

## Superradiant pulses and directed angular momentum states\*†

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We investigate the statistical properties of superradiant pulses emitted by a system of many atoms. By using a representation of the atomic density operator in terms of directed angular-momentum states (which are also known as atomic coherent states), we find that the statistical behavior of the atoms is formally equivalent to that of a harmonic oscillator subject to linear amplitude amplification and driven by Gaussian white noise. The emitted light pulses are then found to exhibit very large quantum fluctuations for atomic initial states corresponding to complete or nearly complete excitation. For all other atomic initial states the superradiant pulses show classical behavior.

## I. INTRODUCTION

It has been well known for quite some time that the emission of radiation by a system of many atoms can, under certain circumstances, lead to exceptionally bright radiation pulses for which the peak intensity is proportional to the square of the number of atoms.<sup>1-7</sup> Such pulses, which are said to be superradiant, have recently received considerable theoretical<sup>1-7</sup> and experimental<sup>8</sup> attention.

For radiation processes at very large wavelengths for which spontaneous emission is usually unimportant and which can therefore be treated classically, superradiance is quite a familiar phenomenon, too familiar indeed to warrant special consideration. However, radiation pulses at wavelengths in the infrared and visible parts of the spectrum are usually not superradiant. The cooperative behavior of the atoms necessary to build up a macroscopic polarization as the source for a macroscopic electric field tends to be destroyed by such effects as collision broadening of the atomic transition frequency, Doppler or crystal-field broadening, destructive interference due to the spatial separation of the atoms, and spontaneous emission. By a careful choice of the atomic system and of the shape of the volume containing the atoms, and by appropriate techniques of preparing the atoms in certain initial states of excitation, however, all of these obstacles can be overcome except for the effects of spontaneous emission. If, with these conditions met, the atoms initially exhibit a macroscopic polarization, they may radiate like a macroscopic dipole, i.e., superradiantly, even at short wavelengths. If, on the other hand, the atoms are initially fully excited and thus display no polarization at all, it is not immediately clear whether or not the emitted pulse will be superradiant. As we shall see, there is a super-

radiant pulse even then which, however, is highly random in character as a result of the highly random character of the spontaneous-emission events that initiate it.

In order to deal with as simple a problem as possible we consider  $N$  identical noninteracting two-level atoms. Since two amplitudes specify the state of a two-level atom, each of them is formally equivalent to a spin of magnitude  $\frac{1}{2}$ . The dynamics of the  $j$ th spin can be described in terms of the familiar vector operator  $\vec{\sigma}_j$ . The  $z$  components  $\sigma_{zj}$  of the spin operators  $\vec{\sigma}_j$  are related to the energy of the atomic system by

$$H_A = \frac{1}{2}N(E_a + E_b) + \frac{1}{2}(E_b - E_a) \sum_{j=1}^N \sigma_{zj}. \quad (1.1)$$

In this equation  $E_a$  and  $E_b$  refer to the energies of the atomic ground state and of the excited state, respectively. The  $x$  components  $\sigma_{xj}$  define the total polarization,

$$p = \frac{1}{2}d \sum_{j=1}^N \sigma_{xj}, \quad (1.2)$$

where  $d$  is the dipole moment of a single atom.

We then assume to have electric-dipole couplings to the electromagnetic field which are equivalent for all atoms, so that only the total polarization  $p$  of all atoms is coupled to the electric field  $E$  according to the Hamiltonian<sup>9</sup>

$$H = H_A + H_F - \frac{1}{2}dE \sum_{j=1}^N \sigma_{xj}, \quad (1.3)$$

where  $H_E$  is the free-field part of  $H$ . Note that the spin variables  $\vec{\sigma}_j$  enter the Hamiltonian only in terms of the total angular-momentum vector

$$\vec{J} = \frac{1}{2} \sum_{j=1}^N \vec{\sigma}_j. \quad (1.4)$$

The components  $J_z$  and  $J_x$  measure the total atomic energy according to Eq. (1.1) and total atomic polarization according to Eq. (1.2), respectively. Since the electric field  $E$  is the same for all atoms, the operator  $\vec{J}^2$  is conserved by the interaction with the field. The behavior of the atoms can therefore be described in terms of the motion of  $\vec{J}$  with the constraint of  $\vec{J}^2$  being fixed. In a classical picture,  $\vec{J}$  would move on a sphere. For a quantum-mechanical treatment, the density operator for the atomic system may be expressed in terms of the set of eigenstates of  $\vec{J}^2$  for a fixed eigenvalue  $j(j+1)$ . If this set of basis states were taken to be the familiar eigenstates of both  $\vec{J}^2$  and  $J_z$ , then the density operator would have a  $(2j+1) \times (2j+1)$  matrix representation. While it is indeed possible to discuss the behavior of the atomic system in terms of such a matrix representation,<sup>2</sup> we shall show that the use of a different basis leads to a simpler and physically more transparent description. The states we shall use as a basis are one which assign a particular direction to the vector  $\vec{J}$  with minimum uncertainty. In view of this property we shall call them the directed-angular-momentum states.<sup>10</sup> They have already been discussed under the names atomic coherent states and Bloch states in several references.<sup>11,12</sup> Since the direction of  $\vec{J}$  can be chosen arbitrarily, these states form a continuous basis and the rules for calculating with them are rather different from those for dealing with the familiar  $(2j+1)$ -fold basis. They are simple and elegant rules, however, and we shall present a self-contained introduction to them in Sec. II. A more detailed account of the states in question can be found in Ref. 11.

Let us think of the composite system of atoms plus field as described in terms of a certain density operator  $\rho_{AF}$ . If we fix our attention for the present on the atoms alone, then all that interests us is the atomic density operator

$$\rho_A = \text{tr}_F \rho_{AF}, \quad (1.5)$$

which is found by summing as indicated over the field variables. The motion of the atomic system thus considered is, of course, irreversible; it tends to lose energy irretrievably to the radiation field. Since  $\vec{J}^2$  is constant, the motion of the atomic system will correspond to a relaxation which will eventually leave the atoms in the state of lowest energy for the given value of  $j$ . In a classical picture, the vector  $\vec{J}$ , whatever its initial orientation, will move on the sphere  $\vec{J}^2 = j^2$  until it comes to rest parallel to the negative  $z$  axis.

The quantum-mechanical description of this relaxation has been shown<sup>2,4,5</sup> to lead to the following equation of motion for the density operator  $\rho_A$ :

$$\frac{d}{dt} \rho_A = \frac{1}{N\tau} ([J_-, \rho_A J_+] + [J_-, \rho_A, J_+]), \quad J_{\pm} = J_x \pm iJ_y. \quad (1.6)$$

In this equation, which has often been called the superradiance master equation, the constant  $\tau$  is the relaxation time for the motion of the vector  $\vec{J}$ . It is proportional to the natural radiative lifetime of the excited state of a single atom and inversely proportional to the number of atoms  $N$ .

The derivation of the master equation (1.6) assumes that the initial state of the field is the vacuum state and that the field radiated by the atoms escapes from the atomic system in a time short compared to the relaxation time  $\tau$ . Indeed, the master equation itself holds only for times  $t$  that are large compared to the radiation escape time.

While the density operator  $\rho_A$  which solves the superradiance master equation (1.6) provides an exhaustive description of the behavior of the radiating atoms, it does not contain any information about the equally interesting behavior of the radiation field. However, the very conditions securing the validity of the master equation, by requiring that the radiation escape time be much smaller than the atomic relaxation time, imply that the field follows the motion of the atoms instantaneously. In order to formulate this statement in a quantitative manner, we introduce the creation and annihilation operators  $a$  and  $a^\dagger$  of photons for the field mode carrying the light pulse. In terms of these operators, the adiabatic relation between atomic and field variables is<sup>2</sup>

$$\langle J_+^l J_-^{l'} \rangle_t = (\text{const}) \times \langle a^{+l} a^{l'} \rangle_t, \quad l, l' = 0, 1, 2, 3, \dots \quad (1.7)$$

The limits of validity of this relation are the same as those for the master equation (1.6). In particular, this adiabatic correspondence holds only for times  $t$  larger than the photon escape time. [If applied literally at  $t=0$ , Eq. (1.7) would contradict the assumption that the field begins in the vacuum state; what happens actually is that the field expectation value  $\langle a^{+l} a^{l'} \rangle_t$  increases from zero to the value given by Eq. (1.7) during a time of the order of the radiation escape time.] Since the field expectation values  $\langle a^{+l} a^{l'} \rangle_t$  for  $l, l' = 1, 2, 3, \dots$  completely specify the behavior of the field, the solution  $\rho_A(t)$  of the master equation (1.6) together with the correspondence law (1.7) provide an exhaustive description of the superradiant light pulse.

## II. DIRECTED ANGULAR-MOMENTUM STATES

The components of the angular-momentum vector  $\vec{J}$  obey the familiar commutation relations

$$[J_z, J_{\pm}] = \pm J_{\pm}, \quad [J_+, J_-] = 2J_z, \quad [\vec{J}^2, J_{\pm}] = 0, \quad (2.1)$$

with

$$J_{\pm} = J_x \pm i J_y,$$

$$\vec{J}^2 = J_x^2 + J_y^2 + J_z^2 = J_z^2 + J_+ J_- - J_z.$$

The well-known angular-momentum eigenstates  $|j, m\rangle$  are defined by

$$\vec{J}^2 |jm\rangle = j(j+1) |jm\rangle,$$

$$J_z |jm\rangle = m |jm\rangle, \tag{2.2}$$

$$J_{\pm} |jm\rangle = [(j \mp m)(j \pm m + 1)]^{1/2} |j, m \pm 1\rangle.$$

For any state  $|jm\rangle$ , the vector  $\vec{J}$  has the variance

$$\langle jm | \vec{J}^2 | jm \rangle - (\langle jm | \vec{J} | jm \rangle)^2 = j(j+1) - m^2. \tag{2.3}$$

The minimum value of this variance for fixed  $j$  is attained for  $m = \pm j$ . The states  $|j, \pm j\rangle$ , in which the angular momentum is aligned along the  $z$  axis, specify the direction of  $\vec{J}$  with minimum uncertainty. It is easily shown that no linear combination of the  $|jm\rangle$  can have a smaller dispersion for  $\vec{J}$ . There do exist, however, many linear combinations of the  $|jm\rangle$  with the same minimum dispersion. These are all simply rotated versions of the state  $|j, j\rangle$ .

We shall call any state which aligns the vector  $\vec{J}$  along a direction  $\vec{e}$  with minimum uncertainty a directed-angular-momentum state. We have seen the state  $|jj\rangle$  to be one particular such state. All other directed-angular-momentum states pertaining to the same eigenvalue of  $\vec{J}^2$  can evidently be generated from the state  $|jj\rangle$  by the unitary rotation operator<sup>11</sup>

$$U(\theta, \varphi) = \exp[i\theta(J_x \sin\varphi - J_y \cos\varphi)]$$

$$= \exp(\gamma J_-) \exp[-J_z \ln(1 + \gamma\gamma^*)] \exp(-\gamma J_+)$$

$$\equiv U(\gamma), \tag{2.4}$$

with  $\gamma = e^{i\varphi} \tan \frac{1}{2}\theta$ . The states so generated,

$$|j, \theta, \varphi\rangle \equiv |j, \gamma\rangle = U(\gamma) |j, j\rangle, \tag{2.5}$$

have the angular momentum oriented in the direction specified by the two angles of rotation,  $\theta$  and  $\varphi$  (see Fig. 1), or, equivalently, by the stereographic projection variable  $\gamma$  (see Fig. 2). While the angles of rotation locate a point on the unit sphere,  $\text{Re } \gamma$  and  $\text{Im } \gamma$  are the Cartesian coordinates of the stereographic projection of this point onto the plane tangent to the sphere in the point  $\theta = 0$ , the projection being taken from the pole opposite to the point of tangency, i.e., from the point  $\theta = \pi$ .

By using the commutation relations (2.1) we easily see that the directed-angular-momentum state  $|j\gamma\rangle$  is an eigenstate of

$$\vec{J} \cdot \vec{e} = U(\gamma) J_z U(\gamma)^{-1}$$

$$= J_z \cos\theta + J_y \sin\varphi \sin\theta + J_x \cos\varphi \sin\theta, \tag{2.6}$$

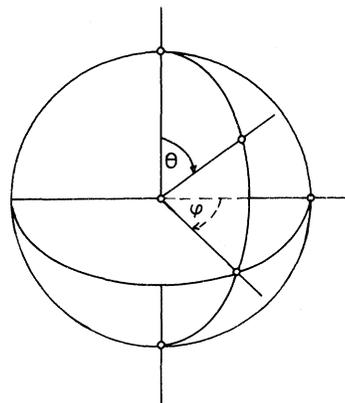


FIG. 1. Angles of rotation  $\theta$  and  $\varphi$ .

with eigenvalue  $j$ . The expansion of  $|j, \gamma\rangle$  in terms of the  $|j, m\rangle$  is obtained by expanding the exponentials in  $U$  in power series and reads

$$|j, \theta, \varphi\rangle = |j, \gamma\rangle = (1 + \gamma\gamma^*)^{-j} \sum_{\nu=0}^{2j} \gamma^{\nu} \left[ \binom{2j}{\nu} \right]^{1/2} |j, j - \nu\rangle. \tag{2.7}$$

From this expansion we find that two different directed-angular-momentum states are, in general, not orthogonal, but rather they have the scalar product

$$\langle j\alpha | j\gamma \rangle = \left( \frac{(1 + \alpha^*\gamma)^2}{(1 + \alpha\alpha^*)(1 + \gamma\gamma^*)} \right)^j. \tag{2.8}$$

It is useful to know explicitly the expectation values of products of angular-momentum operators in a state  $|j\gamma\rangle$ . By using the definition (2.5) and the commutation relations (2.1), we get

$$\langle j\gamma | J_+^l J_z^k J_-^{l'} | j\gamma \rangle$$

$$= (1 + \gamma\gamma^*)^{-2j} \frac{\partial^{l+l'}}{\partial \gamma^{*l} \partial \gamma^{l'}} \left( j - \gamma^* \frac{\partial}{\partial \gamma^*} \right)^k (1 + \gamma\gamma^*)^{2j}. \tag{2.9}$$

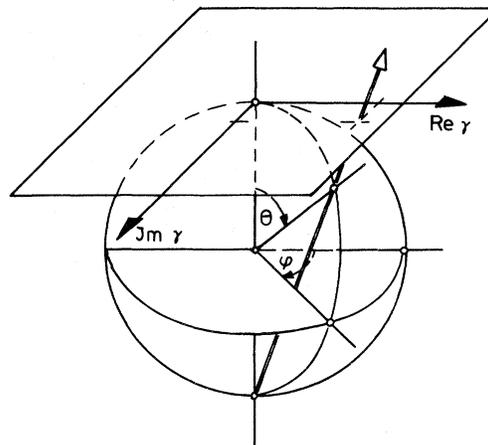


FIG. 2. Stereographic projection variable  $\gamma$ .

Simple and important special cases of this rather complicated expression are

$$\begin{aligned}\langle j\gamma|J_z|j\gamma\rangle &= j(1-\gamma\gamma^*)/(1+\gamma\gamma^*) = j \cos\theta, \\ \langle j\gamma|J_+|j\gamma\rangle &= \langle j\gamma|J_-|j\gamma\rangle^* \\ &= j[2\gamma/(1+\gamma\gamma^*)] = je^{i\varphi} \sin\theta.\end{aligned}\quad (2.10)$$

These formulas again express the geometrical meanings of the angles  $\theta$  and  $\varphi$ .

It is easy to see that the set of directed angular-momentum states for fixed  $j$  and  $0 \leq \theta \leq \pi$ ,  $0 \leq \varphi \leq 2\pi$  (or, equivalently,  $-\infty < \text{Re}\gamma, \text{Im}\gamma < +\infty$ ) form an overcomplete set and can thus be used as a basis. The overcompleteness is evident from the following resolution of the unity operator:

$$\int d^2\gamma |j\gamma\rangle\langle j\gamma| = \sum_{m=-j}^j |jm\rangle\langle jm| = 1, \quad (2.11)$$

where

$$\int d^2\gamma = \int_{-\infty}^{+\infty} d\text{Re}\gamma \int_{-\infty}^{+\infty} d\text{Im}\gamma.$$

This identity is most conveniently verified by inserting in the left-hand side the decomposition (2.7) of  $|j\gamma\rangle$  in terms of the  $|jm\rangle$  and carrying out the integral over the complex  $\gamma$  plane term by

term. Any eigenstate of  $\vec{J}^2$  with eigenvalue  $j(j+1)$  can now be expanded in terms of the directed-angular-momentum states. Moreover and more importantly, any mixture of the states  $|jm\rangle$  with fixed  $j$  can be represented as

$$M = \sum_{mn} M_{mn} |jm\rangle\langle jn| = \int d^2\gamma M(\gamma) |j\gamma\rangle\langle j\gamma|, \quad (2.12)$$

with a suitable chosen weight function  $M(\gamma)$ . Especially if the statistical behavior of the system is described by a density operator  $\rho(t)$ , which remains a mixture of states  $|jm\rangle$  with  $j$  fixed at all times, an equivalent and, as we shall see, more convenient description of the system is provided by a time-dependent weight function  $P(\gamma, t)$  according to

$$\rho(t) = \int d^2\gamma P(\gamma, t) |j\gamma\rangle\langle j\gamma|. \quad (2.13)$$

If the representation (2.13) is used, the expectation values of products of angular-momentum operators can be calculated as averages of the corresponding expectation values in a state  $|j\gamma\rangle$  over the complex variable  $\gamma$  with the weight  $P(\gamma t)$  as

$$\langle J_+^l J_z^k J_-^{l'} \rangle_t = \int d^2\gamma P(\gamma, t) \langle j\gamma | J_+^l J_z^k J_-^{l'} | j\gamma \rangle = \int d^2\gamma P(\gamma, t) (1+\gamma\gamma^*)^{-2j} \frac{\partial^{l+l'}}{\partial \gamma^{*l} \partial \gamma^{l'}} \left( j - \gamma^* \frac{\partial}{\partial \gamma^*} \right)^k (1+\gamma\gamma^*)^{2j}. \quad (2.14)$$

### III. SUPERRADIANCE MASTER EQUATION

We have already outlined in Sec. I that the super-radiant behavior of an atomic system as implied by the master equation (1.6) has the operator  $\vec{J}^2$  conserved. Although the components  $J_x$  and  $J_z$  of the vector  $\vec{J}$  refer to the atomic polarization and to the atomic energy, respectively, the vector  $\vec{J} = (J_x, J_y, J_z)$  formally is an angular-momentum operator. Therefore, if the initial atomic density operator  $\rho_A(t=0)$  is a mixture of states all of which are eigenstates of  $\vec{J}^2$  with one and the same total angular-momentum quantum number  $j$ , then the time-dependent atomic density operator  $\rho_A(t)$  can be given the representation (2.13) at all times. The master equation (1.6) for  $\rho_A(t)$  then implies an equation of motion for the weight function  $P(\gamma t)$  which we shall now construct. The derivation is formulated most conveniently by using a

differently normalized version of the directed-angular-momentum states  $|j\gamma\rangle$ . By introducing the states

$$||\gamma\rangle = (1+\gamma\gamma^*)^j |j\gamma\rangle = e^{\gamma J_-} |j, j\rangle, \quad (3.1)$$

we may write the representation (2.13) of  $\rho_A(t)$  as

$$\rho_A(t) = \int d^2\gamma P(\gamma t) (1+\gamma\gamma^*)^{-2j} ||\gamma\rangle\langle\gamma||. \quad (3.2)$$

The convenience of the states  $||\gamma\rangle$  is that, as we may see from Eqs. (2.4) and (2.5), the operation of  $J_{\pm}$  on them is given simply by

$$J_- ||\gamma\rangle = \frac{\partial}{\partial \gamma} ||\gamma\rangle, \quad J_+ ||\gamma\rangle = \gamma \left( 2j - \gamma \frac{\partial}{\partial \gamma} \right) ||\gamma\rangle. \quad (3.3)$$

Equipped with these properties of the states  $||\gamma\rangle$ , we find that when the representation (3.2) is inserted in the master equation (1.6) the latter becomes, with<sup>13</sup>  $N=2j$ ,

$$\begin{aligned}\dot{\rho}_A(t) &= \int d^2\gamma \dot{P}(\gamma, t) (1+\gamma\gamma^*)^{-N} ||\gamma\rangle\langle\gamma|| = \frac{1}{N\tau} \int d^2\gamma P(\gamma, t) (1+\gamma\gamma^*)^{-N} (2J_- ||\gamma\rangle\langle\gamma|| J_+ - J_+ J_- ||\gamma\rangle\langle\gamma|| - ||\gamma\rangle\langle\gamma|| J_+ J_-) \\ &= \frac{1}{N\tau} \int d^2\gamma P(\gamma t) (1+\gamma\gamma^*)^{-N} \left( \frac{\partial^2}{\partial \gamma \partial \gamma^*} + \frac{\partial^2}{\partial \gamma^2} - (N+2) \frac{\partial}{\partial \gamma} \gamma + \text{c.c.} \right) ||\gamma\rangle\langle\gamma||.\end{aligned}\quad (3.4)$$

In the latter expression for the right-hand side the integrations may be carried out by parts so that the operations of differentiation with respect to  $\gamma$  and  $\gamma^*$  are transferred to the function  $P(\gamma t)$ . The master equation then takes the form

$$\int d^2\gamma \dot{P}(\gamma t) |j\gamma\rangle\langle j\gamma| = \int d^2\gamma |j\gamma\rangle\langle j\gamma| \frac{1}{N\tau} \left( -(N+2) \frac{\partial}{\partial \gamma} \gamma + \frac{\partial^2}{\partial \gamma^2} \gamma^2 + \frac{\partial^2}{\partial \gamma \partial \gamma^*} + \text{c.c.} \right) P(\gamma, t). \quad (3.5)$$

A sufficient condition for Eq. (3.5) to hold is that the weight function  $P(\gamma, t)$  obey the partial differential equation<sup>14</sup>

$$\dot{P}(\gamma, t) = \frac{1}{N\tau} \left( -(N+2) \frac{\partial}{\partial \gamma} \gamma + \frac{\partial^2}{\partial \gamma \partial \gamma^*} + \frac{\partial^2}{\partial \gamma^2} \gamma^2 + \text{c.c.} \right) \times P(\gamma, t). \quad (3.6)$$

Once we have solved Eq. (3.6) subject to appropriate initial conditions, we have in effect solved the master equation. We can then find the statistical properties of the superradiant atoms by using the representation (2.14), which will reduce the problem of evaluating operator averages to one of integration.

#### IV. ASYMPTOTIC BEHAVIOR OF THE WEIGHT FUNCTION

If the second-order derivative terms in Eq. (3.6) were neglected, which is not in general permissible, the resulting first-order equation would be essentially classical in character. It would simply describe the drift of the weight function  $P(\gamma, t)$  away from the origin in the stereographic plane, which corresponds to the classical descent of the vector  $\vec{J}$  to the pole  $\theta = \pi$ . This classical approximation to the radiative relaxation process has been discussed by several authors.<sup>2,3,12,15,16</sup> One characteristic of the classical approximation is that it neglects the development of dispersion for  $\vec{J}$  during the relaxation and therefore also the generation of noise in the radiated light pulse. It is the second-order derivative terms in Eq. (3.6) which take into account the effects of quantum fluctuations and thereby induce a dispersion of  $\vec{J}$ . We shall see that these effects are of considerable importance for certain initial states.

Rather than construct a formally exact solution<sup>12</sup> to Eq. (3.6), we shall find it more economical and interesting to discuss the asymptotic properties of the solution as the number of atoms,  $N$ , becomes large. In doing this we shall show that two of the terms of Eq. (3.6), the terms

$$\frac{1}{N\tau} \left( \frac{\partial^2}{\partial \gamma^2} \gamma^2 + \frac{\partial^2}{\partial \gamma \partial \gamma^*} \gamma^{*2} \right) P, \quad (4.1)$$

are asymptotically small in their effect and may be dropped. The remaining second-order derivative term,

$$\frac{2}{N\tau} \frac{\partial^2}{\partial \gamma \partial \gamma^*} P, \quad (4.2)$$

is then entirely responsible for the description of quantum fluctuations.

In order to identify the contributions of the terms (4.1), it is useful to introduce a factor  $\lambda$  in their coefficient in the differential equation which then reads, for  $N \gg 1$ ,

$$\frac{d}{dt} P(\gamma, t) = (L_0 + \lambda L_1) P(\gamma, t), \quad \lambda = 1, \quad (4.3)$$

with

$$L_0 = \frac{1}{\tau} \left( -\frac{\partial}{\partial \gamma} \gamma - \frac{\partial}{\partial \gamma^*} \gamma^* + \frac{2}{N} \frac{\partial^2}{\partial \gamma \partial \gamma^*} \right), \quad (4.4)$$

$$L_1 = \frac{1}{N\tau} \left( \frac{\partial^2}{\partial \gamma^2} \gamma^2 + \frac{\partial^2}{\partial \gamma^* \partial \gamma^*} \gamma^{*2} \right).$$

Let us now consider the time-dependent behavior of the moments

$$\langle \gamma^n \gamma^{*m} \rangle_t = \int d^2\gamma \gamma^n \gamma^{*m} P(\gamma, t), \quad n, m = 0, \pm 1, \pm 2, \dots \quad (4.5)$$

of the distribution function  $P(\gamma, t)$ . By using Eq. (4.3) for  $P$  we find that these moments obey the coupled set of differential equations

$$\tau \frac{d}{dt} \langle \gamma^n \gamma^{*m} \rangle_t = \left( n + m + \frac{\lambda}{N} [n(n-1) + m(m-1)] \right) \langle \gamma^n \gamma^{*m} \rangle_t + \frac{2nm}{N} \langle \gamma^{n-1} \gamma^{*m-1} \rangle_t, \quad \lambda = 1. \quad (4.6)$$

We shall show by considering some specific examples that the effect of the terms proportional to  $\lambda$  in these equations is negligible in the limit  $N \rightarrow \infty$ , provided we limit our consideration to  $|n|$  and  $|m|$  sufficiently small compared to  $N$  and to times  $t \ll N\tau$ . Since the duration of the superradiant pulse is always much smaller than  $N\tau$ , the latter restriction is not a significant one.

The simplest averages to evaluate are those with either  $n$  or  $m$  equal to zero. If  $m = 0$ , for example, we integrate Eq. (4.6) immediately to find

$$\langle \gamma^n \rangle_t = \langle \gamma^n \rangle_0 \exp\left\{ \left( \frac{t}{\tau} \right) [n + \lambda n(n-1)/N] \right\}. \quad (4.7)$$

The differential equation for averages of the form

$\langle \gamma^n \gamma^{*m} \rangle_t$  is then seen to contain the solution (4.7) as an inhomogeneous term. It is likewise easily solved. By proceeding recursively we can evidently go on to construct all averages of the form  $\langle \gamma^n \gamma^{*m} \rangle_t$  provided either  $n$  or  $m$  is non-negative. A particularly useful result is the variance

$$\langle \gamma \gamma^* \rangle_t - |\langle \gamma \rangle_t|^2 = e^{2t/\tau} (\langle \gamma \gamma^* \rangle_0 - |\langle \gamma \rangle_0|^2) - 1/N. \quad (4.8)$$

The averages (4.7) and (4.8) clearly illustrate the statement made earlier that the terms containing  $\lambda$  are negligible. The same property holds for all of the moments (4.5) evaluated by means of Eq. (4.6). Inasmuch as the physically interesting properties of the superradiant system are determined by finite-order moments of  $P$ , we can drop the term  $L_1$  from the equation of motion (4.3) and construct the weight function  $P(\gamma, t)$  from the much simpler equation

$$\tau \frac{dP}{dt} = \left( -\frac{\partial}{\partial \gamma} \gamma - \frac{\partial}{\partial \gamma^*} \gamma^* + \frac{2}{N} \frac{\partial^2}{\partial \gamma \partial \gamma^*} \right) P. \quad (4.9)$$

This equation implies a remarkably simple physical picture for the statistical behavior of the atomic system. The variable  $\gamma$  behaves like the complex amplitude of a harmonic oscillator subject to linear amplitude amplification and driven by Gaussian white noise. Equation (4.9) differs from the well-known Fokker-Planck equation for a damped harmonic oscillator<sup>17,18</sup> by the sign of the first-order derivative terms. This change of sign turns the damping into an amplification. The solution  $P(\gamma, t | \alpha)$ , which obeys the initial condition

$$P(\gamma, 0 | \alpha) = \delta^{(2)}(\gamma - \alpha), \quad (4.10)$$

is readily verified to have the following Gaussian form:

$$P(\gamma, t | \alpha) = [\pi(e^{2t/\tau} - 1)/N]^{-1} \times \exp\{-|\gamma - \alpha e^{t/\tau}|^2 [(e^{2t/\tau} - 1)/N]^{-1}\}. \quad (4.11)$$

With the help of this special solution we find the solution  $P(\gamma, t)$ , which obeys the more general initial condition

$$P(\gamma, 0) = P_0(\gamma), \quad (4.12)$$

as

$$P(\gamma, t) = \int d^2 \alpha P(\gamma, t | \alpha) P_0(\alpha). \quad (4.13)$$

We see that the weight function  $P(\gamma, t | \alpha)$  drifts, starting from its initial location  $\gamma = \alpha$ , toward infinity in the complex  $\gamma$  plane. This drift is accompanied by an increase of the width of the Gaussian peak. The width is the limit of the variance (4.8) of  $\gamma$  for large  $N$ . It is zero initially and increases

indefinitely as  $t \rightarrow \infty$ . This behavior of the weight function, when projected on the sphere of constant  $\vec{J}^2$ , describes the relaxation of the vector  $\vec{J}$  to the ground state  $\theta = \pi$ .

## V. ASYMPTOTIC MOMENTS OF THE INTENSITY

The evaluation of the radiated intensity and its higher-order moments requires us, according to the adiabatic correspondence (1.7), to calculate the angular-momentum expectation values  $\langle J_+^l J_-^l \rangle_t$  for  $l=1, 2, 3, \dots$ . The latter is indeed proportional to the  $l$ th-order moment of the intensity. The appearance of a superradiant pulse manifests itself in the fact that this moment displays, at least during a time interval near the peak of the pulse, a proportionality to  $N^{2l}$ . Any behavior which is not fully superradiant, on the other hand, would imply that the  $l$ th intensity moment increases with a power of  $N$  less than  $2l$ . We might expect non-superradiant behavior to prevail in the wings of the pulse away from its peak. The asymptotic weight function (4.11) or (4.13) is especially well suited for an investigation of the superradiant part of the pulse.

We obtain the  $l$ th-order moment of the intensity, according to Eq. (2.14), by averaging

$$\langle j\gamma | J_+^l J_-^l | j\gamma \rangle = (1 + \gamma \gamma^*)^{-N} \frac{\partial^{2l}}{\partial \gamma^l \partial \gamma^{*l}} (1 + \gamma \gamma^*)^N \quad (5.1)$$

over  $\gamma$  with the asymptotic weight function (4.11) or (4.13). Insofar as we are mainly interested in the behavior of the pulse while it is superradiant, before doing this average we may also simplify the expression (5.1), by evaluating it asymptotically as  $N \rightarrow \infty$ . By carrying out the differentiations, we see that the right-hand side in Eq. (5.1) is a polynomial of order  $2l$  in  $N$  with coefficients depending on  $\gamma$  and  $\gamma^*$ . Asymptotically, for  $N \rightarrow \infty$ , the polynomial can be replaced by its monomial of order  $2l$ , provided that the modulus of  $\gamma$  is neither infinite nor vanishingly small. The asymptotic form of Eq. (5.1) is easily constructed by noting that each of the  $2l$  differentiations has to generate an explicit factor  $N$  if a term  $N^{2l}$  is to be produced. We find, by using

$$N!/(N-2l)! \rightarrow N^{2l} \text{ as } N \rightarrow \infty, \quad (5.2)$$

$$\begin{aligned} \langle j\gamma | J_+^l J_-^l | j\gamma \rangle &= \left( \frac{N\gamma}{1 + \gamma \gamma^*} \right)^l \left( \frac{N\gamma^*}{1 + \gamma \gamma^*} \right)^l \\ &= \left( \frac{1}{2} N e^{i\vartheta} \sin\theta \right)^l \left( \frac{1}{2} N e^{-i\vartheta} \sin\theta \right)^l \\ &= (\langle j\gamma | J_+ | j\gamma \rangle)^l (\langle j\gamma | J_- | j\gamma \rangle)^l. \end{aligned} \quad (5.3)$$

This asymptotic approximation is not without limits of validity, however. We emphasize that it is correct only if the complex amplitude  $\gamma$  is suffi-

ciently far away from both zero and infinity in the  $\gamma$  plane. For directed angular-momentum states with an amplitude  $\gamma$  asymptotically close to either zero or infinity, i.e., for states asymptotically close to either the fully excited state or the ground state, the right-hand sides in Eq. (5.3) vanish; the left-hand side, however, need not vanish but rather is of lower than  $2l$ th order in  $N$ .

Let us define the normalized  $l$ th moment of the intensity as

$$M_l(t) = \langle J_+^l J_-^l \rangle_t / (\frac{1}{2}N)^{2l}. \quad (5.4)$$

Since we provided  $M_l(t)$  with an explicit factor  $N^{-2l}$ , it will be of order unity during the superradiant part of the radiated pulse. Its asymptotic form as  $N \rightarrow \infty$  is obtained by averaging Eq. (5.3) over  $\gamma$  with the weight function  $P$  as given by (4.11) or (4.13). If the atomic initial state is a directed angular-momentum state  $|j = \frac{1}{2}N, \alpha\rangle$ , we have to use the weight function given by Eq. (4.11), and obtain

$$M_l(\alpha, t) = \int d^2\gamma \left( \frac{2|\gamma|}{1+|\gamma|^2} \right)^{2l} \left( \frac{\pi(e^{2t/\tau} - 1)}{N} \right)^{-1} \\ \times \exp[-|\gamma - \alpha e^{t/\tau}|^2 [(e^{2t/\tau} - 1)/N]^{-1}]. \quad (5.5)$$

This expression is valid for initial amplitudes  $\alpha$  and times  $t$  such that the Gaussian weight function  $P(\gamma t | \alpha)$  occurring in it does not place all of its weight on points too close to either  $\gamma = 0$  or  $\gamma = \infty$  in the  $\gamma$  plane. The superradiant part of the pulse which, as we shall see, occurs while  $P(\gamma t | \alpha)$  concentrates its weight near  $|\gamma| = 0(1)$ , is correctly described by Eq. (5.5). In the nonsuperradiant wings of the field pulse the intensity is quite low. The expression (5.5) is not intended to approximate the intensity and its moments in the nonsuperradiant wings of the pulse, i.e., at very short and very large times.<sup>16</sup>

If the atoms are initially prepared in a mixture of directed angular-momentum states  $|j = \frac{1}{2}N, \alpha\rangle$  with weight  $P_0(\alpha)$ , the time-dependent intensity moments (5.4) are obtained, according to Eq. (4.13), by averaging the expression (5.5) over  $\alpha$  with the weight  $P_0(\alpha)$ .

## VI. STATISTICAL BEHAVIOR OF SUPERRADIANT PULSES

We may expect initial states corresponding to complete or nearly complete atomic excitation to give rise to the most interesting statistical properties of the emitted light pulse. Since nearly fully excited atoms display no or almost no electric polarization, their radiative behavior possesses no classical analog. Indeed, according to the classical theory of radiation, fully excited atoms

are in a state of unstable equilibrium and do not radiate at all. Obviously, therefore, quantum effects play a decisive role in the generation of superradiant light pulses originating from such initial states.<sup>12,15,19</sup>

In evaluating the normalized intensity moments according to Eq. (5.4) for initial states corresponding to an asymptotically small electric polarization, it is convenient to parametrize these states by introducing the variables  $\Delta$  and  $\psi$  by

$$\alpha = e^{i\psi} (\Delta/N)^{1/2}, \quad \Delta = N |\alpha|^2. \quad (6.1)$$

If we choose  $\Delta$  such that  $\Delta/N \rightarrow 0$  as  $N \rightarrow \infty$ , the directed angular-momentum state  $|j = \frac{1}{2}N, \alpha = e^{i\psi} (\Delta/N)^{1/2}\rangle$  corresponds, as is obvious from Eq. (2.10), to the initial atomic inversion

$$\langle j, \alpha | J_z | j, \alpha \rangle = \frac{1}{2}N - \Delta \quad (6.2)$$

and to the initial atomic polarization

$$|\langle j, \alpha | J_{\pm} | j, \alpha \rangle| = (N\Delta)^{1/2}. \quad (6.3)$$

We see that the parameter  $\Delta$  measures the deviation of  $\langle J_z \rangle$  from full inversion and that the initial polarization is asymptotically small relative to the total number of atoms.

In order to cast the expression (5.5) for the intensity moments into a form that is convenient for further evaluation, we note that it predicts the maximum intensity for times where

$$|\gamma| \approx |\alpha| e^{t/\tau} \approx 1. \quad (6.4)$$

This estimate follows from the fact that the integral in Eq. (5.5) draws its most important contributions from the neighborhood of the maximum of the integrand with respect to the modulus of  $\gamma$ . This maximum is constrained to occur near  $|\gamma| = |\alpha| e^{t/\tau}$  by the Gaussian weight function  $P(\gamma, t | \alpha)$  and near  $|\gamma| = 1$  by the expression (5.3) for the expectation value  $\langle j\gamma | J_+^l J_-^l | j\gamma \rangle$  which multiplies  $P(\gamma, t | \alpha)$  in the integrand in Eq. (5.5). By inserting the representation (6.1) for the initial-state amplitude  $\alpha$  into the estimate (6.4), we see that the intensity of the pulse will be large for times where

$$Ne^{-2t/\tau} \approx \Delta. \quad (6.5)$$

It is therefore suggestive to use as a time variable the parameter  $z$ , defined as

$$z = Ne^{-2t/\tau}. \quad (6.6)$$

Obviously,  $z$  will be close to  $\Delta$  while the intensity is large.

We now eliminate  $\alpha$  and  $t$  from Eq. (5.5) in favor of the variables  $\Delta$  and  $z$  and find for the normalized intensity moments

$$M_i(\alpha = e^{i\psi}(\Delta/N)^{1/2}, t = \frac{1}{2}\tau \ln(N/z) \equiv M_i(\Delta, z) = \frac{z}{\pi} \left(1 - \frac{z}{N}\right)^{-1} \int d^2\gamma \left(\frac{2|\gamma|}{1+|\gamma|^2}\right)^{2i} \exp\left[-z\left|\gamma - e^{i\psi}\left(\frac{\Delta}{z}\right)^{1/2}\right|^2 \left(1 - \frac{z}{N}\right)^{-1}\right]. \quad (6.7)$$

For times near the peak of the intensity, we have

$$z/N \approx \Delta/N \approx 1, \quad (6.8)$$

so that Eq. (6.7) simplifies asymptotically, as  $N \rightarrow \infty$ , to read

$$M_i(\Delta, z) = \frac{\pi}{z} \int d^2\gamma \left(\frac{2|\gamma|}{1+|\gamma|^2}\right)^{2i} \times \exp\left[-z\left|\gamma - e^{i\psi}\left(\frac{\Delta}{z}\right)^{1/2}\right|^2\right]. \quad (6.9)$$

The integral over the complex  $\gamma$  plane is most easily evaluated by using polar coordinates,

$$\gamma = |\gamma| e^{i\psi}. \quad (6.10)$$

By invoking the well-known integral representation for the Bessel function of imaginary argument and order zero,<sup>20</sup>

$$I_0(x) = \frac{1}{\pi} \int_0^\pi d\varphi e^{ix \cos \varphi}, \quad (6.11)$$

we can carry out the integral over the phase  $\varphi$  of  $\gamma$  explicitly. The remaining integral over the modulus of  $\gamma$  assumes an especially simple form if we use the integration variable  $y$ , defined as

$$y = (1 + |\gamma|^2)z. \quad (6.12)$$

We find, as our final asymptotic formula for the normalized intensity moments for the case of nearly complete initial atomic excitation,

$$M_i(\Delta, z) = z^i e^z e^{-\Delta} \int_z^\infty dy (y-z)^i y^{-2i} e^{-y} \times I_0(2[\Delta(y-z)]^{1/2}). \quad (6.13)$$

The most important and interesting property of the emitted light pulse is now immediately obvious from this expression. Since the number of atoms,  $N$ , does not occur explicitly in Eq. (6.13), the normalized intensity moments will be of order unity in the limit as  $N \rightarrow \infty$  at least for times near the peak of the pulse, i.e., for  $z \approx \Delta$ . Therefore, the non-normalized intensity moments  $\langle a^{+i} a^i \rangle \sim N^{2i} M_i$  will be of order  $N^{2i}$ . In other words, even from completely or nearly completely excited atomic initial states for which the polarization is zero or asymptotically small, a superradiant pulse will eventually be generated. Furthermore, the shapes of the moments  $M_i$  as functions of the rescaled time are independent of the number of atoms. Let us also note that the moments  $M_i(\Delta, z)$  are independent of the phase  $\varphi$  of the initial state amplitude  $\alpha$ .

For the fully excited atomic initial state, i.e., for  $\Delta=0$ , the result (6.13) reduces to a form conjectured by Degiorgio.<sup>19</sup> For  $\Delta \neq 0$  it is equivalent to a previous result of ours obtained for the states  $|j, m\rangle$  with  $m=j, j-1, j-2, \dots$  as atomic initial states.<sup>15</sup> This equivalence will be demonstrated in the Appendix.

We have numerically integrated the integral in Eq. (6.13) for various initial-state parameters  $\Delta$ . The normalized intensity  $M_1(\Delta, z)$  so obtained is, for all  $\Delta$  with  $\Delta/N \rightarrow 0$  as  $N \rightarrow \infty$ , a bell-shaped function of the time  $z$  with a superradiant maximum value of order unity. This maximum occurs, in agreement with the estimate (6.5), for  $z = z_{\max} \approx \Delta$  corresponding to the real time

$$t = t_{\max} \approx \tau \ln(N/\Delta). \quad (6.14)$$

The precise dependence of  $z_{\max}$  on  $\Delta$  is plotted in Fig. 3. The half-width of the intensity as a function of  $t$  turns out to be insensitive to the initial-state parameter  $\Delta$  and to be close to the relaxation time  $\tau$ . Note that the time  $t_{\max}$  of maximum intensity is appreciably larger than the width  $\tau$  of the superradiant pulse. The normalized maximum intensity  $M_1(\Delta, z_{\max})$  is plotted in Fig. 4. It is a monotonically increasing function of  $\Delta$  and approaches unity as  $\Delta$  increases. Since  $|\langle J_x \rangle| = \frac{1}{2}N$  is the maximum value the atomic polarization can assume according to a classical treatment of the relaxation of the vector  $\vec{J}$  with  $\vec{J}^2$  fixed,  $M_1 = 1$  is the maximum value of the normalized intensity predicted by a classical theory. The numerical result for  $M_1(\Delta, z_{\max})$

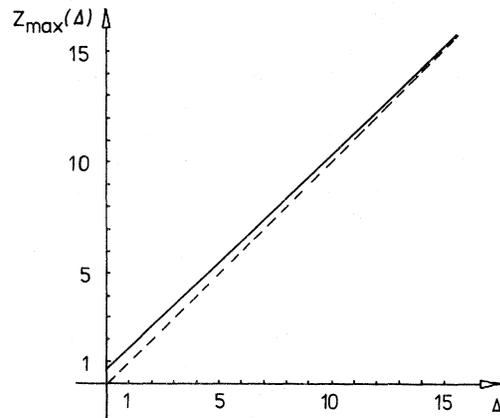


FIG. 3. Time  $z_{\max}(\Delta)$  of maximum intensity as a function of the deviation  $\Delta$  of the initial inversion  $J_z(0)$  from total inversion  $\frac{1}{2}N$ ; the dashed line shows the classical result  $z_{\max, \text{class}}(\Delta) = \Delta$  [from Eq. (6.20)].

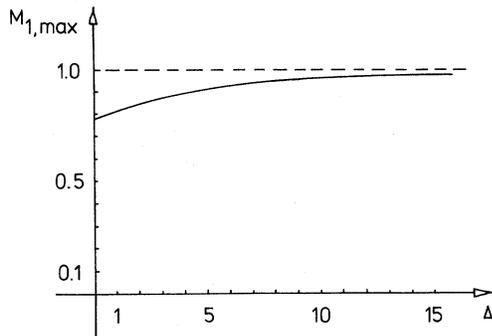


FIG. 4. Maximum intensity  $M_1(\Delta, z_{\max}) \equiv M_{1, \max}$ ; the dashed line gives the classical result  $M_{1, \max} = 1$ .

thus shows that the superradiant system behaves quite nonclassically for nearly completely excited atomic initial states for which  $\Delta$  is close to zero. However, as soon as the initial-state parameter  $\Delta$  becomes large compared to unity, the quantum-mechanical result becomes practically indistinguishable from the classical result.

As a more direct measure of quantum fluctuations in the pulse we have calculated the dispersion of the intensity at the time of maximum intensity,

$$\sigma(\Delta) \equiv \frac{M_2(\Delta, z_{\max}) - M_1(\Delta, z_{\max})^2}{M_1(\Delta, z_{\max})^2}, \quad (6.15)$$

as a function of  $\Delta$ . The result is plotted in Fig. 5. The dispersion  $\sigma(\Delta)$  takes on its largest value for  $\Delta=0$ , i.e., for the fully excited initial state, and then decreases monotonically and rapidly to zero as  $\Delta$  increases. We see that the superradiant pulse exhibits large quantum fluctuations for nearly fully excited initial states while such fluctuations are unimportant for initial states with  $\Delta$  large compared to unity. The appearance of large fluctuations for the case of small  $\Delta$  is, of course, due to the initial spontaneous-emission events which trigger the relaxation of the angular-momentum vector  $\vec{J}$  away from the (classically unstable equilibrium) orientation parallel to the positive  $z$  direction. Since the initial spontaneous-emission events are completely uncorrelated among one another, the pulses triggered by them may be expected, and indeed turn out to be, roughly amplified noise. It is somewhat surprising, however, that even an asymptotically small initial deviation from full excitation of the atoms corresponding to  $\Delta$  large compared to unity but asymptotically small compared to  $N$  suffices to lead to superradiant pulses behaving fully classically.

The classical behavior of the pulses emerging from systems initially prepared in a state with large  $\Delta$ , which we have inferred from the numerical evaluation of the integral in Eq. (6.13) just

discussed, may even be described in an analytically closed form. To do this we could consider the integrals in Eq. (6.13) in the limit as  $\Delta \rightarrow \infty$  and show that, in this limit,

$$M_i(\Delta, z) = M_1(\Delta, z)^i \text{ as } \Delta \rightarrow \infty. \quad (6.16)$$

However, it is simpler and more instructive to discuss the behavior of the  $M_i$  for  $\Delta = |\alpha|^2/N$  large by working with the expression (5.5). The latter will evidently imply classical behavior, i.e., Eq. (6.16), when the width of the Gaussian weight function  $P(\gamma, t | \alpha)$  is negligible. In order to find the conditions under which  $P(\gamma, t | \alpha)$  is effectively equivalent to a  $\delta$  function with respect to the moment  $M_i(\alpha, t)$ , we expand the function of  $|\gamma|$ , multiplying the Gaussian in the integrand in Eq. (5.5) around the point  $\gamma = \alpha e^{t/\tau}$  at which the Gaussian peaks. The resulting expansion of the integral has as its expansion parameter the ratio of the width of the Gaussian to the squared modulus of the amplitude  $\alpha e^{t/\tau}$ ,

$$\frac{e^{2t/\tau} - 1}{N} \frac{e^{-2t/\tau}}{|\alpha|^2} = \frac{1 - e^{-2t/\tau}}{N|\alpha|^2}. \quad (6.17)$$

When this ratio is small compared to unity, i.e., if

$$N|\alpha|^2 = \Delta \gg 1, \quad (6.18)$$

the series for  $M_i(\alpha, t)$  is well approximated by its first term,

$$M_i(\alpha, t) = \left( \frac{|\alpha| e^{t/\tau}}{1 + |\alpha|^2 e^{2t/\tau}} \right)^{2i}. \quad (6.19)$$

This result evidently has the classical property (6.16). It is easily rewritten in the familiar form<sup>2</sup>

$$\begin{aligned} M_1(\alpha, t) &= \text{sech}^2[(t - t_{\max})/\tau], \\ t_{\max} &= -\tau \ln |\alpha| = \tau \ln(N/\Delta), \\ M_i(\alpha, t) &= M_1(\alpha, t)^i. \end{aligned} \quad (6.20)$$

Equation (6.20) are the well-known results of the classical theory of superradiance<sup>2,3</sup> and hold, according to the condition (6.18), when the deviation of the initial atomic inversion  $\langle J_{z0} \rangle$  from full inver-

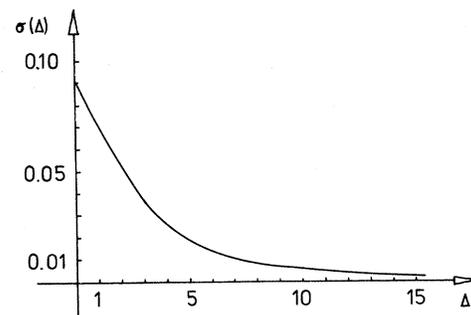


FIG. 5. Dispersion of the intensity  $\sigma(\Delta)$  at the time  $z_{\max}$  of maximum intensity.

sion  $\frac{1}{2}N$  is large compared to unity.

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#### APPENDIX

For angular-momentum states  $|j = \frac{1}{2}N, m = \frac{1}{2}N - \nu\rangle$ , with  $\nu = 0, 1, 2, \dots$  as atomic initial states, we had previously obtained<sup>14</sup> the following expression for the normalized intensity moments:

$$M_1(\nu, z) = z^1 e^z \int_z^\infty dy (y-z)^1 y^{-2} e^{-y} [(y-z)\nu/\nu!], \quad (\text{A1})$$

with  $z$  as defined by Eq. (6.6). From these moments we can construct the ones evaluated in the present paper,  $M_1(\Delta, z)$ , by using the expansion

(2.7) of directed-angular-momentum states  $|j\alpha\rangle$  in terms of the states  $|jm\rangle$ . We obtain, with  $|\alpha|^2 = \Delta/N$  and  $j = \frac{1}{2}N$ ,

$$M_1(\Delta, z) = \left(1 + \frac{\Delta}{N}\right)^{-N} \sum_{\nu=0}^N \binom{N}{\nu} \left(\frac{\Delta}{N}\right)^\nu M_1(\nu, z). \quad (\text{A2})$$

Here we substitute Eq. (A1) and find

$$M_1(\Delta, z) = z^1 e^z \int_z^\infty dy (y-z)^1 y^{-2} e^{-y} \left(1 + \frac{\Delta}{N}\right)^{-N} \times \sum_{\nu=0}^N \frac{N!}{\nu! \nu! (N-\nu)!} \frac{m(y-z)}{N} \nu. \quad (\text{A3})$$

In the limit as  $N \rightarrow \infty$  this expression simplifies, since

$$\left(1 + \frac{\Delta}{N}\right)^{-N} \rightarrow e^{-\Delta}$$

and since the sum over  $\nu$  becomes the well-known power-series expansion of the Bessel function  $I_0[2[m(y-z)]^{1/2}]$ .<sup>20</sup> Therefore, as  $N \rightarrow \infty$ , Eq. (A3) becomes identical with Eq. (6.13).

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† A preliminary account of the results of this paper was presented in a talk at a summer school in Erice, Italy, May 1974 (see Ref. 10).

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