PHYSICAL REVIEW A VOLUME 13, NUMBER 2 FEBRUARY 1976

Excitation of atomic hydrogen in the eikonal-Born-series approximation

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(Received 6 May 1975)

The eikonal-Born-series method, developed recently to analyze elastic scattering of electrons by atomic hydrogen and helium, is extended to inelastic scattering. In this paper we investigate the excitation of atomic hydrogen to the $2s$ or $2p$ state by electron bombardment. Important differences from the elastic case are discussed and a careful comparison with the Glauber method is made. Also, for angles greater than 20° comparison is made with recent experimental data on the sum of the differential cross sections for $2s$ and $2p$ excitation. Very good agreement is found.

I. INTRODUCTION

In a recent paper (hereafter referred to as I), Byron and Joachain[1] have discussed the relationship between the Born series and the Glauber[2] approximation for the case of electron-atom elastic scattering. They pointed out that the Born series is essentially a power series in k_i^{-1}, the wave number of the incident electron, so that if one wishes to achieve a consistent leading-order improvement on the first Born approximation one must evaluate four terms, the real and imaginary parts of the second Born approximation, the real part of the third Born term, and the leading term in the exchange amplitude. They showed that the Glauber approximation when interpreted as a series expansion in k_i^{-1} gives a good approximation to the imaginary part of the second Born approximation via its term of order k_i^{-1} and that in fact this approximation to the imaginary part is good even for large scattering angles and not just for the small angles which are assumed in traditional derivations. They also conjectured on the basis of potential scattering theory[3-5] that the term of order k_i^{-2} in the Glauber series gives a good approximation to the real part of the third Born term at all angles.

Despite these remarkable properties, the Glauber approximation is seriously deficient in its treatment of elastic scattering because, in addition to omitting exchange, an easily remedied omission, it gives no account at all of the real part of the second Born term, which is very important at all angles and which dominates at small angles.

On the other hand, for inelastic scattering it is known from the work of Ghosh and Sil,[6] Tai *et al.*,[7] and Byron[8] that the Glauber approximation gives a good account of the experimentally well-known departures from the first Born approximation for the integrated cross sections for electrons exciting the $2s$ state[9] and the $2p$ state[10] of atomic hydrogen. It is the purpose of this present work to analyze these two problems in the spirit of I to see if the agreement found in Refs. 6 and 7 is fortuitous or if it rests on some further remarkable property of the Glauber approximation. The analysis of I was rather general as regards the comparison between the Born and Glauber series; thus one should expect that in the absence of some rather special circumstances the Glauber approximation should suffer the same kind of difficulties in the case of inelastic scattering as in the case of elastic scattering.

However, there are certain differences between elastic and inelastic scattering which might have an important effect on any theoretical analysis. We mention briefly the two most striking ones:

(i) At small angles, elastic scattering is modified in a profound way by an effect which may be physically described as a polarization potential acting in the elastic channel.[11] The Glauber approximation takes no account of this effect. Clearly, for inelastic scattering this phenomenon will be considerably modified.

(ii) For elastic scattering the first Born term gives the leading approximation to the scattering amplitude in the high-energy limit at all angles. This is not the case for inelastic scattering. It is obvious that because of off-shell elastic scattering in intermediate states the second Born term will dominate inelastic scattering at wide angles (momentum transfer $K > 1$ in atomic units), because the most effective way for an "inelastic" electron to get scattered through a large angle is to elastically scatter through a large angle and then inelastically scatter through a small angle to get into the desired final state (or vice versa). For example, in $1s$-ns transitions such a term falls off as $k_i^{-1} K^{-2}$ for K large, whereas the first Born term falls off as K^{-6}. Since the Glauber approximation gives the imaginary part of the second Born term quite well at all angles in elastic scattering one might hope that it would give the leading term for inelastic scattering at wide angles.

These two questions will be investigated in more detail below.

The plan of this paper is as follows: In Sec. II we summarize and generalize to inelastic scattering some of the basic results of I. In Sec. III we give a detailed analysis of the excitation of atomic hydrogen to the $2s$ state by electron bombardment. In Sec. IV the same analysis will be performed for the $2p$ state. These two cases turn out to be rather different; thus it seems best to discuss them separately. In Sec. V we summarize, compare with experiment, and attempt to draw some conclusions and make suggestions for further work. The notation used throughout is that of I with one or two obvious modifications such as a superscript to distinguish the $2s$ and $2p$ final states.

II. PRELIMINARY RESULTS

In this section we recall a few notations and results of I which will be particularly useful in what follows. We shall write the Born and Glauber series in a fashion nearly identical to that of I, namely, for the Born series

$$f^{2l} = \sum_{n=1}^{\infty} \bar{f}_{Bn}^{2l} \tag{2.1}$$

and for the Glauber series

$$f_G^{2l} = \sum_{n=1}^{\infty} \bar{f}_{Gn}^{2l} \,. \tag{2.2}$$

The quantities $\bar{f}_{Bn}^{2l}$ and $\bar{f}_{Gn}^{2l}$ are trivial modifications of Eqs. (2.13) and (2.21) of I. The superscript l takes on the value of s or p depending on whether we are studying the excitation of the $2s$ or $2p$ state.

The evaluation of the individual terms in Eq. (2.2) is a straightforward numerical matter, but this is not the case for Eq. (2.1), where an exact evaluation of any term save the first will be enormously difficult. Thus the closure approximation has been widely used (see I for a list of the major references). For the problem at hand we have directly from Eq. (2.29) of I

$$\bar{f}_{SB2}^{2l} = \frac{2}{\pi^2} \int d\vec{q} \frac{1}{K_i^2 K_f^2} \frac{1}{q^2 - p^2 - i\epsilon} \times \langle 2l | e^{i\vec{K}\cdot\vec{r}} - e^{i\vec{K}_i\cdot\vec{r}} - e^{-i\vec{K}_f\cdot\vec{r}} + 1 | 1s \rangle \,, \tag{2.3}$$

where the subscript SB2 refers to the "simplified" second Born approximation, $\vec{K} = \vec{k}_i - \vec{k}_f$, $\vec{K}_i = \vec{k}_i - \vec{q}$, and $\vec{K}_f = \vec{k}_f - \vec{q}$. Here $\vec{k}_i$ and $\vec{k}_f$ are the initial and final momenta of the incident electron and $p^2 = k_i^2 - 2\Delta_i$, where Δ_i is the average intermediate-state excitation energy, measured from the ground state. Clearly it is a trivial matter to put in a few intermediate states exactly. Since we expect wide-angle scattering to be dominated by elastic intermediate states, it is natural to include the states $1s$ and $2l$ exactly in $\bar{f}_{B2}^{2l}$. We shall also see that it is reasonable to treat the intermediate $2p$ state exactly in studying the $2s$ excitation process. With an appropriate number of intermediate states inserted exactly we shall call the resulting amplitude the second Born term, although even this is, of course, only an approximation. Thus [see Eq. (2.32) of I]

$$\begin{aligned}\bar{f}_{B2}^{2l} = {} & \frac{2}{\pi^2} \int d\vec{q} \frac{1}{K_i^2 K_f^2 (q^2 - p^2 - i\epsilon)} \langle 2l | e^{i\vec{K}\cdot\vec{r}} - e^{i\vec{K}_i\cdot\vec{r}} - e^{-i\vec{K}_f\cdot\vec{r}} + 1 | 1s \rangle \\ & + \frac{2}{\pi^2} \int d\vec{q} \sum_n \frac{1}{K_i^2 K_f^2 (q^2 - k_n^2 - i\epsilon)} \langle 2l | e^{-i\vec{K}_f\cdot\vec{r}} - 1 | n \rangle \langle n | e^{i\vec{K}_i\cdot\vec{r}} - 1 | 1s \rangle \\ & - \frac{2}{\pi^2} \int d\vec{q} \sum_n \frac{1}{K_i^2 K_f^2 (q^2 - p^2 - i\epsilon)} \langle 2l | e^{-i\vec{K}_f\cdot\vec{r}} - 1 | n \rangle \langle n | e^{i\vec{K}_i\cdot\vec{r}} - 1 | 1s \rangle \,, \end{aligned} \tag{2.4}$$

where $k_n^2 = k_i^2 - 2\Delta_{in}$ and n runs over those states which we wish to treat exactly. The quantity Δ_{in} is just the energy difference between the initial state and the state n. Thus if $n = 1s$, $k_n = k_i$, while if $n = 2s$ or $2p$, $k_n = k_f$.

III. EXCITATION OF ATOMIC HYDROGEN TO THE 2s STATE

A. Basic results

We begin by recalling the well-known expression for the first Born approximation for $1s$-$2s$ excitation. It is given by

$$\bar{f}_{B1}^{2s} = -\frac{8\sqrt{2}}{(K^2 + \frac{9}{4})^3} \,. \tag{3.1}$$

It will be the purpose of this section to study the corrections to this simple expression.

Using Eq. (2.3) it is straightforward to obtain $\bar{f}_{SB2}^{2s}$ from the bound-state wave functions of atomic hydrogen. An elementary calculation yields

$$\bar{f}_{SB2}^{2s} = -\frac{8\sqrt{2}}{\pi^2} \int \left(\frac{1}{K_f^2 (K_i^2 + \alpha^2)^3} + \frac{1}{K_i^2 (K_f^2 + \alpha^2)^3} - \frac{K^2}{(K^2 + \alpha^2)^3} \frac{1}{K_i^2 K_f^2} \right) \frac{d^3 q}{q^2 - p^2 - i\epsilon} \,, \tag{3.2}$$

where $\alpha = \frac{3}{2}$. This can be written as

$$\bar{f}^{2s}_{SB2} = -8\sqrt{2}\left(\frac{1}{2}\frac{d^2J_i(\alpha^2)}{d(\alpha^2)^2} + \frac{1}{2}\frac{d^2J_f(\alpha^2)}{d(\alpha^2)^2} - \frac{K^2}{(K^2+\alpha^2)^3}J_i(0)\right), \tag{3.3}$$

where the key integral $J_{i,f}$ is given by

$$J_{i,f}(\alpha^2) = \frac{1}{\pi^2}\int \frac{d^3q}{K^2_{f,i}(K^2_{i,f}+\alpha^2)(q^2-p^2-i\epsilon)}. \tag{3.4}$$

Using familiar Feynman methods as in I, we have

$$J_{i,f}(\alpha^2) = \int_0^1 \frac{dt}{[\alpha^2 t + t(1-t)K^2]^{1/2}[(\alpha^2 \pm 2\Delta)t + 2\Delta_{f,i} - 2ip(\alpha^2 t + t(1-t)K^2)^{1/2}]}, \tag{3.5}$$

where the plus sign goes with J_i and the minus sign with J_f. In Eq. (3.5) Δ is the energy difference between the $1s$ and $2s$ states ($\frac{3}{8}$ a.u.) and $\Delta_f = \Delta_i - \Delta$ is the average intermediate-state energy measured from the final state. The real and imaginary parts of $J_{i,f}$ are virtually identical to Eqs. (2.37a) and (2.37b) of I with minor modifications necessary to account for the fact that the initial and final states do not have the same energy. As noted in I, this integral can easily be evaluated analytically. In practice we used the analytic expression for $J_{i,f}$ and differentiated it numerically to obtain $\bar{f}^{2s}_{SB2}$ from Eq. (3.3).

Since we are dealing with intermediate and large energies, the limit k_i large (hence p and k_f large) is of interest. In this limit a simple expression can be obtained for $J_{i,f}$. One finds

$$\operatorname{Im} J_{i,f} = \frac{1}{2p(\alpha^2+K^2)} \times \left(\ln\frac{p^2(\alpha^2+K^2)^2}{\alpha^2\Delta^2_{f,i}} - \frac{(\alpha^2+2\Delta_{i,f})^2}{4p^2\alpha^2} + \cdots\right), \tag{3.6a}$$

$$\operatorname{Re} J_{i,f} = \frac{\pi}{2p(\alpha^2+K^2)} + \frac{(\alpha^2 \pm 2\Delta)(\alpha^2+K^2) + 2\Delta_{f,i}(K^2-\alpha^2)}{2p^2\alpha(\alpha^2+K^2)^2} + \cdots. \tag{3.6b}$$

In order to compute $\bar{f}^{2s}_{SB2}$ we also need the quantity $J = J_{i,f}(\alpha^2 = 0)$. We see from these equations that the limit is not straightforward, so we must return to Eq. (3.5) and evaluate this at $\alpha = 0$. This may be done exactly; as in I one finds

$$\operatorname{Im} J \equiv \operatorname{Im} J_{i,f}(0) = \frac{1}{K(p^2K^2+4\Delta_i\Delta_f)^{1/2}} \ln\frac{[(p^2K^2+4\Delta_i\Delta_f)^{1/2}+pK]^2}{4\Delta_i\Delta_f}, \tag{3.7a}$$

$$\operatorname{Re} J \equiv \operatorname{Re} J_{i,f}(0) = \frac{\pi}{K(p^2K^2+4\Delta_i\Delta_f)^{1/2}}. \tag{3.7b}$$

Note that if one takes the elastic limit $\Delta = 0$, $\Delta_i = \Delta_f$ then Eqs. (3.7) reduce to Eqs. (2.39) of I.[12]

With these expressions in hand, a useful large-k_i form for $\bar{f}^{2s}_{SB2}$ can be obtained from Eq. (3.3). Upon differentiating twice Eqs. (3.6a) and (3.6b) we have

$$\operatorname{Im}\bar{f}^{2s}_{SB2} \cong -\frac{8\sqrt{2}}{p}\left[\frac{1}{2\alpha^4(\alpha^2+K^2)} + \frac{1}{\alpha^2(\alpha^2+K^2)^2} - \frac{3}{(\alpha^2+K^2)^3} + \frac{1}{(\alpha^2+K^2)^3}\left(\ln\frac{p^2(\alpha^2+K^2)^2}{\alpha^2\Delta_i\Delta_f} - \frac{pK}{(p^2K^2+4\Delta_i\Delta_f)^{1/2}}\ln\frac{[pK+(p^2K^2+4\Delta_i\Delta_f)^{1/2}]^2}{4\Delta_i\Delta_f}\right)\right]_{\alpha=3/2}, \tag{3.8a}$$

$$\operatorname{Re}\bar{f}^{2s}_{SB2} \cong -\frac{8\sqrt{2}}{p}\left[\frac{\pi}{(\alpha^2+K^2)^3}\left(1 - \frac{pK}{(p^2K^2+4\Delta_i\Delta_f)^{1/2}}\right) - \frac{\alpha^2 - 3(\Delta_i+\Delta_f)}{8p\alpha^5(\alpha^2+K^2)} - \frac{2\alpha^2 - 3(\Delta_i+\Delta_f)}{4p\alpha^3(\alpha^2+K^2)^2} + \frac{\alpha^2 + 3(\Delta_i+\Delta_f)}{p\alpha(\alpha^2+K^2)^3} - \frac{6\alpha(\Delta_i+\Delta_f)}{p(\alpha^2+K^2)^4}\right]_{\alpha=3/2}. \tag{3.8b}$$

For the energies of interest here ($E \geqslant 100$ eV) Eq. (3.8a) approximates the true $\operatorname{Im}\bar{f}^{2s}_{SB2}$ to better than 10% at all angles; Eq. (3.8b) does not represent $\operatorname{Re}\bar{f}^{2s}_{SB2}$ quite as well for reasons which will be discussed below.

B. Imaginary part of $\bar{f}^{2s}_{SB2}$

Let us analyze the imaginary part of $\bar{f}^{2s}_{SB2}$ first. We begin by noting that according to Eq. (3.7a), as soon as K becomes a bit larger than $2(\Delta_i\Delta_f)^{1/2}/p$ $\operatorname{Im} J$ attains the limiting form

$$\operatorname{Im} J \simeq (1/pK^2)\ln(p^2K^2/\Delta_i\Delta_f); \tag{3.9}$$

thus $\operatorname{Im}\bar{f}^{2s}_{SB2}$ takes the form

$$\operatorname{Im}\bar{f}^{2s}_{SB2} \cong -\frac{8\sqrt{2}}{p}\left(\frac{1}{2\alpha^4(\alpha^2+K^2)} + \frac{1}{\alpha^2(\alpha^2+K^2)^2} - \frac{3}{(\alpha^2+K^2)^3} + \frac{1}{(\alpha^2+K^2)^3}\ln\frac{(\alpha^2+K^2)^2}{\alpha^2K^2}\right). \tag{3.10}$$

Note that this is *independent* of the average excitation energy.

In I it was shown that the imaginary part of the second term in the Glauber series (which is in fact purely imaginary) approximates $\mathrm{Im}\bar{f}_{B2}^{2s}$ rather well in the case of elastic scattering, not just for small angles as one would expect from the usual derivations of the Glauber approximation, but in fact at *all* angles. That this is also the case for the excitation of the $2s$ state of atomic hydrogen is readily shown. The second-order Glauber term is most easily evaluated[13] in closed form by transforming the various terms of $\bar{f}_{G2}$ [see Eq. (2.21) of I] into momentum space. One obtains in a completely straightforward manner

$$\bar{f}_{G2}^{2s} = -i\frac{2}{\pi}\frac{\sqrt{2}}{k_i}\frac{d^2}{d(\alpha^2)^2}\int\left(\frac{1}{Q^2[(\vec{Q}-\vec{K})^2+\alpha^2]} + \frac{1}{(\vec{Q}-\vec{K})^2(Q^2+\alpha^2)} - \frac{K^2}{K^2+\alpha^2}\frac{1}{Q^2(\vec{Q}-\vec{K})^2}\right)d^2Q, \tag{3.11}$$

where $\vec{Q}$ is a two-dimensional vector in the plane of the momentum transfer. We note that this is what would be obtained from Eq. (3.2) if one replaced the closure propagator $G=(q^2-p^2-i\epsilon)^{-1}$ by a "Glauber propagator"

$$G_G = [-2k_i(\vec{k}_i-\vec{q})\cdot\hat{K}_\perp - i\epsilon]^{-1}, \tag{3.12}$$

where $\hat{K}_\perp$ is a unit vector perpendicular to $\vec{K}$ lying in the plane defined by the triangle formed by $\vec{k}_i$, $\vec{k}_f$, and $\vec{K}$. The integration of Eq. (3.11) is straightforward; doing the azimuthal integration and making the change of variable $Q^2=y$, we find

$$\bar{f}_{G2}^{2s} = -\frac{2\sqrt{2}}{k_i}\frac{d^2}{d(\alpha^2)^2}\int_0^\infty\left(\frac{1}{y[(y+K^2+\alpha^2)^2-4K^2y]^{1/2}} + \frac{1}{(y+\alpha^2)\,|y-K^2|} - \frac{K^2}{K^2+\alpha^2}\frac{1}{y\,|y-K^2|}\right)dy. \tag{3.13}$$

Note that although there is a singularity in each individual term, there is no singularity in the integrand as a whole. Thus if one integrates on y from ϵ_1 to $K^2-\epsilon_2$ and then from $K^2+\epsilon_3$ to infinity one gets a result which is perfectly finite as ϵ_1, ϵ_2, and ϵ_3 go to zero. The value for $\bar{f}_{G2}^{2s}$ thus obtained is precisely the expression of Eq. (3.10) but with p replaced by k_i. Thus except at very small angles the second Born and second Glauber terms agree to leading order in k_i^{-1}. It is a simple matter to show that for the *imaginary* part of the second Born term the various intermediate-state contributions of Eq. (2.4) do not affect this conclusion.

The solid curve in Fig. 1 shows[14] the quantity $\mathrm{Im}\bar{f}_{B2}^{2s}$ as obtained from Eq. (2.4). The part corresponding to $\mathrm{Im}\bar{f}_{SB2}^{2s}$ was evaluated as discussed above; the terms coming from the finite sum on intermediate states (taken to include $1s$, $2s$, and $2pm$ for reasons discussed below) offer no additional complications beyond those already discussed. The dotted curve gives the Glauber-approximation result of Eq. (3.10). Clearly, the inclusion of the three intermediate states in the finite sum of Eq. (2.4) has a negligible effect. However, they play a much more important role in the calculation of $\mathrm{Re}\bar{f}_{B2}^{2s}$, as we shall see below.

In Fig. 1 we have also included (dashed curve) the contributions to $\mathrm{Im}\bar{f}_{B2}^{2s}$ of the $1s$ and $2s$ intermediate states alone. It is seen that at large angles these two states dominate strongly, but at small angles they deviate from the full amplitude by as much as a factor of 2. This strongly suggests that the distorted-wave Born approximation (DWBA) will not be able to give a useful account of small-angle scattering, since it approximates the second Born term by the contributions of the $1s$ and $2s$ intermediate states coming from the

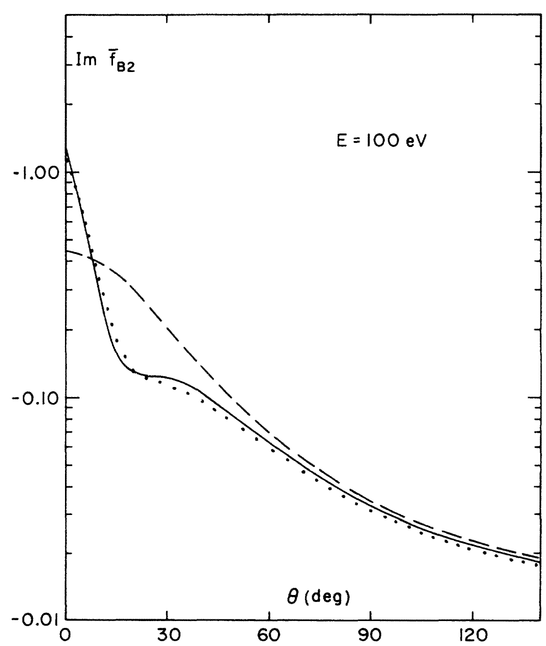

FIG. 1. Imaginary part of the second Born term for the excitation of atomic hydrogen to the $2s$ state by electron bombardment. The solid curve shows the result of this paper, the dotted curve is the Glauber approximation, and the dashed curve is the result obtained by including only the $1s$ and $2s$ intermediate states. The incident electron energy is 100 eV.

static potential acting in the initial and final channels, respectively. Absorption and polarization parts of the initial and final channel optical potentials act only in higher orders in k_i^{-1}.

C. Real part of $\bar{f}_{SB2}^{2s}$

We turn now to the study of the real part of the second Born term. This was found in I to play a very important part in determining the leading correction to the first Born approximation for elastic scattering. The situation here is superficially very similar to the elastic case. Looking at Eq. (3.8b) we see that just as in the elastic case at small angles the real part of $\bar{f}_{SB2}^{2s}$ is dominated by a term of order p^{-1}.

This term of order p^{-1}, which we will denote by $f^{(1)}$, can be written for $K \ll 1$ as

$$f^{(1)} = \frac{\pi \langle 2s|z^2|1s\rangle}{p}\left(1 - \frac{pK}{[(pK)^2 + 4\Delta_i\Delta_f]^{1/2}}\right), \quad (3.14)$$

as has been shown elsewhere[11] for the very similar case of elastic scattering, where the matrix element is $\langle 1s|z^2|1s\rangle$. It is clear that this comes from the sum on intermediate p states, and in fact one can readily obtain the small-angle form without carrying out the intermediate-state sum by closure. Calling this quantity $f^{(2)}$, we have

$$f^{(2)} = \pi \sum_n \frac{1}{p_n}\langle 2s|z|np\rangle\langle np|z|1s\rangle \times\left(1 - \frac{p_n K}{[(p_n K)^2 + 4\Delta_{in}\Delta_{fn}]^{1/2}}\right), \quad (3.15)$$

where $p_n^2 = k_i^2 - 2\Delta_{in}$, and where Δ_{in} and Δ_{fn} are the energy differences between the nth state and the initial and final states, respectively. Note that if one replaces p_n by p and Δ_{in} and Δ_{fn} by Δ_i and Δ_f, respectively, then the sum can be done by closure, and one obtains Eq. (3.14). Clearly, for the $2p$ intermediate state $\Delta_{fn} = 0$; this state does not actually contribute to the sum! Thus it is clearly undesirable to include this state in a closure approximation. In addition, the contribution of the $2p$ intermediate-state term to Eq. (3.14) is negative, while all other terms are positive. In the identity

$$\langle 2s|z^2|1s\rangle = \sum_n \langle 2s|z|np\rangle\langle np|z|1s\rangle$$

the left-hand side is negative because the first term in the sum is negative and all the other positive terms are not large enough to counteract this one term. Thus the expression $f^{(1)}$ is very misleading since it is dominated by a large negative term which in a careful analysis should not contribute at all. For this reason, we put the $2p$ intermediate state exactly [via Eq. (2.4)] in all the work reported here.

In addition, for small angles it makes sense to insert the sum on all states with $n > 2$ via Eq. (3.15) since the matrix elements $\langle 2s|z|np\rangle$ and $\langle np|z|1s\rangle$ are readily evaluated analytically, even for continuum states.[15] This was done by adding to the amplitude of Eq. (2.4) the expression

$$\Delta\bar{f}_{B2}^{2s} = \pi K \sum_{n>2}\langle 2s|z|np\rangle\langle np|z|1s\rangle \times\left(\frac{1}{[(pK)^2 + 4\Delta_i\Delta_f]^{1/2}} - \frac{1}{[(pK)^2 + 4\Delta_{in}\Delta_{fn}]^{1/2}}\right), \quad (3.16)$$

which removes the small-angle average intermediate-state part of $\bar{f}_{SB2}^{2s}$ and inserts approximately the terms with their correct excitation energies. The quantity $\Delta\bar{f}_{B2}^{2s}$ is nothing more than $f^{(2)} - f^{(1)}$ with the term $n = 2$ removed and with p_n approximated by p.

These small-angle effects fall off rapidly outside scattering angles somewhat larger than $(\Delta_i\Delta_f)^{1/2}/E$. For angles larger than this, Eq. (3.8b) takes the form

$$\mathrm{Re}\bar{f}_{SB2}^{2s} \simeq \frac{8\sqrt{2}}{p^2}\left(\frac{\alpha^2 - 3(\Delta_i + \Delta_f)}{8\alpha^5(\alpha^2 + K^2)} + \frac{2\alpha^2 - 3(\Delta_i + \Delta_f)}{4\alpha^3(\alpha^2 + K^2)^2} - \frac{\alpha^2 + 3(\Delta_i + \Delta_f)}{\alpha(\alpha^2 + K^2)^3} + \frac{6\alpha(\Delta_i + \Delta_f)}{(\alpha^2 + K^2)^4}\right). \quad (3.17)$$

The key point to be made about Eq. (3.17) is that because of the magnitudes of α^2, Δ_i, and Δ_f ($\alpha^2 = \frac{9}{4}$, $\Delta_i \simeq \frac{1}{2}$, $\Delta_f \simeq \frac{1}{8}$), $\mathrm{Re}\bar{f}_{SB2}^{2s}$ depends very sensitively on Δ_i and Δ_f, in marked contrast with what was found for $\mathrm{Im}\bar{f}_{SB2}^{2s}$. In addition, this term is much smaller than order-of-magnitude estimates would suggest. For example, the first term in Eq. (3.17), which dominates when K is sufficiently large, is proportional to $\alpha^2 - 3(\Delta_i + \Delta_f)$; for the values of Δ_i and Δ_f given above, $\alpha^2 - 3(\Delta_i + \Delta_f) = \frac{3}{8}$, which is a factor of 5 or 6 smaller than either of the contributing pieces. This cancellation also occurs at small angles. For example, when $K \ll 1$, the last two terms in Eq. (3.17) nearly cancel for values of Δ_i near $\frac{1}{2}$. The result of this is that over most of the angular range the magnitude of $\mathrm{Re}\bar{f}_{SB2}^{2s}$ is much smaller than $\mathrm{Re}\bar{f}_{G3}^{2s}$, even though the individual terms in $\mathrm{Re}\bar{f}_{SB2}^{2s}$ are of the same order of magnitude as $\mathrm{Re}\bar{f}_{G3}^{2s}$.

This situation is illustrated in Fig. 2 where we show $\mathrm{Re}\bar{f}_{B2}^{2s}$ for two different values of Δ_i (0.5 and 0.6) compared with $\mathrm{Re}\bar{f}_{G3}^{2s}$. The values of $\mathrm{Re}\bar{f}_{B2}^{2s}$

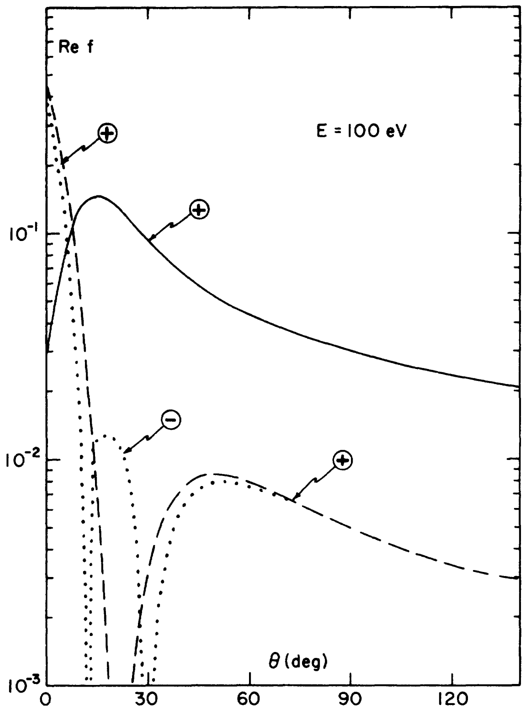

FIG. 2. Solid curve shows the Glauber approximation to the real part of the third Born term for 100-eV electrons exciting atomic hydrogen to the 2s state. The two other curves show two EBS approximations to the real part of the second Born term, the dashed curve for an average excitation energy of 0.5 a.u., the dotted curve for an average excitation energy of 0.6 a.u.

have been obtained via Eq. (2.4), with the 1s, 2s, and 2p intermediate states inserted exactly and with small-angle effects treated via Eq. (3.16). Thus at large angles, where intermediate-state elastic scattering dominates, $\mathrm{Re}\bar{f}_{B2}^{2s}$ is insensitive to Δ_i (unlike $\mathrm{Re}\bar{f}_{SB2}^{2s}$) because the dominant (elastic) terms are put in with their correct excitation energies. However, we see that at small angles, particularly between 10° and 30°, the sensitivity to Δ_i is considerable. The amplitude is going through two zeroes here, so it is very sensitive to fine details. Clearly, one would probably need to put in a great many states correctly in order to get a reasonable approximation to the true functional form. From the practical point of view, since $\mathrm{Re}\bar{f}_{G3}^{2s}$ dominates strongly over $\mathrm{Re}\bar{f}_{B2}^{2s}$ in the region of sensitivity, the dependence on Δ_i is unimportant, since $\mathrm{Re}\bar{f}_{G3}^{2s}$ will dominate the leading real correction to the first Born direct amplitude. In the elastic case, we found $\mathrm{Re}\bar{f}_{B2}\simeq\mathrm{Re}\bar{f}_{G3}$; so the correction due to the second Born term was very important. For 2s excitation this will not be the case except at very small angles where the peak due to long-range forces comes into play. Because $\Delta_i\Delta_f$ is so small for the 2s state of hydrogen the angular extent of this peak is very small. We should point out, however, that for the excitation of the $2\,^1S$ state of helium, where $\Delta_i\simeq 1.3$, $\Delta_f\simeq 0.6$, we would expect these long-range effects to be important out to larger angles.

Finally, before closing this section we remark that the contribution of the 1s and 2s states alone to $\mathrm{Re}\bar{f}_{B2}^{2s}$ follows the same pattern as for $\mathrm{Im}\bar{f}_{B2}^{2s}$. It gives a good account of $\mathrm{Re}\bar{f}_{B2}^{2s}$ at large angles, but as soon as the momentum transfer becomes of order unity or less it bears little resemblance to $\mathrm{Re}\bar{f}_{B2}^{2s}$, other than being of the same general order of magnitude. Thus distorted-wave calculations will probably be inadequate in this region for $\mathrm{Re}\bar{f}_{B2}^{2s}$ just as they were for $\mathrm{Im}\bar{f}_{B2}^{2s}$.

D. Exchange corrections

In I it was pointed out that if we write the exchange amplitude as

$$g^{2s}=\sum_{n=1}^{\infty}\bar{g}_{Bn}^{2s}, \tag{3.18}$$

then the leading term in the exchange amplitude, $\bar{g}_{B1}$, is of order k_i^{-2} and real; thus it contributes to the leading correction to the differential cross section at the same level as do the real parts of the second- and third-order terms of the direct amplitude. The situation is very similar here, but there are two differences.

First, for small values of the momentum transfer the Ochkur approximation to $\bar{g}_{B1}^{2s}$, which is given as in I by

$$g_{\mathrm{Och}}^{2s}=-\frac{2}{k_i^2}\int e^{i\vec{K}\cdot\vec{r}}\phi_{2s}^*(\vec{r})\phi_{1s}(\vec{r})d\vec{r}$$
$$=-\frac{8\sqrt{2}}{k_i^2}\frac{K^2}{(K^2+\frac{9}{4})^3}, \tag{3.19}$$

tends to zero as K^2. Hence it does not contribute significantly to the scattering amplitude at small angles, being in fact of order k_i^{-4} at $\theta=0$ (since $K\simeq\Delta/k_i$ at $\theta=0$). It should be noted that the terms of $\bar{g}_{B1}^{2s}$ which are omitted by g_{Och}^{2s} are also of order k_i^{-4} or higher (for all K); so it is consistent to neglect them.

The second point regarding the exchange amplitude is the fact that for large momentum transfers the Ochkur term falls off as $k_i^{-2}K^{-4}$, just as in the elastic case. One would expect that intermediate-state elastic scattering should produce a second-order imaginary term, $\mathrm{Im}\bar{g}_{B2}^{2s}$, which varies as $k_i^{-3}K^{-2}$ for K large. By interference with $\mathrm{Im}\bar{f}_{B2}^{2s}$ this will give a contribution to the differential cross section which varies like $k_i^{-4}K^{-4}$ when K is large. The leading term in the K-large limit comes from the square of $\mathrm{Im}\bar{f}_{B1}^{2s}$ (*not* from the square of $\bar{f}_{B1}^{2s}$ as in elastic scattering) and varies as $k_i^{-2}K^{-4}$, as is clear from Eq. (3.8a). Thus the second-order exchange term will give a piece of the leading correction (of *relative* order k_i^{-2}) in the large-angle region. Unfortunately, as we shall discuss below, we do not know how to calculate all of the terms which contribute to the leading correction for the direct amplitude. Thus we have not bothered to obtain $\mathrm{Im}\bar{g}_{B2}^{2s}$, although it is possible to do so at large angles.

E. Full scattering amplitude

Let us now turn to the problem of putting together all of the pieces of the scattering amplitude which we have been discussing. We have seen above that the second term in the Glauber series, $\bar{f}_{G2}^{2s}$, agrees very well with the imaginary part of the corresponding term in the Born series, $\mathrm{Im}\bar{f}_{B2}^{2s}$, at all scattering angles. In potential theory,[3-5] various workers have made it appear highly likely that when k_i is large, in addition to agreement for the second term, there is agreement in all orders. That is, for odd-order terms $\bar{f}_{Gn}(K)=\mathrm{Re}\bar{f}_{Bn}(K)$ for all K, while for even-order terms $\bar{f}_{Gn}(K) = i\,\mathrm{Im}\bar{f}_{Bn}(K)$ for all K. As in I, let us conjecture that in third order, by analogy with potential scattering, we have $\mathrm{Re}\bar{f}_{B3}^{2s}(K)=\bar{f}_{G3}^{2s}(K)$ for all K. Under this assumption, we may proceed as in I to write

$$f^{2s}\cong\bar{f}_{B1}^{2s}+\mathrm{Re}\bar{f}_{B2}^{2s}+\bar{f}_{G3}^{2s}+i\,\mathrm{Im}\bar{f}_{B2}^{2s},\quad g^{2s}\cong g_{\mathrm{Och}}^{2s} \tag{3.20}$$

and

$$\frac{d\sigma^{2s}}{d\Omega}=\tfrac{1}{4}|f^{2s}+g^{2s}|^2+\tfrac{3}{4}|f^{2s}-g^{2s}|^2. \tag{3.21}$$

This expression,[16] as explained in I, will give all terms of relative order k_i^{-2} which correct the first Born approximation to the differential cross section, as long as the momentum transfer dependence of all these terms is the same. This was the case in I where all terms in the Born series for elastic scattering fall off as K^{-2} for large K. Here, however, Eq. (3.20) can be used only in the small-to-intermediate momentum transfer region, since $\bar{f}_{B1}^{2s}$ falls off as K^{-6} for large K, whereas the other terms fall off as K^{-2}, as in the elastic case. At large momentum transfers, the *dominant* term in $d\sigma^{2s}/d\Omega$ will be given by $(\mathrm{Im}\bar{f}_{B2}^{2s})^2$.

To see the situation more clearly let us write out in detail the leading terms in the scattering amplitude for large momentum transfers:

$$\bar{f}_{B1}^{2s}\simeq-8\sqrt{2}/K^6, \tag{3.22a}$$

$$\mathrm{Re}\bar{f}_{B2}^{2s}\simeq 3Q/8k_i^2K^2, \tag{3.22b}$$

$$\mathrm{Im}\bar{f}_{B2}^{2s}\simeq -Q/k_iK^2-9Q/4k_iK^4+AQ/k_i^3K^2, \tag{3.22c}$$

$$\mathrm{Re}\bar{f}_{B3}^{2s}\simeq\mathrm{Re}\bar{f}_{G3}^{2s}\simeq Q[2\ln(\tfrac{2}{3}K)+1]/k_i^2K^2, \tag{3.22d}$$

$$\mathrm{Im}\bar{f}_{B3}^{2s}\simeq BQ/k_i^3K^2, \tag{3.22e}$$

$$\mathrm{Im}\bar{f}_{B4}^{2s}\simeq\mathrm{Im}\bar{f}_{G4}^{2s}\simeq Q[2\ln^2(\tfrac{2}{3}K)+2\ln(\tfrac{2}{3}K)+\tfrac{1}{6}\pi^2]/k_i^3K^2, \tag{3.22f}$$

$$\bar{g}_{B1}^{2s}\simeq-8\sqrt{2}/k_i^2K^4, \tag{3.22g}$$

$$\mathrm{Im}\bar{g}_{B2}^{2s}\simeq CQ/k_i^2K^2, \tag{3.22h}$$

where $Q=2^{13/2}/3^4$ and A, B, and C are constants independent of k_i and K. The asymptotic forms of Eqs. (3.22d) and (3.22f) are derived in the Appendix. The constants A and C are fairly straightforward to obtain, coming from elastic intermediate states in the second-order terms of the direct and exchange amplitudes. However, B is by no means so simple. That $\mathrm{Im}\bar{f}_{B3}^{2s}$ must have the form given in Eq. (3.22e) can be readily deduced from the generalized unitarity relation, but a precise evaluation of B does not seem possible.

Using Eqs. (3.20) and (3.22) we can write at large angles

$$\frac{d\sigma^{2s}}{d\Omega}\simeq\frac{2^{13}}{3^8k_i^2K^4}[1+(\tfrac{3}{2}\ln\tfrac{2}{3}K+\tfrac{121}{64}-\tfrac{1}{3}\pi^2-2A-2B-C)k_i^2 +9/2K^2+\cdots]. \tag{3.23}$$

Notice that if we had neglected the overlap between $\mathrm{Im}\bar{f}_{B2}^{2s}$ and $\mathrm{Im}\bar{f}_{B4}^{2s}$, the leading correction term would have been $(4/k_i^2)\ln^2(\tfrac{2}{3}K)$ rather than $(3/2k_i^2)\times\ln(\tfrac{2}{3}K)$, i.e., we would have a spuriously large correction. It should also be noted that although we cannot evaluate B we still get the *weakly dominant* part (proportional to $\ln K$) of the correction to the lowest-order term. It is interesting to constrast Eq. (3.23) with the Glauber result. Using the k_i^{-1} part of Eq. (3.22c) together with Eqs. (3.22d) and (3.22f) we find

$$\frac{d\sigma_G^{2s}}{d\Omega}=\frac{2^{13}}{3^8k^2K^4}\left(1+\frac{1-\pi^2/3}{k_i^2}+\frac{9}{2K^2}+\cdots\right) \tag{3.24}$$

i.e., the Glauber result falls below the asymptotic form at large angles, whereas at sufficiently high energy the EBS result will rise above the asymptotic form.

The discussion above clearly indicates that although Eq. (3.20) is fine for small-angle scattering, it will give spuriously large results at large angles. However, from Eq. (3.21) we see that if we use

$$\frac{d\sigma_{\mathrm{EBS}}^{2s}}{d\Omega}=\tfrac{1}{4}(\mathrm{Re}f^{2s}+g_{\mathrm{Och}}^{2s})^2+\tfrac{3}{4}(\mathrm{Re}f^{2s}-g_{\mathrm{Och}}^{2s})^2 +(\mathrm{Im}\bar{f}_{B2}^{2s})^2+2(\mathrm{Im}\bar{f}_{B2}^{2s})(\mathrm{Im}\bar{f}_{G4}^{2s}), \tag{3.25}$$

where $\mathrm{Re}f^{2s}$ is given by Eq. (3.20), we will have all terms of order 1 and order k_i^{-2} for small angles (along with some terms of higher order) and at larger angles we will have no spurious terms in $\ln K$.

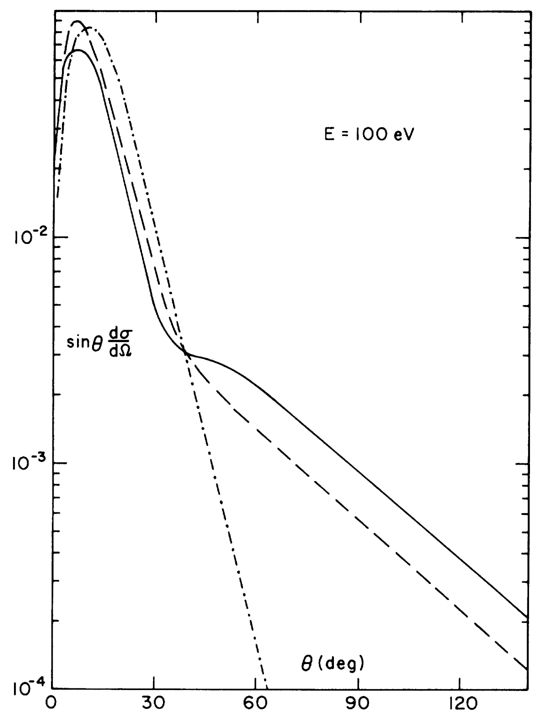

FIG. 3. Differential cross section for the excitation of the 2s state of atomic hydrogen by electron impact at 100 eV, multiplied by $\sin\theta$. The solid curve is the result of this paper, the dashed curve is the Glauber approximation, and the dash-dotted curve is the first Born approximation.

In particular, for large k and large K one would have an expression similar to Eq. (3.23) but with $B=C=0$. It is Eq. (3.25) which we shall use for the EBS cross section in this paper.

In Fig. 3 we illustrate the situation at 100 eV. In addition to the EBS result (solid curve) we show the Glauber approximation (dashed curve) and the first Born approximation (dash-dotted curve). We note that the results of Fig. 3 all contain the overall multiplicative kinematical factor k_f/k_i which has been omitted for the sake of convenience in the cross-section equations given in this section. The dramatic manner in which second-order, off-shell elastic scattering modifies the first Born approximation at large angles is very clear. The EBS result lies significantly higher than the Glauber result at large angles, as one might guess from the presence of the term in $\ln K$ in Eq. (3.23). At smaller angles, the EBS result falls below the Glauber result by 10–20%. This is much less than the difference found in elastic scattering[1] at the same energy. In I the difference between EBS and Glauber was always at least a factor of 2 for scattering angles less than 30°. The reason for this change is twofold: The very small region of long-range effects in $\mathrm{Re}\bar f_{B2}^{2s}$ and the remarkable cancellations in $\mathrm{Re}\bar f_{B2}^{2s}$ outside the small-angle region. Whether or not this cancellation has some significant underlying physical reason is not clear.

Finally, let us comment briefly on the accuracy of the EBS results. Since they derive fron. perturbation theory and since one has obviously stopped at the order in question because one doesn't know how to calculate all of the next-order terms, it is difficult to make reliable estimates of accuracy. The most optimistic estimate would be to say that at small angles one expects corrections of order k_i^{-4}, which would mean an error of only a few percent at 100 eV. However, a more realistic scale factor [see Eqs. (3.3) and (3.6)] would suggest that higher-order terms should be of order $(3/2k_i)^4$, which leads to an estimate of about 10%, certainly a more reasonable figure. At wide angles, similar considerations suggest that the accuracy would be of order $(3/2k_i)^2$, i.e., about 30%. As one goes to higher energies, these errors rapidly become smaller.

Another possible approach would be to calculate the Glauber cross section in a manner analogous to Eq. (3.27), i.e.,

$$\frac{d\sigma_G^{2s}}{d\Omega} = (\bar f_{G1}^{2s} + \bar f_{G3}^{2s})^2 + |\bar f_{G2}^{2s}|^2 + 2(\mathrm{Im}\bar f_{G2}^{2s})(\mathrm{Im}\bar f_{G4}^{2s}) \quad (3.26)$$

and then compare this to the full Glauber result. This estimate agrees fairly well with the one just given, within about 10% at small angles and about 20% at large angles. The region of major sensitivity lies between 30° and 50° where the cross section changes over from being dominated by the first Born term to being dominated by the second Born term. In this region there is considerable cancellation between terms and hence great sensitivity to higher orders.

IV. EXCITATION OF ATOMIC HYDROGEN TO THE 2p STATE

A. Basic results

Proceeding as in Sec. III we have for the $2p$ excitation amplitude

$$\vec f_{SB2}^{2p} = \frac{12\sqrt{2}\, i}{\pi^2}\int \frac{1}{q^2 - p^2 - i\epsilon}\,\frac{1}{K_i^2 K_f^2} \times \left(\frac{\vec K}{(K^2+\alpha^2)^3} - \frac{\vec K_i}{(K_i^2+\alpha^2)^3} + \frac{\vec K_f}{(K_f^2+\alpha^2)^3}\right) d^3q\,, \quad (4.1)$$

where from this vector amplitude the amplitudes for excitation to a particular magnetic sublevel are

$$f_{SB2}^{2p\pm} = \mp (1/\sqrt{2})[(\vec f_{SB2}^{2p})_x \pm i(\vec f_{SB2}^{2p})_y]\,, \quad (4.2a)$$

$$f_{SB2}^{2p0} = (\vec f_{SB2}^{2p})_z\,. \quad (4.2b)$$

In this same notation, the first Born term is given by

$$\vec{f}^{\,2p}_{B1} = -12\sqrt{2}\, i\vec{K}/K^2(K^2+\alpha^2)^3 . \qquad (4.3)$$

To evaluate Eq. (4.1), it is convenient to decompose it via partial fractions to Feynman form. This yields

$$\vec{f}^{\,2p}_{SB2} = \frac{6\sqrt{2}\, i}{\pi^2}\frac{d^2}{d(\alpha^2)^2} \times \int \frac{1}{q^2-p^2-i\epsilon}\left(\frac{\vec{K}_i}{\alpha^2 K_f^2(K_i^2+\alpha^2)} - \frac{\vec{K}_f}{\alpha^2 K_i^2(K_f^2+\alpha^2)} - \frac{K^2}{\alpha^2(K^2+\alpha^2)}\frac{\vec{K}}{K_i^2K_f^2}\right) d^3q .$$

Clearly, all the integrals in the above can be obtained by differentiating a single basic type of integral with respect to α^2. This integral is

$$\vec{I}_{i,f} = \frac{1}{\pi^2}\int \frac{\vec{K}_{i,f}}{(q^2-p^2-i\epsilon)K_{f,i}^2(K_{i,f}^2+\alpha^2)}\, d^3q . \qquad (4.4)$$

In terms of $\vec{I}_{i,f}$, $\vec{f}^{\,2p}_{SB2}$ can be rewritten

$$\vec{f}^{\,2p}_{SB2} = 6\sqrt{2}\, i \frac{d^2}{d(\alpha^2)^2}\left(\frac{1}{\alpha^2}(\vec{I}_i - \vec{I}_f) - \vec{K}\frac{K^2}{\alpha^2(K^2+\alpha^2)}J\right) , \qquad (4.5)$$

where J is given by Eqs. (3.7). By some lengthy but straightforward algebra $\vec{I}_{i,f}$ can be written in terms of the integrals $J_{i,f}$ of Sec. III. One finds

$$\begin{aligned} \mathrm{Re}\vec{I}_i = \frac{\vec{k}_f}{q}\Big[& \pi\frac{K^2-2\Delta}{k_f} + \left(\frac{\pi}{2} + \tan^{-1}\frac{k_i-p}{\alpha} - \tan\frac{\alpha}{k_i+p}\right)\frac{K^2+2\Delta}{k_i} \\ & - 4K\sin^{-1}\frac{K}{(K^2+\alpha^2)^{1/2}} + [(2\Delta-K^2)(\alpha^2+K^2)+4\Delta_f K^2]\,\mathrm{Re}J_i\Big] \\ + \frac{\vec{K}}{q}\Big[& 2k_f\pi - \left(\frac{\pi}{2}+\tan^{-1}\frac{k_i-p}{\alpha} - \tan\frac{\alpha}{k_i+p}\right)\frac{2k_f^2-K^2+2\Delta}{k_i} \\ & + \frac{2(2\Delta-K^2)}{K}\sin^{-1}\frac{K}{(K^2+\alpha^2)^{1/2}} + [2k_f^2(K^2-\alpha^2)-(K^2-2\Delta)^2-2\Delta_f(2\Delta-K^2)]\,\mathrm{Re}J_i\Big] , \end{aligned} \qquad (4.6a)$$

$$\begin{aligned} \mathrm{Im}\vec{I}_i = \frac{\vec{k}_f}{q}\Big(& \frac{K^2+2\Delta}{2k_i}\ln\frac{(k_i+p)^2+\alpha^2}{(k_i-p)^2+\alpha^2} + \frac{K^2-2\Delta}{k_f}\ln\frac{k_f+p}{k_f-p} + [(2\Delta-K^2)(\alpha^2+K^2)+4\Delta_f K^2]\,\mathrm{Im}J_i\Big) \\ + \frac{\vec{K}}{q}\Big(& 2k_f\ln\frac{k_f+p}{k_f-p} - \frac{2k_f^2-K^2+2\Delta}{2k_i}\ln\frac{(k_i+p)^2+\alpha^2}{(k_i-p)^2+\alpha^2} + [2k_f^2(K^2-\alpha^2)-(2\Delta-K^2)^2-2\Delta_f(2\Delta-K^2)]\,\mathrm{Im}J_i\Big) , \end{aligned} \qquad (4.6b)$$

where

$$q = [K^2-(k_i-k_f)^2][K^2-(k_i+k_f)^2] , \qquad (4.6c)$$

and where Δ_i, Δ_f, and Δ are defined as in Sec. III. Note that q vanishes at $\theta=0$ and $\theta=\pi$. Thus if we have done things correctly the zero in the denominators of Eqs. (4.6a) and (4.6b) must be cancelled by a zero in the numerator. This is not obvious by casual inspection, but it does in fact occur. To obtain $\vec{I}_f$, one merely interchanges i and f everywhere in Eqs. (4.6) *and* replaces Δ by $-\Delta$. Since $J_{i,f}$ is readily evaluated analytically one has a complete expression for $\vec{I}_{i,f}$. Upon differentiating this expression, we have all the ingredients for Eq. (4.5).

In practice, since the differentiation process is rather cumbersome to perform analytically, it was done by elementary numerical methods. As an additional check $\vec{I}_{i,f}$ and its derivatives (taken analytically) were also obtained by Feynman integration of Eq. (4.4) using numerical integration with respect to the Feynman parameter t [see, e.g., Eq. (3.5)].

Although Eqs. (4.6a) and (4.6b) are very cumbersome, one point is clear immediately, namely, $\vec{f}^{\,2p}_{SB2}$ is not simply proportional to the vector $\vec{K}$ as in $\vec{f}^{\,2p}_{B1}$, but depends on one additional vector, taken to be $\vec{k}_f$ in Eqs. (4.6a) and (4.6b). In this respect it differs from the Glauber amplitude which is, like $\vec{f}^{\,2p}_{B1}$, proportional to $\vec{K}$ as a result of choosing the trajectory to lie along a direction perpendicular to the momentum transfer. This choice, as is well known, implies a vanishing $m=0$ component of the scattering amplitude in the frame defined by a z axis perpendicular to the momentum transfer. Thus upon rotating back to the frame whose z axis is $\vec{k}_i$, one obtains an amplitude proportional to $\vec{K}$.

The recent experimental work by Eminyan *et al.*,[17] in which e^--γ coincidences are measured following the excitation of the lowest p state in helium, determines parameters $\lambda(\theta)$ and $\chi(\theta)$ which can be interpreted as measuring precisely the extent to which the physical amplitude is or is not simply proportional to $\vec{K}$. The quantity $\lambda(\theta)$ is just the differential cross section for excitation to the $m=0$ magnetic sublevel divided by the full differential cross section, and $\chi(\theta)$ is the phase difference between the complex numbers representing the $m=1$ and $m=0$ amplitudes. Any theory in which the scattering amplitude $\vec{f}^{2p}$ is proportional to $\vec{K}$ will give

$$\chi(\theta)\equiv 0\,, \tag{4.7a}$$

$$\lambda(\theta)=1-k_f^2(\sin^2\theta)/K^2\,. \tag{4.7b}$$

In helium Eminyan *et al.*[17] find major departures from Eqs. (4.7) which show very clearly that the physical amplitude has a significant component which does not lie in the $\vec{K}$ direction. Clearly, the same situation is expected to occur in hydrogen, although experimental verification will be more difficult.

B. Real part of the second Born term

We begin by looking at the real part of the second Born term since apart from an unimportant azimuthal phase factor the second term in the Glauber series will be real for excitation to the $2p$ state. In light of the extraordinary large-angle properties found elsewhere for potential scattering[3] and elastic electron-atom scattering[1] and noted in the previous section for $1s$-$2s$ excitation, it would be desirable to have a more transparent expression for $\mathrm{Re}\,\vec{f}_{SB2}^{2p}$ in the large-angle region. Fortunately Eqs. (4.6a) and (4.6b) simplify considerably under the assumptions $k_i \gg 1$ and $K \gg 1$ which correspond to wide-angle ($\theta \gtrsim 60°$) and intermediate-angle scattering. Using Eqs. (3.6) for $J_{i,f}$ and expanding the remaining terms in Eq. (4.6b) one finds for $\mathrm{Re}\vec{f}_{SB2}^{2p}$ the following expression:

$$\mathrm{Re}\vec{f}_{SB2}^{2p}=\frac{12\sqrt{2}}{\alpha^6}\left[\frac{\alpha^2}{2k_iK^4}+\frac{\Delta_f}{2k_i^3K^2}\left(\ln\frac{k_i^2\alpha^2}{\Delta_f^2}-\frac{3}{2}\right)\right]\vec{K}-\frac{6\sqrt{2}}{\alpha^6k_i^3K^2}\left(\Delta_f\ln\frac{k_i^2\alpha^2}{\Delta_f^2}-\Delta_i\ln\frac{k_i^2\alpha^2}{\Delta_i^2}+\tfrac{3}{2}(\Delta_i-\Delta_f)\right)\vec{k}_i\,. \tag{4.8}$$

Looking at Eq. (2.4), which gives the second Born term with certain intermediate states included exactly, we see that it will be the part of the elastic matrix element which behaves like K_i^{-2} or K_f^{-2} for K_i or K_f large which will dominate the large-angle contribution of the elastic intermediate states to $\vec{f}_{B2}^{2p}$. Given the importance of the Δ_i and Δ_f dependence of Eq. (4.8) we should clearly include these elastic terms exactly. In the large-k_i, large-K limit all of the correction terms can be evaluated analytically. For the correction terms involving the final state as an intermediate state one finds

$$\mathrm{Re}(\vec{f}_{B2}^{2p})_{\mathrm{corr},f}=\frac{12\sqrt{2}}{\alpha^6}\times\left[\frac{\alpha^2}{4k_iK^4}\vec{K}+\frac{\Delta_i}{2k_i^3K^2}\left(\ln\frac{k_i^2\alpha^2}{\Delta_i^2}-\frac{3}{2}\right)\vec{k}_i\right]\,, \tag{4.9a}$$

where $\Delta_i=\Delta$ in the "on-shell" case [second term of Eq. (2.4)] and has its usual value ($2\Delta_i=k_i^2-p^2$) in the "off-shell" case [third term of Eq. (2.4)]. Similarly, for the correction terms involving the initial state as an intermediate state one finds

$$\mathrm{Re}(\vec{f}_{B2}^{2p})_{\mathrm{corr},i}=\frac{12\sqrt{2}}{\alpha^6}\times\left[\frac{\alpha^2}{4k_iK^4}\vec{K}-\frac{\Delta_f}{2k_i^3K^2}\left(\ln\frac{k_i^2\alpha^2}{\Delta_f^2}-\frac{3}{2}\right)\vec{k}_f\right]\,, \tag{4.9b}$$

where $\Delta_f=-\Delta$ for the on-shell case and has its usual value ($2\Delta_f=k_f^2-p^2$) for the off-shell case. If we combine Eqs. (4.9a) and (4.9b) to get the total off-shell contribution of the elastic intermediate states in second order, we obtain precisely Eq. (4.8), as we expect since off-shell elastic scattering should dominate. Thus the true second-order amplitude will be given at large momentum transfers precisely by the sum of Eqs. (4.9a) and (4.9b) with the appropriate on-shell values of Δ_i and Δ_f. We thus obtain

$$\mathrm{Re}\vec{f}_{B2}^{2p}\simeq\frac{6\sqrt{2}}{\alpha^4k_iK^4}\vec{K}-\frac{6\sqrt{2}\Delta}{\alpha^6k_i^3K^2}\left(\ln\frac{k_i^2\alpha^2}{\Delta^2}-\frac{3}{2}\right)\vec{K}+\frac{12\sqrt{2}\Delta}{\alpha^6k_i^3K^2}\left(\ln\frac{k_i^2\alpha^2}{\Delta^2}-\frac{3}{2}\right)\vec{k}_i\,. \tag{4.10}$$

Since Eq. (4.10) contains terms in k_i^{-3} it is clear that it cannot agree with the second-order Glauber term which contains only terms of order k_i^{-1}. The terms of order k_i^{-3} are the same size as the first term when K is large and thus causes $\mathrm{Re}\vec{f}_{B2}^{2p}$ to differ significantly from $\mathrm{Re}\vec{f}_{G2}^{2p}$, in marked contrast to what was found in the case of $2s$ excitation.

For the small- and intermediate-angle region, the situation is somewhat different. In this case a simplified expression for $\mathrm{Re}\vec{f}_{SB2}^{2p}$ is readily obtained. Using Eq. (3.6a) in Eq. (4.6b) one has (when $k_i \gg 1$ and $k_i \gg K$)

$$\mathrm{Re}\vec{f}_{SB2}^{2p}\simeq(1/k_i)T_1(K)\vec{K}+(1/k_i^2)T_2(K)\hat{k}_i\,, \tag{4.11a}$$

where

$$T_1(K) = -\frac{12\sqrt{2}}{\alpha^6}\left[\frac{1}{K^2}\left(1-\frac{\alpha^6}{(\alpha^2+K^2)^3}\right)\left(\ln\frac{k_i^2(\alpha^2+K^2)^2}{\alpha^2\Delta_i\Delta_f} - \frac{k_iK}{(k_i^2K^2+4\Delta_i\Delta_f)^{1/2}}\ln\frac{[(k_i^2K^2+4\Delta_i\Delta_f)^{1/2}+k_iK]^2}{4\Delta_i\Delta_f}\right) - \frac{1}{K^2}\ln\frac{\alpha^2+K^2}{\alpha^2} - \frac{3\alpha^2}{2(\alpha^2+K^2)^2} - \frac{3\alpha^4}{(\alpha^2+K^2)^3}\right], \quad (4.11b)$$

$$T_2(K) = -\frac{12\sqrt{2}}{\alpha^6}\left[\Delta\left(\frac{\alpha^6}{K^2(\alpha^2+K^2)^3}\ln\frac{\alpha^2+K^2}{\alpha^2}+\frac{3}{4(\alpha^2+K^2)}+\frac{\alpha^2}{(\alpha^2+K^2)^2}+\frac{3\alpha^4}{2(\alpha^2+K^2)^3}\right) + \frac{1}{K^2}\left(1-\frac{\alpha^6}{(\alpha^2+K^2)^3}\right)\left(\Delta_f\ln\frac{k_i\alpha}{\Delta_f}-\Delta_i\ln\frac{k_i\alpha}{\Delta_i}\right)\right]. \quad (4.11c)$$

The main point about these equations is the following: Because T_1 and T_2 are of the same order of magnitude when $K \lesssim 1$, when the z component of $\vec{K}$ is of order k_i^{-1}, i.e., when $\theta \sim k_i^{-1}$, the two terms contribute equally to the $m=0$ amplitude. Since the term in T_2 is missing from the Glauber amplitude, this means that the $m=0$ part of $\mathrm{Re}\vec{f}^{2p}_{SB2}$ will be very different from the corresponding Glauber quantity. Because the term in T_2 does not contribute to the $m=\pm 1$ amplitude, it is possible that the $m=\pm 1$ components of the Glauber second-order amplitude will agree with the corresponding part of the $m=\pm 1$ second Born amplitudes in this small- and intermediate-angle region. That this is in fact the case may be shown in a manner nearly identical to that employed in studying the $2s$ amplitude. Using the Glauber propagator of Eq. (3.12) in Eq. (4.1) to replace $(q^2-p^2-i\epsilon)^{-1}$ we obtain

$$\vec{f}^{2p}_{G2} = -\frac{3\sqrt{2}}{\pi k_i}\frac{d^2}{d(\alpha^2)^2}\frac{1}{\alpha^2}\int\left(\frac{\vec{Q}}{(\vec{Q}-\vec{K})^2(Q^2+\alpha^2)} + \frac{\vec{K}-\vec{Q}}{Q^2[(\vec{Q}-\vec{K})^2+\alpha^2]} - \frac{K^2}{K^2+\alpha^2}\frac{\vec{K}}{Q^2(\vec{Q}-\vec{K})^2}\right)d^2Q. \quad (4.12)$$

The integral is readily done by Feynman integration; one finds

$$\vec{f}^{2p}_{G2} = (1/k_i)T_1(K)\vec{K}\,|_{\Delta_i=0}\,, \quad (4.13)$$

which tells us that except at very small angles the $m=\pm 1$ Glauber amplitude agrees to leading order in k_i^{-1} with the $m=\pm 1$ Born amplitude, a result

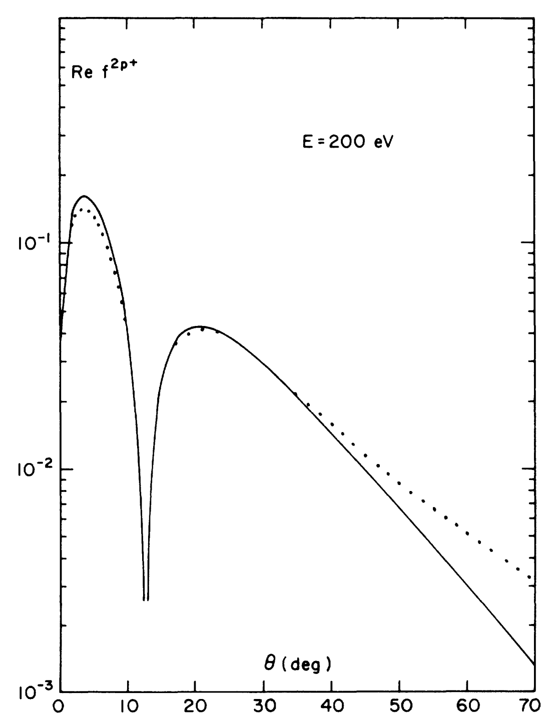

FIG. 4. Real part of the second Born term for the excitation of atomic hydrogen to the $2p_+$ state by electron bombardment at 200 eV. The solid curve is the result of this paper and the dotted curve is the result of the Glauber approximation.

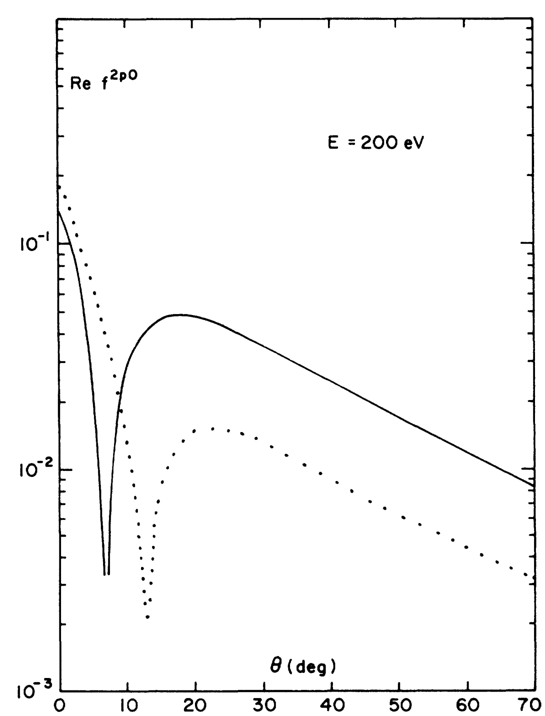

FIG. 5. Same as Fig. 4 but for excitation of the $2p_0$ state.

similar to what we found for the $2s$ amplitude.

The situation at an electron energy of 200 eV is shown in Figs. 4 and 5. Figure 4 gives the real part of the $m=1$ amplitude while Fig. 5 does the same for the $m=0$ amplitude. In all of these figures the dotted curve denotes the Glauber approximation $\mathrm{Re}\bar{f}_{G2}^{2pm}$ and the solid curve shows $\mathrm{Re}\bar{f}_{B2}^{2pm}$ obtained by putting in only the Yukawa part of the average potentials in the initial- and final-state elastic matrix elements in Eq. (2.4). In other words, in computing quantities like

$$\langle \vec{k}_f, 2pm \,|\, V \,|\, \vec{q}, 2pm \rangle$$

or

$$\langle \vec{q}, 1s \,|\, V \,|\, \vec{k}_i, 1s \rangle$$

in Eq. (2.4) only the Yukawa part of $\langle 1s|V|1s\rangle$ and $\langle 2pm|V|2pm\rangle$ is retained. This is because of the fact that the $2pm$ average potential contains a non-spherically-symmetric part which is computationally cumbersome to include in Eq. (2.4). The Yukawa part is spherically symmetric and poses no problems. Of course the Yukawa part will dominate at large angles because of the r^{-1} singularity at the origin.

The difference between the Born and Glauber results in Fig. 5 for the $m=0$ amplitude at small angles is in striking contrast with the situation shown in Fig. 4 for the $m=1$ amplitude. Clearly the Glauber choice of trajectory gives a good representation of the $m=1$ second order amplitude but is much less successful with the $m=0$ amplitude.[8] It should be remembered, however, that in the small-angle region the first Born amplitude is very large, and this term is given exactly in the Glauber approximation. Finally, we remark that the results of Figs. 4 and 5 were obtained using a value of 0.5 a.u. for Δ_i. Moderate variations about this value have only a minor effect on the amplitude.

C. Imaginary part of the second Born term

The quantity $\mathrm{Im}\vec{f}_{SB2}^{2p}$ is much simpler to analyze than was $\mathrm{Re}\vec{f}_{SB2}^{2p}$. Using Eq. (3.6) one finds that in the limit of large k_i and large K Eq. (4.6a) simplifies greatly; thus Eq. (4.5) gives

$$\mathrm{Im}\vec{f}_{SB2}^{\,2p} = \tfrac{16}{27}\sqrt{2}\,\vec{K}/k_i^2K^2 + \cdots . \tag{4.14}$$

This is independent of the average excitation energy and so is unaffected at large angles by the insertion of the extra terms contained in Eq. (2.4). Thus we may write

$$\mathrm{Im}\vec{f}_{B2}^{\,2p} \simeq \mathrm{Im}\vec{f}_{SB2}^{\,2p} \simeq \tfrac{16}{27}\sqrt{2}\,\vec{K}/k_i^2K^2 + \cdots . \tag{4.15}$$

Comparing this with Eq. (4.3) for $\vec{f}_{B1}^{2p}$ and with Eq. (4.10) for $\mathrm{Re}\vec{f}_{B2}^{2p}$ at large angles, we see that contrary to what we found in Sec. III for excitation to the $2s$ state the non-Glauber term $\mathrm{Im}\vec{f}_{B2}^{2p}$ dominates wide-angle scattering. Thus the differential cross section given by the eikonal-Born-series method will look very different at larger scattering angles than will the Glauber differential cross section. Indeed, we see from Eq. (4.15) that for large scattering angles the EBS cross section varies as

$$\frac{d\sigma_{\mathrm{EBS}}^{2p}}{d\Omega} \sim k_i^{-4}K^{-2} , \tag{4.16}$$

whereas the Glauber cross section will vary like

$$\frac{d\sigma_{G}^{2p}}{d\Omega} \sim k_i^{-2}K^{-6} , \tag{4.17}$$

as is seen from Eq. (4.13). Thus since $K \sim k_i$ in the large-angle region, we see that the Born series differential cross section will be larger than the Glauber differential cross section by a factor of k_i^2.

It is interesting to compare this with what we found for excitation to the $2s$ state. From Eqs. (3.8) and (3.10) we see that at large scattering angles

$$\frac{d\sigma_{\mathrm{EBS}}^{2s}}{d\Omega} \simeq \frac{d\sigma_{G}^{2s}}{d\Omega} \sim k_i^{-2}K^{-4} . \tag{4.18}$$

This means that in the large-angle region both the $2s$ and $2p$ differential cross sections are of the same order of magnitude according to the EBS method, whereas the Glauber method will yield a $2p$ differential cross section which is a factor of k_i^{-2} smaller than that for $2s$ excitation. In atomic hydrogen there is no direct experimental information to clarify this point. However, in helium where the $2\,^1S$ and $2\,^1P$ states are easily distinguished in energy-loss spectra, Suzuki and Takayanagi[18] have found that for 200 eV electrons the differential cross sections for the excitation of these two states are nearly equal in the large-angle region.

D. Discussion

At this point we can proceed exactly as in Sec. III to construct the full amplitude by adding exchange as in Sec. III D and using the third-order Glauber term to approximate the third Born term as explained in Sec. III E. There are, however, several difficulties with this approach beyond those already discussed in Sec. III. First, we have seen that the second-order Glauber term suffers serious difficulties at small scattering angles since it misses a piece of the amplitude proportional to $\hat{k}_i$, the Glauber amplitude being proportional to $\vec{K}$ alone. It seems likely that the third Glauber term will suffer from the same difficulty. Second, at wide angles the second-order Glauber term does not give the leading piece of the second Born term. This occurs because of the fact that although it is nominally of order k_i^{-1} whereas the non-Glauber piece is nominally of order k_i^{-2}, it falls off as K^{-3} for large K, whereas the non-Glauber piece falls off only as K^{-1}. In third order the same situation may again occur.

However, in order to give the reader a general picture of the systematics of $2p$ excitation we have decided to show the result obtained by following the procedure of Sec. III. Clearly, these results are open to question on certain points, but we can indicate several reasons why we feel they are worthwhile. Regarding the large-angle region, if $\mathrm{Im}\vec{f}_{G3}^{2p}$ (see the Appendix), which is of order $k_i^{-2}K^{-3} \times \ln K$, gives at least the order of magnitude of $\mathrm{Im}\vec{f}_{B3}^{2p}$ at large angles in the same manner found in second order, then $\mathrm{Im}\vec{f}_{B3}^{2p}$ is negligible at large angles compared to $\mathrm{Im}\vec{f}_{B2}^{2p}$. Furthermore, even if $\mathrm{Re}\vec{f}_{B3}^{2p}$ is of order $k_i^{-3}K^{-1}$ (i.e., if it has a slower momentum transfer falloff than $\mathrm{Im}\vec{f}_{B3}^{2p}$) the imaginary part of the second Born term will still dominate the differential cross section. Of course, just as for $2s$ excitation this will give only the leading term, the first Born term being completely negligible in the large-angle region. The first correction term remains inaccessible for reasons similar to those discussed in Sec. III E. At small angles, if one expects the Glauber results to be satisfactory for the third-order $m = \pm 1$ amplitude, then the procedure of Sec. III would be expected to give reliable results for our $m = \pm 1$ EBS amplitude. Clearly, for reasons mentioned above the $m = 0$ Glauber amplitude in third order must be viewed with skepticism in the small-angle region, thereby rendering the $m = 0$ EBS amplitude suspect. However, it is important to remember that at small angles the first Born amplitude is very strongly dominant due to the K^{-1} behavior of $\vec{f}_{B1}^{2p}$ for small K. This means that errors in higher-order terms are much less serious at small angles in $2p$ excitation than they would be in, say, $2s$ excitation.

Figures 6 and 7 show, respectively, the two differential cross sections $d\sigma^+/d\Omega$ and $d\sigma^0/d\Omega$ for an incident energy of 100 eV. Note that as one moves outside the scattering angle of about 20° the difference between the Glauber and EBS results becomes very striking. We may remark that the large differences seen in Figs. 6 and 7 are not reflected in the summed, integrated cross section for excitation to the $2p$ state. Even at energies as low as 100 eV the vast bulk of the integrated cross section falls in the angular range from 0° to 10°,

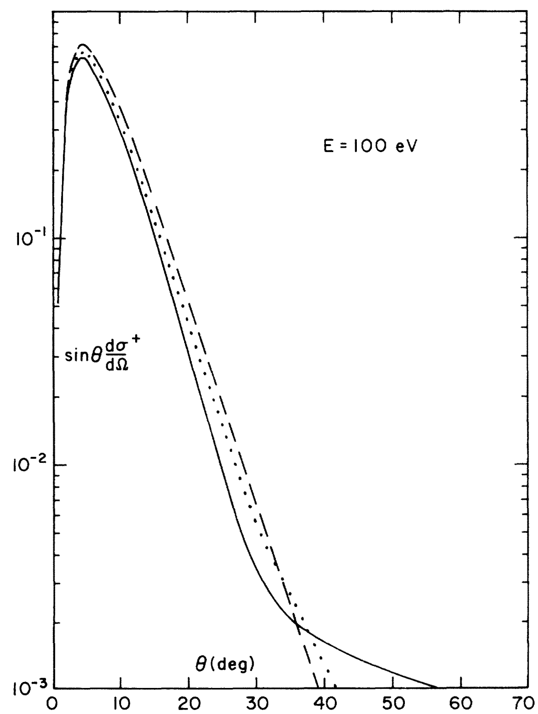

FIG. 6. Differential cross section for the excitation of the $2p_+$ state of atomic hydrogen by electron bombardment at 100 eV, multiplied by $\sin\theta$. The solid curve is the result of this paper, the dotted curve is the Glauber approximation, and the dashed curve is the first Born approximation.

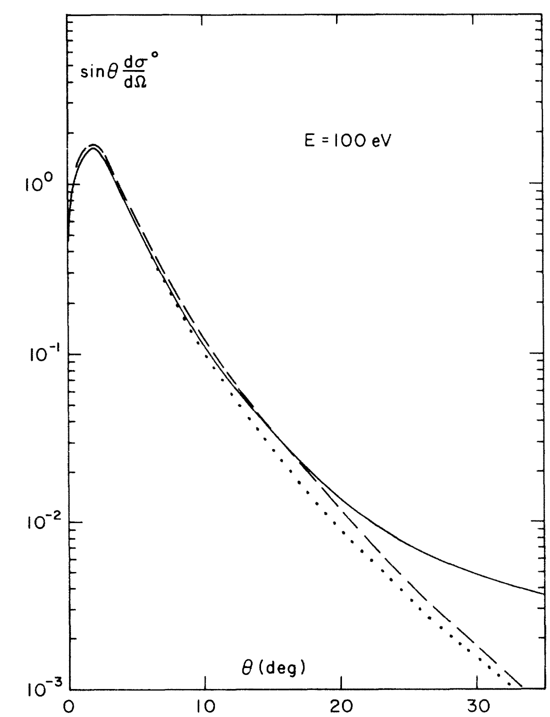

FIG. 7. Same as Fig. 6 but for excitation to the $2p_0$ state.

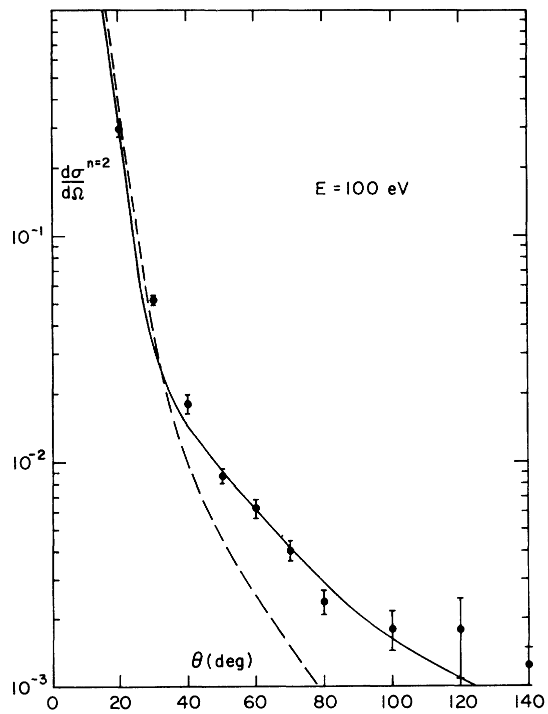

FIG. 8. Differential cross section for the excitation of the $n=2$ state of atomic hydrogen by electron bombardment at 100 eV. The solid curve is the result of this paper and the dashed curve is the Glauber approximation. The experimental points are those of Ref. 19.

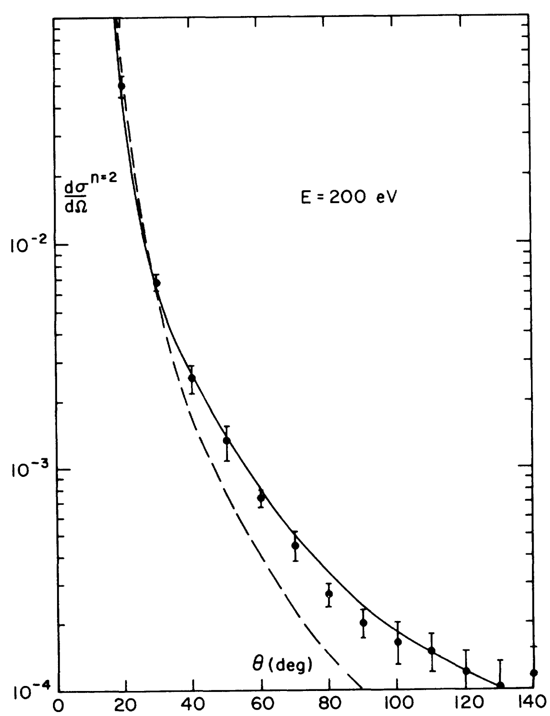

FIG. 9. Same as Fig. 8, but at 200 eV.

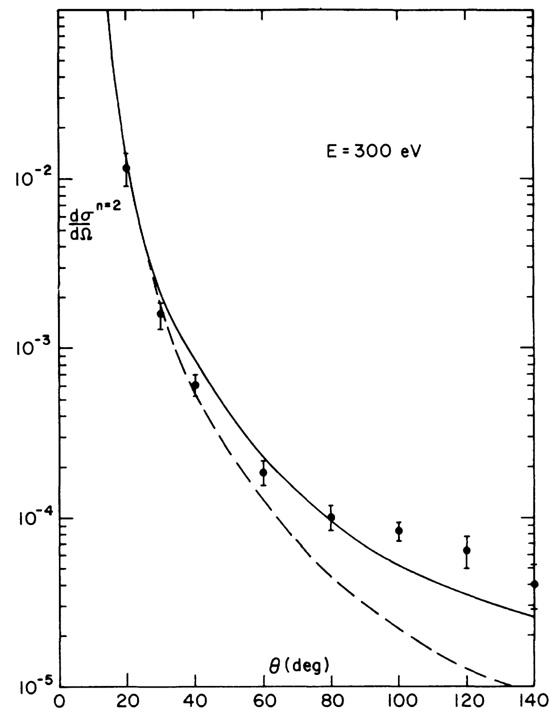

FIG. 10. Same as Fig. 8, but at 300 eV.

where the two methods offer only rather small corrections to the first Born approximation.

When one integrates the results of Figs. 6 and 7 over all angles and sums over magnetic sublevels, one finds that the Glauber results for σ^{2p} is less than the first Born result by about 10% while the EBS results falls 13% below the first Born result. The difference between the EBS and Glauber integrated cross sections is thus not experimentally significant. In the region between 100 and 200 eV, the polarization of the light emitted following excitation to the $2p$ state by electron bombardment is passing through zero, and hence large-percentage differences between Glauber and EBS results are found, being of the order of magnitude of 50% at both 100 and 200 eV. However, the absolute values are small, and the differences again appear not to be experimentally significant.

V. CONCLUSIONS AND COMPARISON WITH EXPERIMENT

With the differential cross sections for $1s$-$2s$ and $1s$-$2p$ excitation in hand, it is a simple matter to obtain a differential cross section for the excitation of the $n=2$ level of atomic hydrogen. Since the $2s$ and $2p$ states appear essentially identical in energy-loss spectra it is very difficult to measure these two differential cross sections separately. Only by looking in coincidence at the scattered electrons and the Lyman-α photons from the

$2p$ state can one disentangle the two. At present we have available the experimental work of Williams and Willis[19] at various energies up to 680 eV incident energy in an angular range from 20° to 140°. At the energies of interest in this paper ($E \geq 100$ eV) this corresponds to momentum transfers greater than 1 a.u. Hence the vast bulk of the integrated cross section comes from angles smaller than those studied by Williams and Willis. Thus the integrated-cross-section measurements are, roughly speaking, complementary to those discussed here.

Figures 8–10 show the experimental results at 100, 200, and 300 eV, respectively; we compare these results with the EBS results of this paper and with the Glauber results. The values obtained by using the first Born approximation are not included since at the angles of interest here they bear almost no resemblance to the experiments, the experimental results being dominated by elastic scattering in intermediate states. Even with the relatively large error bars on the experimental points, particularly at the larger angles, one sees clearly the failure of the Glauber approximation in the wide-angle region. Here the $2s$ and $2p$ states contribute roughly equally in the EBS theory, while in the Glauber approximation only the $2s$ state contributes significantly. As mentioned above, in helium the results of Suzuki and Takayanagi[18] already point strongly towards nearly equal $2\,^1S$ and $2\,^1P$ excitation differential cross sections. As experiments of the type shown in Figs. 8–10 become increasingly refined one will of course be able to say more meaningful things about the relationship between experiment and theory. At present, one may say that the overall picture given by the EBS method is very satisfactory.

It must be emphasized that in the large-angle regime the EBS method is basically a leading-order method [somewhat more than this for $2s$ excitation; see Sec. III E], rather analogous to the first Born approximation at small angles. With more refined experimental results one will be able to learn something about the leading corrections. From the theoretical point of view much remains to be done regarding these corrections, particularly in the case of $2p$ excitation where the momentum transfer dependence of the third-order term is not known with certainty. Also, the question of higher exchange effects remains to be studied for $2p$ excitation. Similar problems, although not so severe, are present in the case of $2s$ excitation, as pointed out in Sec. III.

Finally, let us comment briefly on other theoretical approaches. The distorted-wave Born approximation,[20] since it is able to treat elastic scattering in second order exactly, should give a leading term in the wide-angle region very similar to what is obtained in the EBS method. Since it treats elastic scattering to all orders, it will also, of course, give an approximation to each order of perturbation theory, not just the lowest orders. Whether or not these higher orders are given correctly is by no means clear, but one may perhaps be optimistic in light of the importance of the central Coulomb singularity. For similar reasons, a $1s$-$2s$-$2p$ full-wave close-coupling calculation with exchange included should do well in the large-angle region. A glance at Fig. 1, however, suggests that such methods will not offer notable improvements to the first Born approximation for small momentum transfers.

An approximation which incorporates both second-order effects and the close-coupling method is that of Bransden and Coleman.[21] In a full-wave treatment it should also be expected to do rather well at all angles, although suffering somewhat in the small-angle region from an inadequate treatment of the third Born term. Recently the suggestion of Byron[8] that for inelastic collisions at small angles one should not use the Glauber trajectory but rather a trajectory along the incident direction has been extended by Gau and Macek[22] to investigate scattering at all angles. For small angles, this expression has some intuitive appeal on kinematical grounds, as pointed out in Ref. 8. However, at larger angles, this attractiveness is by no means obvious; and in fact the dependence on k_i and K away from the small-angle region is quite different than that found from an analysis of the second Born term. In addition this approach will give a real (imaginary) part of the second-order term for $2s$ ($2p$) excitation which is of order k_i^{-1} for k_i large, also in disagreement with the second Born prediction.

ACKNOWLEDGMENTS

One of us (F.W.B.) is deeply grateful to the Fulbright Commission for its support during the academic year 1973–1974 when much of this work was done. It is also a pleasure to thank Professor C. J. Joachain both for his hospitality at the Université Libre de Bruxelles during this same period and for many interesting discussions concerning the EBS method.

APPENDIX

The large-K behavior of the Glauber amplitude is readily obtained as follows: The z integration on the product of the initial- and final-bound-state wave functions can be done analytically in terms of K-Bessel functions, and the integration on the azimuthal coordinate of the incident electron is also straightforward. Then using ρ to denote the ratio of the impact-parameter coordinate of the bound electron to that of the incident electron one finds in the case of $2s$ excitation

$$f_G^{2s} = -\frac{2^{7/2}k_i}{3^6 i\pi}\int J_0(\tfrac{2}{3}K\beta)\left(K_0(\beta\rho) - \frac{6}{\beta\rho}K_1(\beta\rho) + K_2(\beta\rho)\right) \times \exp\left(\frac{i}{k_i}\ln(1-2\rho\cos\phi+\rho^2)\right) \times \beta^5\rho^3 d\beta d\rho d\phi. \tag{A1}$$

Integrals of products like $J_l(ax)K_m(bx)x^n$ can be done in terms of the ratio of two polynomials.[23] One finds in this case

$$f_G^{2s} = -\frac{2^{19/2}k_i}{3^6\delta^2 i\pi} \times \int_0^\infty dy \int_0^{2\pi} d\phi \frac{y^3(y^4-7y^2+4)}{(y^2+1)^5} \times \exp\left(\frac{i}{k_i}\ln(1-2\delta y\cos\phi+\delta^2y^2)\right), \tag{A2}$$

where $\delta = \frac{2}{3}K$ and $y = \rho/\delta$. Expanding when δ is large (K large) and making the change of variable $x = y^2$, we obtain

$$f_G^{2s} \simeq -\frac{2^{19/2}k_i}{3^6\delta^2 i}\exp\left(\frac{2i}{k_i}\ln\delta\right) \times \int_0^\infty \frac{x(x^2-7x+4)}{(x+1)^5}\exp\left(\frac{i}{k_i}\ln x\right)dx. \tag{A3}$$

This integral is readily done by contour integration, yielding

$$f_G^{2s} \simeq i\,\frac{2^{13/2}}{3^4K^2k_i}\,\frac{(\pi/k_i)(1+i/k_i)}{\sinh(\pi/k_i)}\exp\left(\frac{2i}{k_i}\ln(\tfrac{2}{3}K)\right). \tag{A4}$$

Upon expanding Eq. (A4) in powers of k_i^{-1}, one obtains the various orders of the Glauber series, beginning with $\bar{f}_{G2}^{2s} \simeq -i2^{13/2}/3^4K^2k_i$. The nth term in the Glauber series is clearly of order $K^{-2}k_i^{-(n-1)}$ times a polynomial in $\ln K$. The term $\bar{f}_{G1}^{2s}$ is missing, since it is of order K^{-6}.

The procedure outlined above can be used to obtain the large-K form of $\vec{f}_G^{2p}$ with equal ease. One finds

$$\vec{f}_G^{2p} \simeq \tfrac{32}{27}\sqrt{2}\,\frac{\vec{K}}{k_iK^4}\,\frac{(\pi/k_i)(1+1/k_i^2)}{\sinh(\pi/k_i)}\exp\left(\frac{2i}{k_i}\ln(\tfrac{2}{3}K)\right). \tag{A5}$$

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