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Plasma screening effect in ion-ion scattering

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When a plasma is extremely dense or hot, the electrons lose their efficiency for stopping heavy charged particles. The main process for their slowing down then is the small-angle ion-ion scattering, where the bare Coulomb cross section becomes infinite as θ^{-4}. The purpose of this paper is to evaluate the collective screening effects which make the ion Coulomb cross section finite. The calculations are performed in the random-phase approximation and show that the Coulomb cross section is essentially cut off for energy transfers smaller than $\hbar\omega_p$, where ω_p is the plasma frequency of the ions.

I. INTRODUCTION

The interaction of fast charged particles with matter has been studied for many years and is now fairly well understood. However, the extremely high compressions which may soon be available with lasers or electron beams bring an entirely new regime for the slowing down of charged particles and necessitate the revision of some classical formulas.

The purpose of this work is to compute the rate of loss of energy, $-dE/dx$, of a fast ion in a dense plasma, due to collisions with other ions. Usually, most of the energy is lost to electrons. However, when an electron gas becomes very dense or very hot, it loses its efficiency for stopping heavy charged particles.[1] This occurs when the velocity of the incident ions falls below the velocity of the electrons at the Fermi surface (for degenerate electrons) or below the thermal velocity of the electrons, if they are nondegenerate.

Under these conditions, energetic ions, such as those resulting from nuclear reactions, will give their energy mostly to other ions, by nuclear collisions. They can thus produce showers of fast ions, some of which may undergo further fusion reactions.[2]

An important factor in the behavior of a fast ion in such a plasma is the small-angle ion-ion scattering, where the *bare* Coulomb cross section becomes infinite as θ^{-4}. Actually, screening effects due to other particles make it finite. A reliable estimate of these collective effects is therefore necessary to compute $-dE/dx$.

In Sec. II of this paper, we show how to compute the energy loss of a fast ion in a dense plasma, by means of the random-phase approximation. A closed expression is derived, for which various approximations are obtained in Sec. III, in the limits of small or of large energy transfer. Some numerical examples are given in Sec. IV, and further approximations (of practical importance) are given in Sec. V.

II. ION SCATTERING IN A DENSE PLASMA

We consider a high-density plasma with n electrons per unit volume. The ion charge is Ze and the ion density n/Z. For simplicity, we shall first assume that there is only one kind of ion in the plasma. At the end of this section, we show how to generalize the calculation to the case of several ion species.

By "high density," we mean that the number of particles in an electron Thomas-Fermi sphere, or in a Debye sphere at high temperature, is large:

$$n(E/4\pi e^2 n)^{3/2} \gg 1\,. \tag{1}$$

Here, E is a measure of the particle energy, namely, $E=\frac{2}{3}E_f$ (where E_f is the Fermi energy), at low temperature, and $E=kT$ at high temperature T.

Our problem is the scattering, by the plasma, of an "external" ion of mass M_i, charge $Z_i e$, and initial energy $E_i = M_i \vec{v}_i^{\,2}/2$. In the present work we discuss only the *ion-ion* scattering: The electrons are taken into account indirectly, via their interaction with the *plasma* ions, but their *direct* interaction with the incident ion is not considered here.

For a high-density plasma, the Born approximation is quite accurate, so that the differential cross section for ion-ion scattering can be written as

$$\frac{d^2\sigma}{d\Delta\, dq} = \frac{Z_i^2 M_i}{E_i}\frac{2e^4}{\hbar^3 q^3} S_{-\vec{q}}(\Delta/\hbar)\,, \tag{2}$$

where Δ and $\hbar\vec{q}$ are the energy and momentum transferred by the incident ion to the system, and where

$$S_{\vec{q}}(\omega) = \int_0^\infty dt\, e^{i\omega t}\langle \rho_{\vec{q}}(t)\rho_{-\vec{q}}(0)\rangle \tag{3}$$

is the plasma-ion form factor. Here, $\rho_{\vec{q}}$ denotes the Fourier transform of the ion density $\rho(\vec{r})$ and the angular brackets denote the statistical ensemble average.

The energy lost by an incident ion per unit length is then given by

$$\frac{dE}{dx}=-\frac{1}{V}\int d\Delta\,\Delta\frac{d\sigma}{d\Delta},$$

where V is the volume of the plasma. We are therefore interested in obtaining an expression for the "cross section per unit energy loss"

$$\frac{1}{V}\frac{d\sigma}{d\Delta}=\frac{1}{V}\int dq\,\frac{d^2\sigma}{d\Delta dq}\,. \tag{4}$$

We shall calculate the plasma-ion form factor $S_{\vec{q}}(\omega)$ in the random-phase approximation,[3] which must be valid at high densities. The density-density correlation $\langle\rho_{\vec{q}}(t)\rho_{-\vec{q}}(0)\rangle$—the response of the system to a charge fluctuation (here, the incident ion)—is intimately connected with the dielectric function of the system, $\epsilon_{\vec{q}}(\omega)$. It is convenient to write

$$S_{\vec{q}}(\omega)=\frac{2\hbar V}{v_q}\frac{1}{1-e^{-\hbar\omega/kT}}\,\mathrm{Im}\frac{1}{\epsilon_{\vec{q}}(\omega)}\,, \tag{5}$$

where $v_q=4\pi e^2/q^2$ is the Fourier transform of the Coulomb interaction.

In the random-phase approximation, the plasma dielectric function is given by[3]

$$\epsilon_{\vec{q}}(\omega)=1-\sum_j P^j_{\vec{q}}(\omega)\,, \tag{6}$$

where $P^j_{\vec{q}}(\omega)$, the polarization function for particles of type j (ions or electrons), is

$$P^j_{\vec{q}}(\omega)=\frac{v_q}{V}Z_j^2\sum_{\vec{p}}\frac{f^j_{\vec{p}+\vec{q}}-f^j_{\vec{p}}}{E^j_{\vec{p}+\vec{q}}-E^j_{\vec{p}}+\hbar\omega}\,. \tag{7}$$

Here, $f^j_{\vec{p}}$ is the distribution function for particles of type j (charge Z_je, mass M_j, kinetic energy $E^j_p=\hbar^2p^2/2M_j$). We shall henceforth use the index e for electrons, and no index at all for ions. The imaginary part of $\epsilon_{\vec{q}}(\omega)$ is obtained by substituting $\omega+i\eta$ for ω and then going to the limit $\eta\to 0$.

To obtain the plasma-ion form factor, we write

$$\mathrm{Im}\frac{1}{\epsilon_{\vec{q}}(\omega)}=-\frac{\mathrm{Im}\epsilon_{\vec{q}}(\omega)}{|\epsilon_{\vec{q}}(\omega)|^2}\,, \tag{8}$$

and take into account only the contribution of the ions to $\mathrm{Im}\epsilon_{\vec{q}}(\omega)$. (The electrons still contribute indirectly by their effect on $|\epsilon_q(\omega)|^2$ in the denominator.)

We now turn to the calculation of the "ionic" dielectric function. Since for practical purposes the ion temperature T is much higher than the ionic Fermi temperature, the distribution function for ions is Maxwellian:

$$f_{\vec{p}}=(2\pi)^3\frac{n}{Z}\left(\frac{\hbar^2}{2\pi MkT}\right)^{3/2}e^{-\hbar^2\vec{p}^2/2MkT}\,. \tag{9}$$

On the other hand, the electron temperature may be either lower or higher than the electronic Fermi temperature. Therefore we take for electrons the Fermi distribution

$$f^e_{\vec{p}}=(1+e^{(E_{\vec{p}}-\mu_e)/kT_e})^{-1}\,, \tag{10}$$

where μ_e is the electron chemical potential.

The imaginary part of $P_{\vec{q}}(\omega)$ for Maxwellian ions can be obtained from Eq. (7),

$$\mathrm{Im}P_q(\omega)=\frac{v_qZ^2}{(2\pi)^3}\int d\vec{p}\,(f_{\vec{p}+\vec{q}}-f_{\vec{p}})[-\pi\delta(E_{\vec{p}+\vec{q}}-E_{\vec{p}}+\hbar\omega)]\,, \tag{11}$$

whence

$$\mathrm{Im}\epsilon_q(\omega)=\frac{\pi}{4}\hbar^2\omega_p^2\left(\frac{2M}{\hbar^2q^2}\right)^{3/2}\frac{1-e^{-\hbar\omega/kT}}{(\pi kT)^{1/2}}\times\exp\left[-\frac{\hbar^2}{2MkT}\left(\frac{q}{2}-\frac{M\omega}{\hbar q}\right)^2\right]\,, \tag{12}$$

where

$$\omega_p^2=4\pi e^2nZ/M$$

is the square of the plasma frequency for ions of mass M and charge Ze. (Note that the number of ions per unit volume is n/Z.)

Equation (12) can be reduced to the high-temperature classical plasma result, by taking formally the $\hbar\to 0$ limit, namely,

$$\mathrm{Im}\epsilon_q(\omega)\to\frac{\omega_p^2}{kT}\left(\frac{\pi M\omega^2}{2kTq^2}\right)^{1/2}e^{-M\omega^2/2kTq^2}\,. \tag{13}$$

However, in the present work, we shall use the more general expression in Eq. (12), since we wish to consider the whole spectrum of energy and momentum transfers $\hbar\omega$ and $\hbar q$.

The real part of the ionic $P_q(\omega)$ is readily calculated as

$$\mathrm{Re}P_q(\omega)=\frac{v_qZ^2}{(2\pi)^3}\mathrm{P}\int d\vec{p}\,\frac{f_{\vec{p}+\vec{q}}-f_{\vec{p}}}{E_{\vec{p}+\vec{q}}-E_{\vec{p}}+\hbar\omega} \tag{14}$$

$$=\tfrac{1}{4}\hbar^2\omega_p^2\left(\frac{2M}{\hbar^2q^2}\right)^{3/2}\frac{1}{(\pi kT)^{1/2}}\times\mathrm{P}\int_{-\infty}^{\infty}dx\,e^{-x^2}\left(\frac{1}{x-x_+}-\frac{1}{x-x_-}\right), \tag{15}$$

where

$$x_\pm=\left(\frac{\hbar^2}{2MkT}\right)^{1/2}\left(\frac{q}{2}\pm\frac{M\omega}{\hbar q}\right). \tag{16}$$

However, since $\mathrm{Re}P_q(\omega)$ appears only in the denominator of Eq. (5), which gives the ionic form factor, we can approximate it by the classical high-temperature limit ($\hbar\to 0$) and write

$$\mathrm{Re}P_q(\omega) = -\frac{M\omega_p^2}{kTq^2} G\left[\left(\frac{M}{2kT}\right)^{1/2}\frac{\omega}{q}\right], \tag{17}$$

where

$$G(z) = \pi^{-1/2}\mathrm{P}\int_{-\infty}^{\infty}\frac{e^{-x^2}dx}{(x-z)} \tag{18}$$

is the well known Maxwellian plasma dispersion function.

Finally, we turn to the electron contribution to $|\epsilon_q(\omega)|^2$. Its only significant part comes from $\mathrm{Re}P_q^e(\omega)$ for $\hbar\omega$ much smaller than the electron energies. This is due to the fact that the electrons are much faster than the plasma ions: They "follow" the ions almost instantaneously, so that their screening can be considered as static on an ionic time scale. We can therefore take the $\omega \to 0$ limit in

$$\mathrm{Re}P_q^e(\omega) = \frac{2v_q}{(2\pi)^3}\mathrm{P}\int d\vec{p}\,\frac{f^e_{\vec{p}+\vec{q}} - f^e_{\vec{p}}}{E^e_{\vec{p}+\vec{q}} - E^e_{\vec{p}} + \hbar\omega}\ . \tag{19}$$

As the main contribution comes from small q, we obtain

$$\mathrm{Re}P_q^e(\omega) = \frac{4e^2}{\pi q^2}\int_0^\infty p^2\,dp\,\frac{\partial}{\partial E_p}f_p^e = -\frac{\kappa_e^2}{q^2}\ , \tag{20}$$

where κ_e is the inverse electronic screening length, which depends on the electron temperature T_e and on the density. For low temperature, we get the Thomas-Fermi result

$$\kappa_e^2 = 6\pi e^2 n/E_f, \quad kT_e \ll E_f \tag{21}$$

and for high temperature, we get the Debye result

$$\kappa_e^2 = 4\pi e^2 n/kT_e, \quad kT_e \gg E_f\ . \tag{22}$$

We can now make some physical approximations to evaluate qualitatively the screening effect. (A more quantitative, but physically less transparent treatment, will follow in Sec. III and IV.)

First, let us assume that $|P^j_\omega(q)| \ll 1$, so that we can replace $|\epsilon_q(\omega)|^2$ by 1 in the denominator. (This approximation will be denoted by an index 0.) We get

$$\left(\frac{1}{V}\frac{d\sigma}{d\Delta}\right)_0 = \frac{Z_i^2 M_i e^2}{E_i}\frac{\omega_p^2}{\Delta^3}\frac{2kT}{\pi^{1/2}} \times\int\frac{dq}{q^2}\exp\left[-\frac{\hbar^2}{2MkT}\left(\frac{M\Delta}{\hbar^2 q} - \frac{q}{2}\right)^2\right]. \tag{23}$$

Introducing the notations

$$\xi = \Delta/2kT \quad\text{and}\quad \eta = \hbar\omega_p/\Delta\ , \tag{24}$$

we can write

$$\left(\frac{1}{V}\frac{d\sigma}{d\Delta}\right)_0 = \frac{Z_i^2 M_i e^2}{E_i}\frac{\eta^2}{\hbar^2}\frac{\eta^2}{4}I_0(\xi)\ , \tag{25}$$

where, with the substitution

$$x = \left(\frac{M}{2kT}\right)^{1/2}\frac{\Delta}{\hbar q}\ ,$$

we have

$$I_0(\xi) = \frac{4}{\pi^{1/2}}\frac{1}{|\xi|}\int dx\,x^2\exp[-x^2(1-\xi/2x^2)^2]\ . \tag{26}$$

Note that the coefficient of I_0 in (25) is the bare two-body Coulomb cross section.

The integration limits in (26) are related to those of Eq. (23), namely,

$$|K - K'| < q < K + K'\ , \tag{27}$$

where $\vec{K}$ and $\vec{K}'$ denote the wave vectors of the incident ion before and after the collision, i.e.,

$$\vec{q} = \vec{K} - \vec{K}'. \tag{28}$$

Since $E_i = \hbar^2K^2/2M_i$ and $\Delta = E_i - \hbar^2K'^2/2M_i$, we have

$$|K \pm K'| = K[1 \pm (1-\Delta/E_i)^{1/2}]\ , \tag{29}$$

so that the integration limits in Eq. (26) are

$$x = \frac{(M\xi\delta/2M_i)^{1/2}}{1 \pm (1-\delta)^{1/2}}\ , \tag{30}$$

where

$$\delta = \Delta/E_i$$

is the fraction of energy transferred in the collision.

Now, Eq. (25) shows that the cross section is large for small δ (large η), so that we can effectively take the integration limits as 0 and ∞, and obtain

$$I_0(\xi) = 1 + 1/\xi\ . \tag{31}$$

We thus see that the screening factor tends to 1 whenever $\xi = \Delta/2kT$ is large.

A less drastic approximation is to neglect the imaginary part, but not the real part of $P^j_q(\omega)$ in the denominator of Eq. (8). From (17) and (20), we have

$$\mathrm{Re}\epsilon_q(\omega) = 1 + \left(\frac{\kappa_e}{q}\right)^2 + \frac{M\omega_p^2}{kTq^2}G\left[\left(\frac{M}{2kT}\right)^{1/2}\frac{\omega}{q}\right]. \tag{32}$$

For small energy transfers ($\omega \to 0$) we have $G \simeq 1$ and we can thus write

$$\mathrm{Re}\epsilon_q(\omega) = 1 + \kappa^2/q^2\ , \tag{33}$$

where κ^2 is the total inverse square screening length

$$\kappa^2 = \kappa_e^2 + \kappa_D^2\ . \tag{34}$$

Here,

$$\kappa_D^2 = 4\pi Z e^2 n/kT \tag{35}$$

is the ionic Debye factor.

We thus see that at low momentum transfer, the denominator in Eq. (8) diverges as q^{-4}: This is just the screening, by electrons *and* ions, which cancels the q^{-4} divergence in the two-body Coulomb cross section.

On the other hand, for large energy transfer, the argument of G in Eq. (32) is large, and we have approximately

$$G \simeq -kTq^2/M\omega^2 \tag{36}$$

(see Appendix B). We obtain

$$\mathrm{Re}\,\epsilon_q(\omega) = 1 + \left(\frac{\kappa_e}{q}\right)^2 - \left(\frac{\omega_p}{\omega}\right)^2 ; \tag{37}$$

i.e., we see that a sound wave with

$$\omega = \omega_p q/(\kappa_e^2 + q^2)^{1/2} \tag{38}$$

can be excited by the incident ion.

Actually, the last term of (37) is important only if $\omega \lesssim \omega_p$. For larger energy transfer, we see that the main contribution to the screening is due to the electrons, and the denominator in Eq. (8) can be approximated as

$$|\epsilon_q(\omega)|^2 \simeq (1 + \kappa_e^2/q^2)^2 . \tag{39}$$

To conclude, we may also write that denominator exactly. We obtain

$$\frac{1}{V}\frac{d\sigma}{d\Delta} = \frac{Z_i^2 M_i}{E_i}\frac{e^2\eta^2}{4\hbar^2} I , \tag{40}$$

where the coefficient of I is the bare two-body Coulomb cross-section. The screening factor I is given by

$$I = \frac{4}{\pi^{1/2}|\xi|}\int \frac{Cx^2\,dx}{\{1+\eta^2x^2[3\tau H(\tau_e)/Z + 2G(x)]\}^2 + [\eta^2\pi^{1/2}(1-e^{-2\xi})\xi^{-1}x^3C]^2} , \tag{41}$$

where the integration limits are given by Eq. (30) and where the following notations have been used:

$$H(z) = \int_0^\infty dx\,\{1+\exp[(x^2-\mu_e)/z]\}^{-1}$$

is the normalized electron screening factor,

$$\tau_e = kT_e/E_f$$

and

$$\tau = kT/E_f$$

are the normalized temperatures, and

$$C \equiv \exp[-x^2(1-\xi/2x^2)^2] .$$

Finally, we note that if we have in the plasma several species at the same temperature T, the above results can be generalized, to a good approximation, by taking

$$\omega_p^2 = 4\pi e^2 \sum_j \frac{n_j Z_j^2}{M_j} , \tag{42}$$

where n_j is the number of ions of type j (mass M_j, charge $Z_j e$) per unit volume. Indeed, it readily follows from Eqs. (40) and (41) that, if the denominator in (41) is set equal to expression (39), i.e., depends only on the electrons (we have seen that usually most of the screening is due to the electrons), then the plasma ions enter in (40) only through η^2, which is proportional to ω_p^2, and we simply have to take the sum of the squares of all the ionic plasma frequencies.

III. EVALUATION OF THE SCREENING FACTOR

Although the integral in Eq. (41) looks formidable, it can be solved numerically—with some care, especially if ξ is large. Moreover fairly good analytic estimates can be obtained for it, for most values of the parameters (see Appendix A).

Taking $Z=1$, for simplicity, it is convenient to define

$$\alpha = \hbar\omega_p/4kT$$

and

$$\nu = 3\hbar\omega_p H(\tau_e)/4E_f .$$

These expressions do not depend on the incident ion, but only on the properties of the plasma (density and temperatures). Note that α can be large or small, but ν is always small. We have

$$\nu/\alpha \to 3kT/E_f$$

for degenerate electrons, and

$$\nu/\alpha \to 2T/T_e$$

for nondegenerate electrons. It is shown in Appendix A that for $\eta > 1$, one has approximately

$$I = \{[\tfrac{1}{2}(\alpha/\gamma^3)^{1/2} - (\alpha^3/\gamma)^{1/2}]\pi^{1/2}e^{\gamma\alpha}\,\mathrm{erfc}(\gamma\alpha)^{1/2} + \alpha/\gamma\}/\eta^2 + 2/\eta^2(1-e^{-2\xi})[(1-\eta^2)^2 + 6\eta^4\nu/\alpha]^{1/2} + 2.3438/\eta^4 , \tag{43}$$

where

$$\gamma = 2\alpha + \nu .$$

For very large η (i.e., for $\Delta \ll h\omega_p$) the first term dominates and becomes

$$I \to (\pi\alpha/\gamma^3)^{1/2}/2\eta^2 \quad \text{if } \alpha \ll 1 , \tag{44a}$$

or

$$I \to 1/\eta^2\gamma^2 \quad \text{if } \alpha \gg 1 . \tag{44b}$$

We thus see that the screening factor behaves as Δ^2, so that the effective cross section tends to a finite limit for $\Delta \to 0$.

On the other hand, for small $|\eta|$ (namely, $|\eta| < 1$ if $\alpha < 1$, or $|\eta| < \alpha^{-1/3}$ if $\alpha > 1$), we have approximately

$$I = \frac{1 + \eta/2\alpha}{(1 + \nu\eta - \eta^2)^2} \quad \text{if } \Delta > 0\,, \tag{45a}$$

and

$$I = e^{4\alpha/\eta} \frac{1 - \eta/2\alpha}{(1 + \nu\eta - \eta^2)^2} \quad \text{if } \Delta < 0\,. \tag{45b}$$

Finally, if $\alpha \gg 1$, the screening integral behaves in a rather curious way in the region $\alpha^{-1/3} < \eta < 1$, where it may have three sharp maxima (see below).

IV. NUMERICAL EXAMPLES

We have evaluated the screening factor numerically and compared the results with the above estimates, for two extreme cases of DT plasmas.

Ultrahigh-density, hot plasma. We took $n = 5.1 \times 10^{27}$ cm^{-3} (10^5 times the solid density) and $kT = 1$ keV. We then obtain $\hbar\omega_p = 40$ eV, so that $\alpha = 0.01$. The electron Fermi energy is $E_f = 10804$ eV and we have $\nu = 0.002\,776\,7$. The results of the numerical integration are given in Fig. 1. We see that our asymptotic estimates describe fairly well the behavior of I over the whole range of η.

Solid DT. We took $n = 5.1 \times 10^{22}$ cm^{-3} and $kT = 0.001\,58$ eV. This gives $\hbar\omega_p = 0.1264$ eV and $\alpha = 20$. The electron Fermi energy is $E_f = 5.0148$ eV and we have $\nu = 0.0189$. Under these conditions, we cannot expect the present theory to be quite accurate, because the actual electron Fermi surface departs appreciably from a sphere. Moreover, the *ion* Fermi energy

$$\frac{E_f}{2}\left(\frac{M_e}{M_D} + \frac{M_e}{M_T}\right)$$

is 0.001 14 eV, which is not *much* smaller than kT. There is therefore also some *ion* degeneracy, which our theory did not take into account.

Nevertheless, our results should be at least qualitatively correct. The result of the numerical integration is given in Fig. 2. It is seen that the asymptotic estimates describe extremely well the behavior of I for $\eta < 0.1$ and $\eta > 1$. In the region $0.1 < \eta < 0.4$, Eq. (45a) gives the correct order of magnitude, while in the region $0.4 < \eta < 0.78$, Eq. (A2) is roughly correct.

On the other hand, the value of I behaves wildly for $0.78 < \eta < 1$, as can be seen in Fig. 3. In particular, there is at $\eta = 0.7796$ an extremely sharp peak which cannot be evaluated numerically.[4] If we take seriously Eqs. (A4) and (A5), the height of the peak is 2.3×10^{27} and its width is 1.4×10^{-53}. These ridiculous values are of course unphysical: As they depend on α, the peak will be erased by any inhomogeneity in n or in kT.

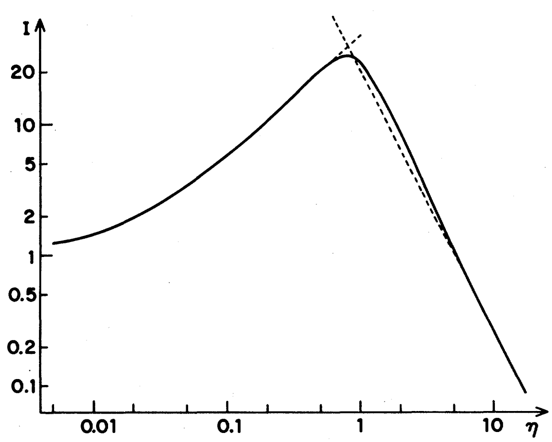

FIG. 1. Screening factor for $kT = 25\hbar\omega_p$. The broken lines represent the asymptotic limits.

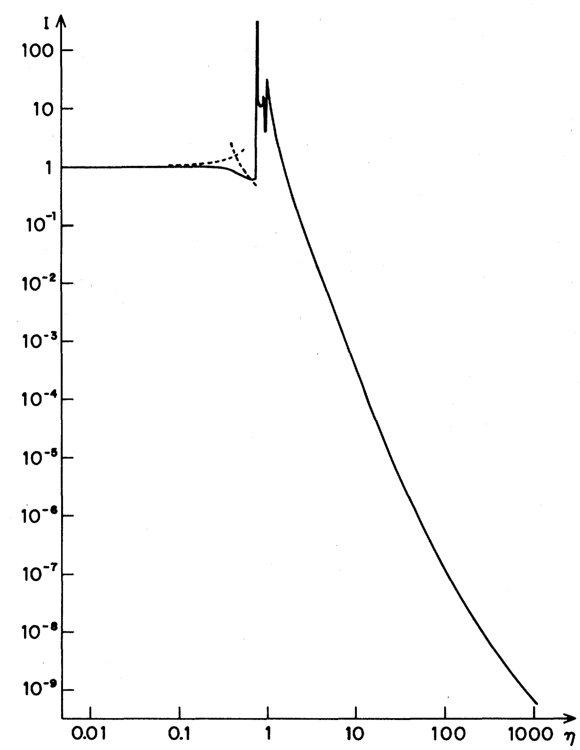

FIG. 2. Screening factor for $\hbar\omega_p = 80\,kT$. The asymptotic limits, represented by broken lines, essentially coincide with the results of the numerical integration for $\eta < 0.1$ and $\eta > 1$. The region $0.78 < \eta < 1.03$ is given with more detail in Fig. 3.

V. AVERAGE SLOWING DOWN

We are now ready to evaluate the average frictional force

$$-\frac{dE_i}{dx}=n\int\frac{d\sigma}{d\Delta}\Delta\,d\Delta\,, \tag{46}$$

where

$$\frac{d\sigma}{d\Delta}=\frac{\pi M_i Z_i^2 e^4 I}{ME_i\Delta^2}\,. \tag{47}$$

We have seen that for $\Delta\ll\hbar\omega_p$, the screening factor I behaves as Δ^2, so that $d\sigma/d\Delta$ tends to a constant. Its limiting value is not relevant, since its contribution to Eq. (46) is negligible, as long as $\hbar\omega_p\ll E_i$.

We can therefore simply cut off the integration range in the interval $-\hbar\omega_p<\Delta<\hbar\omega_p$, and use Eqs. (45a) and (45b). For larger Δ, i.e., small η, a good approximation is simply

$$I=1+2kT/\Delta \quad \text{if } \Delta>\hbar\omega_p$$

and

$$I=(1-2kT/\Delta)e^{\Delta/kT} \quad \text{if } \Delta<-\hbar\omega_p\,.$$

Collecting the various terms, we obtain:

$$-\frac{dE_i}{dx}=\frac{\pi M_i Z_i^2 e^4}{ME_i}\int_{\hbar\omega_p}^{\Delta_{\max}}(1-e^{-\Delta/kT})\left(1+\frac{2kT}{\Delta}\right)\frac{d\Delta}{\Delta}\,, \tag{48}$$

where $\Delta_{\max}\simeq E_i$ is the value of Δ for which the integration limits in Eq. (29) start moving appreciably toward each other. The integral in Eq. (48) can easily be evaluated[5] as

$$\ln\left(\frac{\Delta}{\omega}\right)+\frac{2(1-e^{-\omega})}{\omega}-\frac{2(1-e^{-\Delta})}{\Delta}+E_1(\omega)-E_1(\Delta)\,,$$

where

$$\Delta\equiv\Delta_{\max}/kT\,, \quad \text{and} \quad \omega\equiv\hbar\omega_p/kT\,.$$

For $\Delta_{\max}\gg kT\gg\hbar\omega_p$, which is the limit encountered in practice, this becomes approximately

$$1.422+\ln[kT\Delta_{\max}/(\hbar\omega_p)^2]-2kT/\Delta_{\max}\,, \tag{49}$$

which multiplies the coefficient in the right-hand side of Eq. (48).

Note that this result is valid only under the approximations mentioned above. If kT/E_i is not negligible, we must introduce a correction related to the fact that the upper integration limit in Eq. (A2) is not infinity, but

$$\frac{M\delta/2M_i}{[1-(1-\delta)^{1/2}]^2}\simeq\frac{4M}{\delta M_i}-\frac{M}{2M_i}\,.$$

It can easily be shown that the truncation of (A2) at the above value of z *reduces* expression (49) by terms of the order of

$$(ME_i/M_ikT)^{3/2}\exp(-ME_i/M_ikT)\,,$$

so that, when the temperature is high enough, $-dE_i/dx$ decreases and finally becomes negative, as expected.

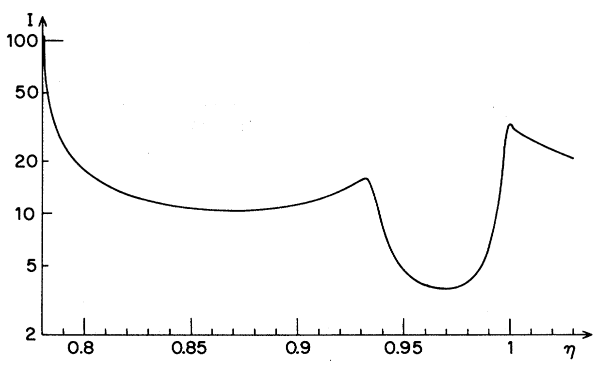

FIG. 3. This is a detail of Fig. 2, for $0.78<\eta<1.03$.

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APPENDIX A: ASYMPTOTIC LIMITS OF THE SCREENING FACTOR

In this Appendix, we shall derive the formulas which were listed in Sec. III. Let us define

$$c=\xi/2=\alpha/\eta, \quad z=x^2/c\,,$$

and

$$V(cz)=V(x^2)=-2x^2G(x)\,.$$

Note that c and z are negative if $\Delta<0$, i.e., if the incident ion gains energy during the collision. The properties of the function V are discussed in Appendix B. With these notations, we can write

$$I=\pm\int\frac{dz}{z(AF^2+B^2/A)}\,,$$

where

$$A=(\pi/cz^3)^{1/2}\exp[c(z+z^{-1}-2)]\,,$$

$$B=\tfrac{1}{2}\pi\eta^2e^{1-4c}\,,$$

$$F=1+\nu\eta z-\eta^2V(cz)\,,$$

and the $\pm$ sign is the same as that of Δ.

The integration limits, for $\delta=\Delta/E_i\ll1$ (this is the only interesting case, since for large-angle scattering the screening becomes negligible), are approximately $M\delta/4M_i$ and $4M/\delta M_i$. They can be taken as 0 and $\pm\infty$ for practical purposes, because extremely small or extremely large values of z make A so large that their contribution to the integral is negligible.

The asymptotic limits of I when η is very large or very small can be evaluated as follows:

Small energy transfer. When η is very large, the main contribution to the screening integral comes from small z, of the order of $1/\eta$. We then have $cz=\alpha\eta^{-2}\eta z \ll 1$, whence (see Appendix B) $V(cz)\simeq -2cz$. Therefore $F\simeq 1+(\nu+2\alpha)\eta z$. On the other hand

$$A \simeq \eta^2[\pi/\alpha(\eta z)^3]^{1/2}e^{\alpha/\eta z}.$$

It follows that $AF^2 \gg B^2/A$. Let us define

$$\gamma = 2\alpha+\nu$$

and $y=1/\eta z$. We obtain

$$I \simeq (\alpha/\pi)^{1/2}\eta^{-2}\int \frac{e^{-\alpha y}y^{-1/2}\,dy}{(y+\gamma)^2}\,.$$

The integral can be written as[6]

$$-\frac{\partial}{\partial\gamma}\int_0^\infty \frac{e^{-\alpha y}y^{-1/2}\,dy}{(y+\gamma)} = -\frac{\partial}{\partial\gamma}[\pi\gamma^{-1/2}e^{\gamma\alpha}\,\mathrm{erfc}(\gamma\alpha)^{1/2}]\,,$$

whence the first term of Eq. (43) follows.

When $|\eta|>1$ but is not *very* large, the main contribution to the screening integral comes from the points where $F=0$, namely, where

$$V(cz)=\eta^{-2}+(\nu/\alpha)cz\,. \tag{A1}$$

It is easily seen from Fig. 4 that there will be two such points, if $|\eta|>0.78$. In the vicinity of each one, we can write $F=F'(z-z_0)$ and replace all other z by z_0. We obtain for each point

$$\frac{1}{Az_0}\int \frac{dz}{[F'(z-z_0)]^2+(B/A)^2}\,.$$

If $B \ll A|F'z_0|$, we have a sharp "resonance," the contribution of which to I is $\pi/B|z_0F'|$. It can easily be shown that the second resonance ($cz_0>3.826$) is always sharp. For large $|\eta|$, and therefore large cz_0, its contribution can be evaluated as follows: We replace $V(cz)$ by its asymptotic value $1+3/2cz$ (see Appendix B), solve Eq. (A1) for cz, and compute $V'(cz)=-3/2c^2z^2$ at that point. The result is the second term in Eq. (43).

The first resonance, on the other hand, is sharp only if $|c|$ is large, i.e., if $\alpha \gg |\eta|$. If this condition holds, its contribution to I behaves as η^{-4} (for large η, it tends to $2.3438/\eta^4$).

Large energy transfer. When η is small (i.e., $\Delta \gg \hbar\omega_p$, but of course $\Delta \ll E_i$) we can neglect $B \ll AF$, unless $\alpha>\eta^{-3}$ (see below). The main contribution to the integral will then come from $z\simeq 1$ (especially if $c=\alpha/\eta$ is large), so that we can replace F by a constant:

$$F\simeq 1+\nu\eta-\eta^2V(\alpha/\eta)\simeq 1+\nu\eta-\eta^2\,.$$

We then obtain

$$I\simeq (c/\pi)^{1/2}F^{-2}e^{2c}\int_0^\infty z^{1/2}e^{-cz-c/z}\,dz\,, \tag{A2}$$

which can be solved exactly.[7] The result is given by Eqs. (45a) and (45b).

Intermediate case: $\alpha^{-1/3}<\eta<0.78$. When α is very large, there is a range of η where F cannot vanish and both terms in the denominator are of the same order of magnitude. Since c is large, the main contribution comes from $z\simeq 1$. We write $z=1+x$, whence $z+z^{-1}-2=x^2+O(x^3)$, and

$$F=(1+\nu\eta-\eta^2-3\eta^2/2c)+x(\nu\eta+3\eta^2/2c)+O(x^2)\,.$$

Unless

$$F_0=1+\nu\eta-\eta^2-3\eta^2/2c$$

is very small, we can neglect the other terms in F and write

$$I\simeq\int\frac{dx}{F_0^2(\pi/c)^{1/2}e^{cx^2}+\pi^2\eta^4/4(\pi/c)^{1/2}e^{cx^2}}\,,$$

$$=\frac{2}{\pi F_0(\alpha\eta^3)^{1/2}}\int\frac{dy}{e^{y^2}/R+R/e^{y^2}}\,,$$

where $y^2=cx^2$ and

$$R=(\pi\alpha\eta^3)^{1/2}/2F_0\,.$$

Note that R is large (since $\alpha\eta^3$ is large). We now write

$$y=(\ln R)^{1/2}+u/2(\ln R)^{1/2}\,,$$

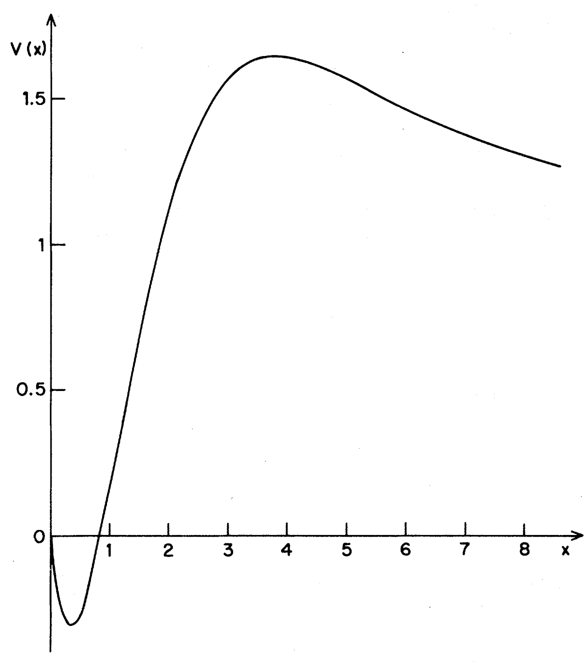

FIG. 4. Function $V(x)$ for $x<8.5$. For large x, see Eq. (B1).

whence

$$I = \frac{1}{2RF_0^2(\pi \ln R)^{1/2}} \times \int \frac{du}{\exp(u + u^2/4\ln R) + \exp(-u - u^2/4\ln R)} .$$

We see that the integrand is minimum when $u \simeq 0$ or when $u \simeq -4\ln R$. Both contributions are equal and the integral is approximately $\int du/\cosh u = \pi/2$. We thus obtain

$$I \simeq \frac{1}{F_0^2}\left(\frac{\pi/2}{(\pi\alpha\eta^3/4F_0^2)\ln(\pi\alpha\eta^3/4F_0^2)}\right)^{1/2} . \qquad \text{(A3)}$$

Singular case: $\alpha \gg 1$, $0.78 < \eta < 1$. When $0.78 < \eta < 1$, we have a situation similar to the intermediate case discussed above, but with some complicating features: (a) There are two points at which F vanishes, the contribution of which must be added to that of the region where $AF^2 + B^2/A$ is minimum. (b) When η is close to 1, then F_0 itself may vanish and we must consider the explicit dependence of F on x. (c) One of the zeros of F may coincide with a minimum of $AF^2 + B^2/A$. Then I becomes quite large (see Fig. 3). (d) For $\eta \simeq 0.78$, both zeros of F coincide and the value of I becomes exceedingly large.

The last feature is the only one readily amenable to an analytic treatment. In $F = 1 + \eta^2[(\nu/\alpha)cz - V(cz)]$, we neglect ν/α and write, near the maximum of V (see Fig. 4),

$$V(cz) = V_0 - V_2(cz - x_0)^2 .$$

The numerical values are $V_0 = 1.6452$, $V_2 = 0.8453$, $x_0 = 3.826$. Thus

$$F = \alpha^2 V_2(z - z_0)^2 + 1 - \eta^2 V_0 ,$$

where $z_0 = x_0/c$.

On the other hand, we can write

$$I = \frac{1}{B}\operatorname{Im}\int \frac{dz/z}{F - iB/A}$$

$$\simeq \frac{1}{Bz_0\alpha^2 V_2}\operatorname{Im}\int \frac{dz}{(z - z_0)^2 - \theta - i\beta} ,$$

where $\theta = (\eta^2 V_0 - 1)\alpha^2 V_2$ and $\beta = 2B/A\alpha^2 V_2$. Note that β is very small, because

$$A = (\pi/cz_0^3)^{1/2}\exp[c(z_0 + 1/z_0 - 2)]$$

is very large, because $c/z_0 = c^2/x_0$ is large.

The last integral has two poles at $z = z_0 \pm (\theta + i\beta)^{1/2}$ and we obtain

$$I \simeq \frac{\pi}{\alpha^2 V_2 B z_0}\operatorname{Re}(\theta + i\beta)^{-1/2}$$

$$= \frac{\pi}{\alpha^2 V_2 B z_0}\left[\frac{\theta + (\theta^2 + \beta^2)^{1/2}}{2(\theta^2 + \beta^2)}\right]^{1/2} , \qquad \text{(A4)}$$

which is maximum for $\theta = \beta/3^{1/2}$. The square bracket then is $[0.6459/\beta]^{1/2}$.

Note that when $|\theta| \gg \beta$, the behavior of I is quite different on both sides of the peak. For $\theta \gg \beta$, we have $[\,]^{1/2} \to \theta^{-1/2}$, while for $-\theta \gg \beta$, we have $[\,]^{1/2} \to \beta/2|\theta|^{3/2} \simeq 0$ (the main contribution to I does not come then from $\eta \sim V_0^{-1/2}$).

The width of the peak (defined by the values of θ for which I is one-half of its maximum) is about 6β. This corresponds to

$$\delta\eta = 3\beta\alpha^2 V_2/\eta = 3.25\beta\alpha^2 . \qquad \text{(A5)}$$

APPENDIX B: THE FUNCTION $V(x)$

We shall now derive the main properties of the function

$$V(x^2) = -2x^2 G(x) ,$$

where

$$G(x) = \pi^{-1/2} P\int_{-\infty}^{\infty} \frac{t\,dt\,e^{-t^2}}{t - x} ,$$

$$= 1 - \pi^{-1/2} x e^{-x^2}\int_{-\infty}^{\infty} dt\,e^{-t^2}(\sinh 2tx)/t .$$

The last expression is obtained by a shift of the origin and by keeping only the even part of the integrand. The result of the numerical integration is given in Fig. 4.

For small x, we have $G \simeq 1$ and therefore $V(x) \simeq -2x$. Further terms of a Taylor series can easily be obtained by expanding e^{-x^2} and $\sinh 2tx$ in powers of x.

For large x, an asymptotic series can be obtained as follows: We note that

$$(\sinh 2tx)/t = 2\int_0^x \cosh 2ty\,dy ,$$

whence

$$G(x) = 1 - 2\pi^{-1/2} x e^{-x^2}\int_0^x dy\int_{-\infty}^{\infty} e^{-t^2}\cosh 2ty\,dt$$

$$= 1 - 2xe^{-x^2}\int_0^x e^{y^2}\,dy ,$$

$$= 1 - \int_0^{2x^2} e^{-z}e^{z^2/4x^2}\,dz ,$$

where $z = 2x(x - y)$. Thus

$$V(x) = 1 - 2x\left(1 - \int_0^{2x} e^{-z}e^{z^2/4x}\,dz\right) .$$

We can now replace the upper limit of the integral by ∞ (the error is of order e^{-2x}) and expand $e^{z^2/4x}$ in power of x^{-1}. We obtain

$$V(x) = 1 + \frac{3}{2x} + \frac{15}{4x^2} + \frac{105}{8x^3} + \cdots . \qquad \text{(B1)}$$

The ratio of consecutive terms is $(2n+1)/2$.

APPENDIX C: THE DEBYE SCREENING LIMIT

In standard calculations,[8] the screened potential of a *fixed* charge is

$$\phi = \phi_c \exp(-\kappa_D r),$$

where ϕ_c is the pure Coulomb potential and where

$$\kappa_D^2 = 4\pi n e^2\left(\frac{1}{kT} + \frac{1}{kT_e}\right).$$

Note that the electrons are treated as nondegenerate. It follows that the scattering amplitude[9] becomes proportional to $(q^2 + \hbar^2\kappa_D^2)^{-1}$, rather than to q^{-2}. For a scatterer *initially at rest*, we can write $q^2 = 2M\Delta$, so that

$$I = (1 + \hbar^2\kappa_D^2/2M\Delta)^{-2} = (1 + \eta\gamma)^{-2},$$

where $\gamma = 2\alpha + \nu$. For large α or η, we have $I \to 1/\eta^2\gamma^2$, in agreement with Eq. (44b). However, the present derivation requires $kT \ll \alpha$ (i.e., $\eta \gg \alpha$), while (44b) was obtained under less restrictive assumptions.

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