(j,0)\int_0^1 \rho^{2p+1}(1-\rho^2)^{\alpha/2}P_m^{(0,\alpha/2)}(1-2\rho^2)\,d\rho. \quad \text{(A15)}$$

By Rodrigues's formula we have[21]

$$(1-\rho^2)^{\alpha/2}P_m^{(0,\alpha/2)}(1-2\rho^2)=\frac{1}{\Gamma(m+1)}\left(\frac{d}{d\rho^2}\right)^m \rho^{2m}(1-\rho^2)^{m+\alpha/2}.$$

After substituting this formula into Eq. (A15) and applying integration by parts, we obtain

$$\int_0^1 \rho^{2p} f_{j,0}(\rho)\rho\, d\rho = 2^{-\alpha/2}\sum_{m=0}^{\infty}\frac{(-1)^m c_m(j,0)}{\Gamma(m+1+\frac{1}{2}\alpha)}\int_0^1 \rho^{2m+1}(1-\rho^2)^{m+\alpha/2}\left(\frac{d}{d\rho^2}\right)^m \rho^{2p}\,d\rho$$

$$=\frac{\Gamma^2(p+1)}{2^{\alpha/2+1}}\sum_{m=0}^{p}\frac{(-1)^m c_m(j,l=0)}{\Gamma(p-m+1)\Gamma(m+p+2+\frac{1}{2}\alpha)}. \quad \text{(A16)}$$

Integrals containing an eigenfunction and a function which is expressible as a convergent power series can be evaluated in terms of the expansion coefficients by means of Eq. (A16).

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[14] Note that $\ln(k_0R/\gamma_j\sqrt{\pi})\sim\ln(k_0R/\sqrt{\pi})$ for j fixed.

[15] If the volume is infinite, then we have a continuum of eigenvalues given by $1-\hat{K}(k)$, $0\le k<\infty$. If the volume is finite but large, then the spectrum of eigenvalues becomes discrete, but some resemblance with the infinite case must persist, so that we expect that the eigenvalues will be equal to $1-\hat{K}(k)$ where now k, $0\le k<\infty$, ranges over some discrete set of values. Considering very small values of k and using Eqs. (5a) or (5b), we see that k must be equal to $\gamma_j R^{-1}$. This is an intuitive argument, but something has also been proved. By considering an integral equation of the type of Eq. (1) defined for a slab of thickness L, V. Hutson [Proc. Edinb. Math. Soc. 14, 5 (1964)] has proved that Eq. (14) applies for $k_0L\gg 1$ ($R=\frac{1}{2}L$). The conditions imposed on the behavior of the kernel are not met in our case, but they do not seem to be essential.

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